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A note on G_δ ideals of compact sets

Maya Saran

Abstract. Solecki has shown that a broad natural class of G_δ ideals of compact sets can be represented through the ideal of nowhere dense subsets of a closed subset of the hyperspace of compact sets. In this note we show that the closed subset in this representation can be taken to be closed upwards.

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Let E be a compact Polish space and let $\mathcal{K}(E)$ denote the hyperspace of its compact subsets, equipped with the Vietoris topology. A set $I \subseteq \mathcal{K}(E)$ is an *ideal* of compact sets if it is closed under the operations of taking subsets and finite unions. An ideal I is a σ*-ideal* if it is also closed under countable unions whenever the union itself is compact. Ideals of compact sets arise commonly in analysis out of various notions of smallness; see [3] for a survey of results and applications.

Following [4], we say that an ideal I has *property* $(*)$ if, for any sequence of sets $K_n \in I$, there exists a G_δ set G such that $\bigcup_n K_n \subseteq G$ and $\mathcal{K}(G) \subseteq I$. Property $(*)$ holds in a broad class of G_δ ideals that includes all natural examples, including the ideals of compact meager sets, measure-zero sets, sets of dimension $\leq n$ for fixed $n \in \mathbb{N}$, and Z-sets. (See [4] for these and other examples and a discussion of property $(*)$.) Solecki has shown in [4] that any ideal in this class can be represented via the meager ideal of some closed subset of $\mathcal{K}(E)$. The following definition is essential for the representation: for $A \subseteq E$,

$$A^* = \{K \in \mathcal{K}(E) : K \cap A \neq \emptyset\}.$$

Theorem 1 (Solecki)**.** *Suppose I is coanalytic and non-empty. Then I has property $(*)$ iff there exists a closed set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that, for any $K \in \mathcal{K}(E)$,*

$$K \in I \iff K^* \cap \mathcal{F} \text{ is meager in } \mathcal{F}.$$

This representation is analogous to a result of Choquet [1] that establishes a correspondence between alternating capacities of order ∞ on E and probability Borel measures on $\mathcal{K}(E)$.

Note that the set $\mathcal{F}$ in Theorem 1 is not unique. We hope to determine properties for $\mathcal{F}$ that make it a canonical representative, perhaps up to some notion of equivalence. One property of interest is that of being *closed upwards*, i.e., for any $A, B \in \mathcal{K}(E)$, if $B \supseteq A \in \mathcal{F}$ then $B \in \mathcal{F}$. This property ensures that the map $K \mapsto K^* \cap \mathcal{F}$, a fundamental function in this context, is continuous. In some examples of G_δ ideals with property $(*)$, the natural choice of the set $\mathcal{F}$ is in fact closed upwards. For example, let μ be an atomless finite probability measure on E and let I be the σ-ideal of compact μ-null sets. Assume that $\mu(U) > 0$ for all non-empty open $U \subseteq E$, so that all sets in I have empty interior. Fix a countable basis of the topology on E and let $s \in (0,1)$ be chosen so that it is not the measure of any finite union of basic sets. Then the set $\mathcal{F} = \{K \in \mathcal{K}(E) : \mu(K) \geq s\}$ works to characterize membership in the ideal.

In the following result we show that as long as the ideal I in Theorem 1 contains only meager sets, we may always find an $\mathcal{F}$ representing it that is closed upwards. We use the following notation in the proof: if $A \subseteq E$ and $\delta > 0$, $A + \delta$ denotes the set $\bigcup_{x \in A} B(x, \delta)$. $\mathrm{Int}(A)$ denotes the interior of A in E.

