Perturbed Hammerstein integral inclusions with solutions that change sign

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Dedicated to Professor Espedito De Pascale on the occasion of his retirement.

Abstract. We establish new existence results for nontrivial solutions of some integral inclusions of Hammerstein type, that are perturbed with an affine functional. In order to use a theory of fixed point index for multivalued mappings, we work in a cone of continuous functions that are positive on a suitable subinterval of $[0,1]$. We also discuss the optimality of some constants that occur in our theory. We improve, complement and extend previous results in the literature.

Keywords: fixed point index, cone, nontrivial solution

Classification: Primary 45G10; Secondary 34A60, 34B10, 47H04, 47H10, 47H30

1. Introduction

Quite often differential inclusions arise in the study of problems in applied mathematics, engineering and economics, since some mathematical models utilize multivalued maps instead of single-valued maps [6], [12]. One approach to finding solutions of a boundary value multivalued problem is to re-write the problem as an integral inclusion and then to investigate the existence of solutions via different tools of nonlinear analysis, see for example [2], [18], [19], [36], [37].

When the problem is to find positive solutions of the corresponding integral operator, some authors use various generalizations [1], [3], [38], [39] of the well-known Guo-Krasnosel'skiĭ theorem on cone-compressions and expansions (see for example [17]). This is usually done by seeking the solutions in the cone of positive functions or in smaller cone included in it. This idea relies on the fact that the integral operator leaves the cone of positive functions invariant.

Here we establish new results for the existence of nontrivial solutions of integral inclusion of the type

$$u(t) \in \gamma(t)\alpha[u] + \int_0^1 k(t,s)F(s,u(s))\,ds, \tag{1.1}$$

where $\alpha[u]$ is a positive functional and both γ and k are allowed to change sign, so a positive solution need not exist. The integral equation corresponding to (1.1) has been studied in [24], motivated from the fact that it arises in the study of some heat flow problem.

Our approach relies on the theory of fixed point index for multivalued mappings [14], which is an extension of the well-known classical fixed point index. For our index calculations we work on the cone of functions that are positive on a suitable interval $[a,b]\subset[0,1]$, but are allowed to change sign elsewhere. This type of cone has never been used before in the multivalued case, not even in the simpler case of integral Hammerstein inclusions, that is when $\alpha[u]\equiv 0$. In order to obtain the existence of nontrivial solutions for multivalued boundary value problems we extend the theoretical results of [24] to the context of set-valued mappings.

We also establish new results for the multivalued, nonlocal BVP

$$-u''(t)\in F(t,u(t)),\quad \text{a.e. on}\quad [0,1], \tag{1.2}$$

$$u'(0)+\alpha[u]=0,\ \beta u'(1)+u(\eta)=0,\ \eta\in[0,1], \tag{1.3}$$

when $\alpha[u]$ is a positive functional given by $\alpha[u]=A_0+\int_0^1 u(s)\,dA(s)$, involving a Lebesgue-Stieltjes integral. This type of BC includes

$$\alpha[u]=\sum_{i=1}^{m}\alpha_i u(\xi_i)\quad\text{and}\quad \alpha[u]=\int_0^1\alpha(s)u(s)\,ds,$$

that is, multi-point and integral BCs, that are widely studied objects both in the single-valued (see for example [26], [35], [43]) and in the multivalued case (see for example [7], [8], [9], [10], [13]). We point out that the BVP (1.2)–(1.3) occurs, in the single-valued case, in the study of a thermostat model [24], [25], [41], [42].

By a more careful analysis of some of the constants that occur in the theory, we are able to obtain values that are, in some sense, optimal for this method. This follows the study done, for different ranges of the parameters and different BCs, for the single-valued case, in [30], [31], [32], [41], [42].

Our results are an improvement valid also in the single-valued case.

2. Preliminaries

We recall some definitions and facts here and refer to [5], [6], [11], [12], [14] for further information. Let $(X,\|\cdot\|)$ be a Banach space and $\mathcal{KC}(X)$ be the set of non-empty compact convex subsets of X.

A multivalued function $F:X\multimap X$ is said to be upper semicontinuous (u.s.c.) on X if for each $x\in X$, the set $F(x)$ is a non-empty closed subset of X and for each open set $V\subset X$, with $F(x)\subset V$, there exists an open neighborhood U of x such that $F(U)\subset V$. A multivalued map F is said to have closed graph if $x_n\to x$, $y_n\to y$, $y_n\in F(x_n)$ imply that $y\in F(x)$.

It is well-known that for a multivalued map F with non-empty compact values, the upper semicontinuity of F is equivalent to the fact that F maps compact sets into relatively compact sets and has closed graph.

Let J be a closed bounded interval of the real line. A multivalued map $F:J\to\mathcal{KC}(X)$ is said to be measurable if for each $x\in X$ the function $D:J\to\mathbb{R}$ defined by $D(t):=d(x,F(t))=\inf\{\|x-z\|\ z\in F(t)\}$ is measurable.

A multivalued map $F : J \times \mathbb{R} \multimap \mathbb{R}$ is said to be an upper-Carathéodory map if for each $u \in \mathbb{R}$, $t \mapsto F(t,u)$ is measurable and for almost every $t \in J$, $u \mapsto F(t,u)$ is upper semicontinuous.