Theorem 2. *For a non-empty closed set $\mathcal{F} \subseteq \mathcal{K}(E)$, the following are equivalent:*

(1) $\forall K \in \mathcal{K}(E)$, *$K$ has non-empty interior* $\Rightarrow K^*$ *non-meager in* $\mathcal{F}$*;*
(2) $\exists \mathcal{F}' \subseteq \mathcal{K}(E)$, *non-empty, closed and closed upwards, such that*

$$\forall K \in \mathcal{K}(E), K^* \text{ non-meager in } \mathcal{F}' \iff K^* \text{ non-meager in } \mathcal{F}.$$

Proof: It is clear that (2)$\Rightarrow$(1), simply because, if $\mathcal{F}' \subseteq \mathcal{K}(E)$ is non-empty and closed upwards, and $U \subseteq E$ is non-empty and open, then $\mathcal{F}' \cap U^*$ is non-empty and open in $\mathcal{F}'$. To prove the other direction, let

$$I = \{K \in \mathcal{K}(E) : K^* \text{ is meager in } \mathcal{F}\}.$$

I is a σ-ideal with property $(*)$. Let $\{\mathcal{V}_n\}$ be a basis of non-empty sets for the relative topology on $\mathcal{F}$, and let $\mathcal{K}_n = \overline{\mathcal{V}_n}$. We now have:

$$K \in I \quad \Rightarrow \quad \forall n,\ K^* \text{ meager in } \mathcal{K}_n; \tag{1}$$

$$K \notin I \quad \Rightarrow \quad \exists n,\ \mathcal{K}_n \subseteq K^*. \tag{2}$$

Assume that I contains some infinite set. In this case, we fix a sequence $\{x_i\}$ and a point $x \in E$ such that the x_i are all distinct, $x_i \to x$, $\{x\} \in I$ and each $\{x_i\} \in I$. (We can just pick the x_i from some fixed infinite set in I.) Let U_i' be open such that $x_i \in U_i'$, $\overline{U_i'} \to \{x\}$ and the sets $\overline{U_i'}$ are pairwise disjoint. We will pick a subsequence U_{n_i}' and define sets (U_i, F_i, W_i), $i \in \mathbb{N}$, satisfying each of these conditions:

- U_i, W_i are open,
- $U_i \subseteq U_{n_i}'$, so the sets $\overline{U_i}$ are pairwise disjoint,
- $F_i \in \mathcal{K}_i$,
- $F_i \subseteq W_i$,
- if $j \leq i$ then $\overline{W_j} \cap \overline{U_i} = \emptyset$.

Let $n_0 = 0$ and note that since $\{x, x_0\} \in I$, $\mathcal{K}_0 \nsubseteq \{x, x_0\}^*$. Let F_0 be a set in $\mathcal{K}_0 \nsubseteq \{x, x_0\}^*$. Let W_0 be an open superset of F_0 such that $x, x_0 \notin \overline{W_0}$, and let $U_0 \subseteq U_0'$ be an open set containing x_0 such that $\overline{U_0} \cap \overline{W_0} = \emptyset$.

Pick $n_1 > 0$ such that for every $m \geq n_1$, $\overline{W_0} \cap \overline{U_m'} = \emptyset$.

To define (U_i, F_i, W_i) for $i > 0$, consider $\mathcal{K}_i$ and U_{n_i}'. Again, we may pick $F_i \in \mathcal{K}_i \setminus \{x, x_{n_i}\}^*$. Let $W_i \supseteq F_i$ be open such that $x, x_{n_i} \notin \overline{W_i}$. Let $U_i \subseteq U_{n_i}'$ be an open set containing x_{n_i} and such that $\overline{U_i} \cap \overline{W_i} = \emptyset$. Pick $n_{i+1} > n_i$ such that for any $m \geq n_{i+1}$, $\overline{W_i} \cap \overline{U_m'} = \emptyset$.

Now note that

$$\begin{aligned} K \in I \quad &\Rightarrow \quad \forall\, n,\ K^* \text{ meager in } \mathcal{K}_n \cap \mathcal{K}(W_n); \\ K \notin I \quad &\Rightarrow \quad \exists\, n,\ \mathcal{K}_n \cap \mathcal{K}(W_n) \subseteq K^*. \end{aligned}$$

In other words, conditions (1) and (2) hold with the sets $\mathcal{K}_n$ replaced by the sets $\mathcal{K}_n \cap \mathcal{K}(W_n)$. Therefore we may simply assume that $\mathcal{K}_n \subseteq \mathcal{K}(W_n)$.