For each $u \in C(J,\mathbb{R})$, the set of L^1-selections of a multivalued map $F : J\times\mathbb{R} \multimap \mathbb{R}$ is given by

$$S^1_{F,u} := \{f_u \in L^1(J,\mathbb{R}) : f_u(t) \in F(t,u(t)) \text{ for a.e. } t \in J\}.$$

The following result is due to Lasota-Opial [33].

Lemma 2.1. *Let J be a closed bounded interval of the real line, $F : J \times \mathbb{R} \to \mathcal{KC}(\mathbb{R})$, F be an upper-Carathéodory map, $S^1_{F,u}$ be non-empty for each $u \in C(J,\mathbb{R})$ and $\Theta : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ be a linear continuous operator. Then $\Theta \circ S^1_{F,\cdot} : C(J,\mathbb{R}) \multimap C(J,\mathbb{R})$, $u \mapsto \Theta(S^1_{F,u})$ is a closed graph map in $C(J,\mathbb{R}) \times C(J,\mathbb{R})$.*

Let K be a cone in a Banach space X, that is, K is a closed convex set such that $\lambda x \in K$ for $x \in K$ and $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When Ω is an open bounded subset of X we write $\Omega_K = \Omega \cap K$, an open subset of K.

We utilize an extension of the well-known fixed point index for single-valued compact maps (see for example [4], [17]) to the case of multivalued maps due to Fitzpatrick and Petryshyn [14]. The formal definition of this index is rather technical and involves a topological degree introduced by Ma [34]. We refer to the paper [14] for the details regarding the construction of the index and summarize in Theorem 2.2, in a similar way as in [40], its main properties. Fixed point indices for multivalued mappings in more general settings can be found in the monograph by Andres and Górniewicz [5].

Theorem 2.2. *Let K be a cone in X and let Ω be a bounded open set of X such that $\Omega_K \neq \emptyset$. Let $T : \overline{\Omega}_K \multimap K$ be an upper semicontinuous compact map. Suppose that $x \notin T(x)$ for all $x \in \partial_K\Omega$, the boundary of Ω relative to K. The fixed point index has the following properties:*

- (P_1) *(Existence) If $i_K(T,\Omega_K) \neq 0$, then T has a fixed point in Ω_K.*
- (P_2) *(Normalisation) If $u \in \Omega_K$, then $i_K(\hat{u},\Omega_K) = 1$, where $\hat{u}(x) = u$ for $x \in \overline{\Omega}_K$.*
- (P_3) *(Additivity) If V_1, V_2 are disjoint relatively open subsets of Ω_K such that $x \notin T(x)$ for $x \in \overline{\Omega}_K \setminus (V_1 \cup V_2)$, then*

$$i_K(T,\Omega_K) = i_K(T,V_1) + i_K(T,V_2).$$

- (P_4) *(Homotopy) Let $h : [0,1]\times\overline{\Omega}_K \multimap K$ be an upper semicontinuous compact map such that $x \notin h(t,x)$ for $x \in \partial_K\Omega$ and $t \in [0,1]$. Then*

$$i_K(h(0,.),\Omega_K) = i_K(h(1,.),\Omega_K).$$

Moreover, from the properties (P_1)–(P_4), *we have that*

(1) *if there exists* $e \in K \setminus \{0\}$ *such that* $x \notin Tx + \lambda e$ *for all* $x \in \partial_K \Omega$ *and all* $\lambda > 0$, *then* $i_K(T, \Omega_K) = 0$,

(2) *if* $\lambda x \notin Tx$ *for all* $x \in \partial_K \Omega$ *and all* $\lambda > 1$, *then* $i_K(T, \Omega_K) = 1$.

3. Existence of nontrivial solutions of perturbed Hammerstein integral inclusions

We study the existence of nonzero solutions of the integral inclusion

$$u(t) \in \gamma(t)\alpha[u] + \int_0^1 k(t,s)F(s,u(s))\,ds, \tag{3.1}$$

in the space $C[0,1]$ of continuous functions, endowed with the usual supremum norm. Although it is not standard, we use the notation

$$\lceil \Upsilon \rceil := \sup \Upsilon, \ \lfloor \Upsilon \rfloor := \inf \Upsilon, \quad \text{where} \quad \Upsilon \subset \mathbb{R},$$

because we believe that this improves the readability of the paper.

From now on, we assume that F, α, γ and the kernel k have the following properties:

(C_1) $F : [0,1] \times \mathbb{R} \to \mathcal{KC}([0,+\infty))$ is an upper-Carathéodory map such that for every $r > 0$, there exists a L^1 function $g_r : [0,1] \to [0,\infty)$ such that

$$\lceil F(t,u) \rceil \le g_r(t) \quad \text{for almost all} \quad t \in [0,1] \quad \text{and all} \quad u \in [-r,r].$$

(C_2) $k : [0,1] \times [0,1] \to \mathbb{R}$ is measurable, and for every $\tau \in [0,1]$ we have

$$\lim_{t \to \tau} \int_0^1 |k(t,s) - k(\tau,s)| g_r(s)\,ds = 0.$$

(C_3) There exist $[a,b] \subset [0,1]$, a L^∞ function $\Phi : [0,1] \to [0,\infty)$ and a constant $c_1 \in (0,1]$ such that

$$\begin{aligned} |k(t,s)| \le \Phi(s) \quad &\text{for} \quad t \in [0,1] \quad \text{and almost every} \quad s \in [0,1] \\ k(t,s) \ge c_1 \Phi(s) \quad &\text{for} \quad t \in [a,b] \quad \text{and almost every} \quad s \in [0,1]. \end{aligned}$$

(C_4) $\gamma : [0,1] \to \mathbb{R}$ is continuous and there exist a constant $c_2 \in (0,1]$ such that

$$\gamma(t) \ge c_2 \|\gamma\| \quad \text{for} \quad t \in [a,b].$$

(C_5) $\alpha : K \to \mathbb{R}^+$ is a continuous functional with

$$\alpha[u] = A_0 + \int_0^1 u(s)\,dA(s),$$

where dA is a Lebesgue-Stieltjes measure with $A_1 := \int_0^1 dA(s) < \infty$.

(C_6) The function $t \mapsto k(t,s)$ is integrable with respect to the measure dA, that is

$$\mathcal{K}(s) := \int_0^1 k(t,s)\,dA(t)$$

is well defined.

$$K = \{u \in C([0,1]) : \min\{u(t) : t \in [a,b]\} \geq c\|u\|\}, \tag{3.2}$$

where $c = \min\{c_1, c_2\}$. This type of cone was introduced in [23] and later used in [15], [16], [20], [21], [22], [24].

This is similar to, but larger than, a cone of non-negative functions used by Lan [28], which is a type of cone first used by Krasnosel'skiĭ, see e.g. [27], and D. Guo, see e.g. [17]. Note that functions in K are positive on the interval $[a,b]$ but may change sign on $[0,1]$. We write $K_r = \{u \in K : \|u\| < r\}$, $\overline{K}_r = \{u \in K : \|u\| \leq r\}$, and define

$$\Gamma = \int_0^1 \gamma(t)\,dA(t).$$

We consider now the multivalued map $T : K \multimap C([0,1])$ defined for $u \in K$ by

$$T(u) := \Big\{h \in C([0,1]) :\ \text{there exists } f_u \in S^1_{F,u} \text{ such that,} \\ \text{for every } \ t \in [0,1],\ h(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s) f_u(s)\,ds\Big\}.$$

We point out that T is well-defined, since (C_1) implies that the set $S^1_{F,u}$ is non-empty (see [12]).

Theorem 3.1. *If the hypotheses (C_1)–(C_6) hold for some $r > 0$, then T maps $\overline{K}_r$ into K. When these hypotheses hold for each $r > 0$, T maps K into K. Moreover, T is an u.s.c. compact map with non-empty convex compact values.*

Proof: Let $u \in \overline{K}_r$, $t \in [0,1]$ and $h \in T(u)$, Then we have,

$$|h(t)| \leq |\gamma(t)|\alpha[u] + \int_0^1 |k(t,s)| f_u(s)\,ds \leq |\gamma(t)|\alpha[u] + \int_0^1 \Phi(s) f_u(s)\,ds$$

so that

$$\|h\| \leq \|\gamma\|\alpha[u] + \int_0^1 \Phi(s) f_u(s)\,ds. \tag{3.3}$$

Also

$$\min_{t\in[a,b]}\{h(t)\} \geq c_2\|\gamma\|\alpha[u] + c_1\int_0^1 \Phi(s)f_u(s)\,ds$$
$$\geq c\Big[\|\gamma\|\alpha[u] + \int_0^1 \Phi(s)f_u(s)\,ds\Big] \geq c\|h\|,$$

where $c = \min\{c_1, c_2\}$. Hence $Tu \subset K$ for every $u \in \overline{K}_r$.

To see that Tu is a convex set for all $u \in K$, let $h_1, h_2 \in Tu$. Then there are f_u^1 and f_u^2 in $S^1_{F,u}$ such that for all $t \in [0,1]$

$$h_1(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)f_u^1(s)\,ds$$

and

$$h_2(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)f_u^2(s)\,ds.$$

Let $\lambda \in (0,1)$. We have

$$\lambda h_1(t) + (1-\lambda)h_2(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)[\lambda f_u^1(s) + (1-\lambda)f_u^2(s)]\,ds,$$

with $\lambda f_u^1(s) + (1-\lambda)f_u^2(s) \in S^1_{F,u}$ because $S^1_{F,u}$ is convex (see [33]).

Now, we show that the multimap T is compact. Firstly, we show that T sends bounded sets into bounded sets. It is enough to see that $T(\overline{K}_r)$ is bounded. Let $u \in \overline{K}_r$ and $h \in T(u)$. Then, for all $t \in [0,1]$ and some $f_u \in S^1_{F,u}$, from (3.3) we have

$$\|h\| \leq \|\gamma\|\alpha[u] + \int_0^1 \Phi(s)g_r(s)\,ds \leq \|\gamma\|(A_0 + rA_1) + M_r,$$

for some $0 \leq M_r < \infty$.

We prove now that T sends bounded sets into equicontinuous sets. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in \overline{K}_r$ and $h \in T(u)$. Then

$$|h(t_1) - h(t_2)| \leq |\gamma(t_1) - \gamma(t_2)|\alpha[u] + \int_0^1 |k(t_1,s) - k(t_2,s)|f_u(s)\,ds$$
$$\leq |\gamma(t_1) - \gamma(t_2)|\alpha[u] + \int_0^1 |k(t_1,s) - k(t_2,s)|g_r(s)\,ds.$$

Then $|h(t_1) - h(t_2)| \to 0$ when $t_1 \to t_2$. By the Ascoli-Arzelà Theorem we can conclude that T is a compact map.

Finally, we show that T has closed graph. Let $u_n, u_0 \in K$, $u_n \to u_0$, $h_n \in T(u_n)$, $h_n \to h_0$. We have to prove that $h_0 \in T(u_0)$, that is, there exists $f_{u_0} \in S^1_{F,u_0}$ such that for all $t \in [0,1]$, $h_0(t) = \gamma(t)\alpha[u_0] + \int_0^1 k(t,s)f_{u_0}(s)\,ds$. Since $h_n \in$

$T(u_n)$, there exists $f_{u_n} \in S^1_{F,u_n}$ such that for all $t \in [0,1]$, $h_n(t) = \gamma(t)\alpha[u_n] + \int_0^1 k(t,s) f_{u_n}(s)\, ds$. From Lemma 2.1, applied to the integral operator

$$\Theta : L^1([0,1], R) \to C([0,1], R), \quad w \mapsto \int_0^1 k(t,s) w(s)\, ds,$$

we obtain that the operator

$$\Theta \circ S^1_{F,\cdot} : C([0,1], R) \multimap C([0,1], R), \quad u \mapsto \left\{ \int_0^1 k(t,s) f_u(s)\, ds : f_u \in S^1_{F,u} \right\}$$

has closed graph. So, because $h_n - \gamma\alpha[u_n] \in \Theta \circ S^1_{F,u_n}$ and $h_n - \gamma\alpha[u_n] \to h_0 - \gamma\alpha[u_0]$, we have that $h_0 - \gamma\alpha[u_0] \in \Theta \circ S^1_{F,u_0}$, that is, there exists $f_{u_0} \in S^1_{F,u_0}$ such that for all $t \in [0,1]$,

$$h_0(t) = \gamma(t)\alpha[u_0] + \int_0^1 k(t,s) f_{u_0}(s)\, ds.$$

□

Let $q : C([0,1]) \to \mathbb{R}$ denote the continuous function $q(u) = \min\{u(t) : t \in [a,b]\}$. We shall use the open set $V_\rho = \{u \in K : q(u) < \rho\}$. V_ρ is equal to the set called $\Omega_{\rho/c}$ in [22]. Note that $K_\rho \subset V_\rho \subset K_{\rho/c}$.

Firstly we prove that the index is 0 on the set V_ρ; this is an extension of Lemma 2.4 of [24] to the multivalued case.

Lemma 3.2. *Assume that there exists $\rho > 0$ such that $u \notin Tu$ for $u \in \partial V_\rho$ and*

(I^0_ρ) *there exists a measurable function $\psi_\rho : [a,b] \to \mathbb{R}_+$ such that*

$$\lfloor F(t,u) \rfloor \geq \rho\, \psi_\rho(t) \quad \text{for all} \quad u \in [\rho, \rho/c] \quad \text{and almost all} \quad t \in [a,b],$$

$$\alpha[u] \geq \alpha_0 \rho \quad \text{for} \quad u \in \partial V_\rho, \tag{3.4}$$

and

$$c_2 \|\gamma\| \alpha_0 + \inf_{t \in [a,b]} \int_a^b k(t,s) \psi_\rho(s)\, ds \geq 1. \tag{3.5}$$

Then we have $i_K(T, V_\rho) = 0$.

Proof: Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $e \in K$. We prove that

$$u \notin T(u) + \lambda e \quad \text{for all} \quad u \in \partial V_\rho \quad \text{and} \quad \lambda > 0,$$

which ensures, by Theorem 2.2, that the index is 0 on the set V_ρ. In fact, if not, there exist $u \in \partial V_\rho$ and $\lambda > 0$ such that $u - \lambda e \in Tu$. Then there exists $f_u \in S^1_{F,u}$

such that for all $t \in [a,b]$

$$\begin{aligned} u(t) &= \gamma(t)\alpha[u] + \int_0^1 k(t,s) f_u(s)\,ds + \lambda \\ &\geq c_2 \|\gamma\| \alpha_0 \rho + \rho \int_a^b k(t,s)\psi_\rho(s)\,ds + \lambda. \end{aligned}$$

By (I_ρ^0), this implies that $q(u) \geq \rho + \lambda > \rho$ contradicting the fact that $u \in \partial V_\rho$. $\square$

We next prove that the index is 1 on the set K_ρ; this is an extension of Lemma 2.6 of [24] to the multivalued case.

Lemma 3.3. *Suppose* $\Gamma < 1$ *and assume that there exists* $\rho > 0$ *such that* $u \notin Tu$ *for all* $u \in \partial K_\rho$ *and*

(I_ρ^1) *there exists a measurable function* $\phi_\rho : [0,1] \to \mathbb{R}_+$ *such that* $\mathcal{K}\phi_\rho \in L^1([0,1])$,

$$\lceil F(t,u) \rceil \leq \rho\, \phi_\rho(t) \quad \textit{for all } u \in [-\rho,\rho] \quad \textit{and almost all } t \in [0,1],$$

and

$$\frac{A_0\|\gamma\|}{(1-\Gamma)\rho} + \frac{\|\gamma\|}{(1-\Gamma)} \int_0^1 \mathcal{K}(s)\phi_\rho(s)\,ds + \sup_{t\in[0,1]} \int_0^1 |k(t,s)|\phi_\rho(s)\,ds \leq 1. \tag{3.6}$$

Then we have $i_K(T, K_\rho) = 1$.

Proof: We show that $\lambda u \notin Tu$ for every $u \in \partial K_\rho$ and for every $\lambda > 1$; this ensures, by Theorem 2.2, that the index is 1 on K_ρ. In fact, if there exists $\lambda > 1$ and $u \in \partial K_\rho$ such that $\lambda u \in Tu$ then for some $f_u \in S^1_{F,u}$

$$\lambda u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s) f_u(s)\,ds. \tag{3.7}$$

Therefore we have

$$\lambda \int_0^1 u(t)\,dA(t) = \alpha[u]\Gamma + \int_0^1 \mathcal{K}(s) f_u(s)\,ds. \tag{3.8}$$

Hence

$$(\lambda - \Gamma)\alpha[u] = \lambda A_0 + \int_0^1 \mathcal{K}(s) f_u(s)\,ds.$$

Substituting into (3.7) gives

$$\lambda u(t) = \frac{\lambda A_0 \gamma(t)}{\lambda - \Gamma} + \frac{\gamma(t)}{\lambda - \Gamma} \int_0^1 \mathcal{K}(s) f_u(s)\,ds + \int_0^1 k(t,s) f_u(s)\,ds.$$

Taking the absolute value and then the supremum for $t \in [0,1]$ yields

$$\lambda\rho \leq \frac{\lambda A_0\|\gamma\|}{\lambda-\Gamma} + \frac{\|\gamma\|}{\lambda-\Gamma}\int_0^1 \mathcal{K}(s)f_u(s)\,ds + \sup_{t\in[0,1]}\int_0^1 |k(t,s)|f_u(s)\,ds.$$

Thus we have, since $\lambda > 1$,

$$\rho < \frac{A_0\|\gamma\|}{1-\Gamma} + \frac{\|\gamma\|}{1-\Gamma}\int_0^1 \mathcal{K}(s)\rho\phi_\rho(s)\,ds + \sup_{t\in[0,1]}\int_0^1 |k(t,s)|\rho\phi_\rho(s)\,ds. \tag{3.9}$$

This contradicts (3.6) and proves the result. □

We can now state the following new result on the existence of multiple nonzero solutions for equation (3.1).

Theorem 3.4. *Equation* (3.1) *has a nonzero solution in K if either of the following conditions hold.*

- (H_1) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that* $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$, *$u \notin Tu$ for $u \in \partial V_{\rho_2}$.*
- (H_2) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < c\rho_2$ such that* $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$, *$u \notin Tu$ for $u \in \partial K_{\rho_2}$.*

Equation (3.1) *has two nonzero solutions in K if one of the following conditions hold.*

- (D_1) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < \rho_2$ and $\rho_2 < c\rho_3$ such that* $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$, *$u \notin Tu$ for $u \in \partial V_{\rho_2}$ and* $(\mathrm{I}^1_{\rho_3})$ *hold.*
- (D_2) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < c\rho_2 < c\rho_3$ such that* $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$, *$u \notin Tu$ for $u \in \partial K_{\rho_2}$ and* $(\mathrm{I}^0_{\rho_3})$ *hold.*

Moreover, if in (D_1), strict inequality holds in $(\mathrm{I}^1_{\rho_1})$, *then equation* (3.1) *has a third solution $u_0 \in K_{\rho_1}$ (possibly zero).*

We omit the proof which follows simply from properties of fixed point index, for details of similar proofs see [22], [28].

Remark 3.5. It is possible to state results for three or more nonzero solutions by expanding the lists in conditions $(D_1), (D_2)$, see [29] for the type of result that may be stated.

4. Nonzero solutions of some BVP

We now consider the BVP

$$-u''(t) \in F(t,u(t)), \text{ a.e. on } [0,1], \tag{4.1}$$

with boundary conditions

$$u'(0) + \alpha[u] = 0,\ \beta u'(1) + u(\eta) = 0,\ \eta \in [0,1]. \tag{4.2}$$

The solution of $-u''=y$ under these BCs can be written

$$u(t)=(\beta+\eta-t)\alpha[u]+\beta\int_0^1 y(s)\,ds+\int_0^\eta(\eta-s)y(s)\,ds-\int_0^t(t-s)y(s)\,ds.$$

By a solution of the BVP (4.1)–(4.2) we mean a solution $u\in C[0,1]$ of the corresponding integral inclusion

$$u(t)\in(\beta+\eta-t)\alpha[u]+\int_0^1 k(t,s)F(s,u(s))\,ds,$$

where

$$k(t,s)=\beta+\begin{cases}\eta-s, & s\le\eta\\ 0, & s>\eta\end{cases}-\begin{cases}t-s, & s\le t\\ 0, & s>t.\end{cases}\tag{4.3}$$

Note that $k(t,s)$ in (4.3) is the kernel for the special case $\alpha[u]\equiv 0$, studied in the case of $F(t,u)$ single-valued in [25].

Here we discuss the case $\beta>0$ and $\beta+\eta<1$. When $0<\beta+\eta<1$ there cannot exist positive solutions for all positive right-hand sides, but there are nonzero solutions that are positive on an interval $[0,b]$ for any b with $0<b<\beta+\eta<1$.

We point out that, with the same technique one can prove, for different ranges of the parameter β the existence of solutions that are positive, negative and negative on an interval, see Remark 3.4 of [24] for the results that can be stated.

Upper bounds. Upper bounds for $|k(t,s)|$ and $\gamma(t)$ were given in [24], [25] as follows

$$\Phi(s)=\|\gamma\|=\begin{cases}\beta+\eta, & \text{for }\ \beta+\eta\ge\frac12,\\ 1-(\beta+\eta), & \text{for }\ \beta+\eta<\frac12.\end{cases}$$

Lower bounds. We take $[a,b]$ with $0\le a<b<\eta+\beta$. Note that in $[a,b]$, $\gamma(t)$ is a decreasing function of t and $\min_{t\in[a,b]}\gamma(t)=\beta+\eta-b$. A simple calculation shows that

$$\min_{t\in[a,b]}k(t,s)=k(b,s)\ge\begin{cases}\beta, & \text{for }\ b\le\eta,\\ \beta+\eta-b, & \text{for }\ b>\eta.\end{cases}$$

Thus we can choose

$$c=\begin{cases}\beta/(\beta+\eta), & \text{for }\ b\le\eta,\ \beta+\eta\ge\frac12,\\ \beta/(1-(\beta+\eta)), & \text{for }\ b\le\eta,\ \beta+\eta<\frac12,\\ (\beta+\eta-b)/(\beta+\eta), & \text{for }\ b>\eta,\ \beta+\eta\ge\frac12,\\ (\beta+\eta-b)/(1-(\beta+\eta)), & \text{for }\ b>\eta,\ \beta+\eta<\frac12.\end{cases}\tag{4.4}$$

Henceforth we work on the cone

$$K = \{u \in C[0,1],\ \min_{t\in[a,b]} u(t) \geq c\|u\|\},$$

with c as in (4.4).

For brevity and clarity, we state a result for the existence of one nontrivial solution when $F(t,u) = g(t)H(u)$, for which the hypotheses are easier to check. Of course there are more general results, including existence of multiple nonzero solutions, analogous to Theorem 3.4.

When $F(t,u) = g(t)H(u)$ where $\Phi g \in L^1[0,1]$ and H is upper semicontinuous, then $\psi_\rho(s) = g(s)H_{\rho,\rho/c}$, where $H_{\rho,\rho/c} = \inf\{\lfloor H(u)\rfloor/\rho : \rho \leq u \leq \rho/c\}$. Then (3.5) reads more simply

$$c_2\|\gamma\|\alpha_0 + H_{\rho,\rho/c}\cdot\frac{1}{M} \geq 1, \tag{4.5}$$

where

$$\frac{1}{M} = \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds. \tag{4.6}$$

Moreover we have $\phi_\rho(s) = g(s)H^{-\rho,\rho}$ where $H^{-\rho,\rho} = \sup\{\lceil H(u)\rceil/\rho : -\rho \leq u \leq \rho\}$. Then (3.6) reads more simply

$$\frac{A_0\|\gamma\|}{\rho(1-\Gamma)} + \Big(\frac{\|\gamma\|}{1-\Gamma}\int_0^1 \mathcal{K}(s)g(s)\,ds + \frac{1}{m}\Big)H^{-\rho,\rho} \leq 1, \tag{4.7}$$

where

$$\frac{1}{m} = \sup_{t\in[0,1]} \int_0^1 |k(t,s)|g(s)\,ds. \tag{4.8}$$

Remark 4.1. We point out that all the constants that appear in (4.5) and (4.7) can be computed. This was shown, in a single-valued case, for the four-point BVP corresponding to $\alpha[u] = \alpha u(\xi)$ with $\xi \in [0,b]$, see Remark 3.1 and Example 3.2 of [24].

Theorem 4.2. *Let $0 \leq a < b < \beta + \eta < 1$, and suppose that $\int_a^b \Phi(s)g(s)\,ds > 0$. Let c be as in (4.4). Let m be as in (4.8) and M as in (4.6). Then the BVP (4.1)–(4.2) has at least one nonzero solution, positive on $[0,b]$, if either one of the following conditions hold.*

- (S_1) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $H^{-\rho_1,\rho_1}$ satisfies (4.7) and $H_{\rho_2,\rho_2/c}$ satisfies $(I^0_{\rho_2})$, with (3.5) replaced by (4.5).*
- (S_2) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < c\rho_2$ such that $H_{\rho_1,\rho_1/c}$ satisfies $(I^0_{\rho_1})$, with (3.5) replaced by (4.5) and $H^{-\rho_2,\rho_2}$ satisfies (4.7).*

5. Optimal constants

We now assume that $g \equiv 1$ and we seek the 'optimal' $[a,b]$ for which $M(a,b)$ is a minimum. This type of problem has been tackled in the past for different BCs [30], [31], [32], [41], [42]. In particular, Webb [41], [42] studied the case when $\beta+\eta \geq 1$, that leads to positive solutions. Here we assume that $\beta+\eta<1$, $\beta>0$ and $\eta>0$. The value $1/m$ was given in [24], as follows

$$1/m = \sup_{t\in[0,1]} \int_0^1 |k(t,s)|\,ds = \max\Big\{\beta+\frac{1}{2}\eta^2, \beta^2-\beta+\frac{1}{2}(1-\eta^2)\Big\}.$$

For arbitrary $0 \leq a < b < \beta+\eta$, the kernel $k(t,s)$ is a positive, non-increasing function of t. Thus

$$1/M(a,b) = \min_{t\in[a,b]} \int_a^b k(t,s)\,ds = \int_a^b k(b,s)\,ds.$$

Note that $\inf_{0\leq a<b} M(a,b) = M(0,b)$ and

$$1/M(0,b) = \int_0^b k(b,s)\,ds = \begin{cases} -b^2+b\beta+\eta b, & \text{if } \ b\leq\eta, \\ -b^2/2+\beta b+\eta^2/2, & \text{if } \ b\geq\eta. \end{cases}$$

Now we have

$$\max_{0<b\leq\eta} \{\beta b+\eta b-b^2\} = \max\{(\beta+\eta)^2/4, \beta\eta\} = \begin{cases} (\beta+\eta)^2/4, & \text{if } \ \beta\leq\eta, \\ \beta\eta, & \text{if } \ \beta\geq\eta, \end{cases}$$

and

$$\max_{0<\eta\leq b} \{-b^2/2+\beta b+\eta^2/2\} = \max\{(\beta^2+\eta^2)/2, \beta\eta\} = \begin{cases} \beta\eta, & \text{if } \ \beta\leq\eta, \\ (\beta^2+\eta^2)/2, & \text{if } \ \beta\geq\eta. \end{cases}$$

This implies that

$$1/M_{opt} = \begin{cases} (\beta+\eta)^2/4, & \text{if } \ \beta\leq\eta, \\ (\beta^2+\eta^2)/2, & \text{if } \ \beta\geq\eta, \end{cases}$$

and therefore we choose

$$[a,b] = \begin{cases} [0,(\beta+\eta)/2], & \text{if } \ \beta\leq\eta, \\ [0,\beta], & \text{if } \ \beta\geq\eta. \end{cases}$$

Finally, in the following table, we compute some values for the constants m, M_{opt}, c for the optimal $[a, b]$, when η and β vary.

η	β	m	M_{opt}	$[a,b]$	c
1/5	1/5	25/8	25	$[0,1/5]$	1/3
	2/5	50/21	10	$[0,2/5]$	1/3
	3/5	50/31	5	$[0,3/5]$	1/4
2/5	1/5	25/7	100/9	$[0,3/10]$	1/2
	2/5	25/12	25/4	$[0,2/5]$	1/2
3/5	1/5	50/19	25/4	$[0,2/5]$	1/4

References

[1] Agarwal R.P., Avery R., Henderson J., O'Regan D., *The five functionals fixed point theorem generalized to multivalued maps*, J. Nonlinear Convex Anal. **4** (2003), 455–462.

[2] Agarwal R.P., O'Regan D., *A note on the existence of multiple fixed points for multivalued maps with applications*, J. Differential Equations **160** (2000), 389–403.

[3] Agarwal R.P., O'Regan D., *A generalization of the Petryshyn-Leggett-Williams fixed point theorem with applications to integral inclusions*, Appl. Math. Comput. **123** (2001), 263–274.

[4] Amann H., *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[5] Andres J., Górniewicz L., *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer Academic Publishers, Dordrecht, 2003.

[6] Aubin J.P., Cellina A., *Differential Inclusions*, Springer, Berlin, 1984.

[7] Benchohra M., Ntouyas S.K., *A note on a three point boundary value problem for second order differental inclusions*, Math. Notes (Miskolc) **2** (2001), 39–47.

[8] Benchohra M., Ouahab A., *Upper and lower solutions method for differential inclusions with integral boundary conditions*, J. Appl. Math. Stoch. Anal. **2006**, Art. ID 10490, 10 pp.

[9] Dhage B.C., Graef J.R., *On boundary-value problems for second order perturbed differential inclusions*, Appl. Anal. **84** (2005), 953–970.

[10] Dhage B.C., Ntouyas S.K., Cho Y.J., *On the second order discontinuous differential inclusions*, J. Appl. Funct. Anal. **1** (2006), 469–476.

[11] Deimling K., *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[12] Deimling K., *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.

[13] Erbe L., Ma R., Tisdell C.C., *On two point boundary value problems for second order differential inclusions*, Dynam. Systems Appl. **15** (2006), 79–88.

[14] Fitzpatrick P.M., Petryshyn W.V., *Fixed point theorems and the fixed point index for multivalued mappings in cones*, J. London Math. Soc. **12** (1975/76), 75–85.

[15] Franco D., Infante G., O'Regan D., *Positive and nontrivial solutions for the Urysohn integral equation*, Acta Math. Sin. (Engl. Ser.) **22** (2006), 1745–1750.

[16] Franco D., Infante G., O'Regan D., *Nontrivial solutions in abstract cones for Hammerstein integral systems*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **14** (2007), 837–850.

[17] Guo D., Lakshmikantham V., *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[18] Hong S., Wang L., *Existence of solutions for integral inclusions*, J. Math. Anal. Appl. **317** (2006), 429–441.
[19] Hong S., *Multiple positive solutions for a class of integral inclusions*, J. Comput. Appl. Math. **214** (2008), 19–29.
[20] Infante G., *Eigenvalues of some non-local boundary-value problems*, Proc. Edinb. Math. Soc. **46** (2003), 75–86.
[21] Infante G., *Nonzero solutions of second order problems subject to nonlinear BCs*, Dynamic systems and applications. Vol. 5, Dynamic, Atlanta, GA, (2008), 222–226.
[22] Infante G., Webb J.R.L., *Nonzero solutions of Hammerstein integral equations with discontinuous kernels*, J. Math. Anal. Appl. **272** (2002), 30–42.
[23] Infante G., Webb J.R.L., *Three point boundary value problems with solutions that change sign*, J. Integral Equations Appl. **15** (2003), 37–57.
[24] Infante G., Webb J.R.L., *Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations*, Proc. Edinb. Math. Soc. **49** (2006), 637–656.
[25] Infante G., Webb J.R.L., *Loss of positivity in a nonlinear scalar heat equation*, NoDEA Nonlinear Differential Equations Appl. **13** (2006), 249–261.
[26] Karakostas G.L., Tsamatos P.Ch., *Existence of multiple positive solutions for a nonlocal boundary value problem*, Topol. Methods Nonlinear Anal. **19** (2002), 109–121.
[27] Krasnosel'skiĭ M.A., Zabreĭko P.P., *Geometrical Methods of Nonlinear Analysis*, Springer, Berlin, 1984.
[28] Lan K.Q., *Multiple positive solutions of Hammerstein integral equations with singularities*, Diff. Eqns and Dynam. Syst. **8** (2000), 175–195.
[29] Lan K.Q., *Multiple positive solutions of semilinear differential equations with singularities*, J. London Math. Soc. **63** (2001), 690–704.
[30] Lan K.Q., *Positive characteristic values and optimal constants for three-point boundary value problems*, Differential & Difference Equations and Applications, 623–633, Hindawi Publ. Corp., New York, 2006.
[31] Lan K.Q., *Properties of kernels and multiple positive solutions for three-point boundary value problems*, Appl. Math. Lett. **20** (2007), 352–357.
[32] Lan K.Q., Yang G.C., *Optimal constants for two point boundary value problems*, Discrete Contin. Dyn. Syst., **suppl.** (2007), 624–633.
[33] Lasota A., Opial Z., *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **13** (1965), 781–786.
[34] Ma T., *Topological degrees of set-valued compact fields in locally convex spaces*, Dissertationes Math. (Rozprawy Mat.) **92** (1972), 1–42.
[35] Ma R., Castaneda N., *Existence of solutions of nonlinear m-point boundary value problems*, J. Math. Anal. Appl. **256** (2001), 556–567.
[36] Marino G., *Nonlinear boundary value problems for multivalued differential equations in Banach spaces*, Nonlinear Anal. **14** (1990), 545–558.
[37] O'Regan D., *Integral inclusions of upper semi-continuous or lower semi-continuous type*, Proc. Amer. Math. Soc. **124** (1996), 2391–2399.
[38] O'Regan D., Zima M., *Leggett-Williams norm-type fixed point theorems for multivalued mappings*, Appl. Math. Comput. **187** (2007), 1238-1249.
[39] O'Regan D., Zima M., *Leggett-Williams theorems for coincidences of multivalued operators*, Nonlinear Anal. **68** (2008), 2879–2888.
[40] Webb J.R.L., *Positive solutions of some three point boundary value problems via fixed point index theory*, Nonlinear Anal. **47** (2001), 4319–4332.
[41] Webb J.R.L., *Multiple positive solutions of some nonlinear heat flow problems*, Discrete Contin. Dyn. Syst., suppl. (2005), 895–903.

[42] Webb J.R.L., *Optimal constants in a nonlocal boundary value problem*, Nonlinear Anal. **63** (2005), 672–685.

[43] Webb J.R.L., *Fixed point index and its application to positive solutions of nonlocal boundary value problems*, Seminar of Mathematical Analysis, Univ. Sevilla Secr. Publ., Seville, 2006, pp. 181–205.

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