We now define $\mathcal{L} \subseteq \mathcal{K}(E)$. For $n, j \in \mathbb{N}$, first define closed sets

$$A_{n,j} = \begin{cases} \overline{U_j} & \text{if } \ j < n, \\ E \setminus \bigcup_{i<n} (U_i + 1/j) & \text{if } \ j \geq n. \end{cases}$$

Also, for every $n \in \mathbb{N}$, let $U_{n,j}$, $j \in \mathbb{N}$, be non-empty disjoint open subsets of U_n. (This is possible because, since $\{x_n\}$ is not open, it must be a limit point of E.)

Define sets $\mathcal{L}_{n,j}$ as follows: for $L \in \mathcal{K}(E)$,

$$L \in \mathcal{L}_{n,j} \iff \exists\, F \in \mathcal{K}_n \ \text{ such that } \ F \cap A_{n,j} \subseteq L \ \textit{ and } \ L \ \text{ intersects } \ U_{n,j}.$$

Let $\mathcal{L} = \bigcup_{n,j} \mathcal{L}_{n,j}$. Since each $\mathcal{L}_{n,j}$ is closed upwards, so is $\mathcal{L}$.

Claim: $K \in I \iff K^*$ is nowhere dense in $\mathcal{L}$.

Let $K \in I$. We want to show that $\mathcal{L} \setminus K^*$ is dense in $\mathcal{L}$. Let $L_1 \in \mathcal{L}_{n,j}$, i.e., L_1 intersects $U_{n,j}$ and there exists a set $F \in \mathcal{K}_n$ such that $F \cap A_{n,j} \subseteq L_1$. Let $L \supseteq L_1$ be close to L_1, satisfying $L_1 \subseteq \mathrm{Int}(L)$ and $\overline{\mathrm{Int}(L)} = L$. Note that L is non-meager in $U_{n,j}$.

Consider the set $\mathcal{D} = \mathcal{K}_n \cap \{F : F \cap A_{n,j} \subseteq \mathrm{Int}(L)\}$. $\mathcal{D}$ is a non-empty open subset of $\mathcal{K}_n$. (Openness follows from this easily checked fact about $\mathcal{K}(E)$: if $A \subseteq E$ is closed and $U \subseteq E$ is open, then $\{F \in \mathcal{K}(E) : F \cap A \subseteq U\}$ is open.) Since $K \in I$, K^* is meager in $\mathcal{K}_n$. So $\mathcal{D} \nsubseteq K^*$. Let $F_1 \in \mathcal{D} \setminus K^*$. Now we can remove from L an open $U \supseteq K$ where U is chosen small enough so that $U \cap F_1 = \emptyset$ and $L \setminus U$ is still non-meager in $U_{n,j}$. The set $L \setminus U$ is in $\mathcal{L}_{n,j} \setminus K^*$ and is close to L.

Conversely, suppose $K \notin I$. We want to show that there exists an open set $\mathcal{U} \subseteq \mathcal{K}(E)$ such that $\emptyset \neq \mathcal{U} \cap \mathcal{L} \subseteq K^*$.

Let $C = \bigcup_n \overline{U_n} \cup \{x\}$, a closed set. Write $K \setminus C = \bigcup_j K_j$, where $K_j = K \setminus (C + 1/j)$, which is closed. Now,

$$K = (K \cap \{x\}) \cup \bigcup_n (K \cap \overline{U_n}) \cup \bigcup_j K_j.$$

Since I is a σ-ideal and $\{x\} \in I$, we have two possible cases: either some $K \cap \overline{U_n} \notin I$ or some $K_j \notin I$.

Case 1: There exists n such that $K \cap \overline{U_n} \notin I$.

In this case we fix such an n, and fix m such that $\mathcal{K}_m \subseteq (K \cap \overline{U_n})^*$. If $m \leq n$ then $\overline{U_n} \cap \overline{W_m} = \emptyset$. So $m > n$. This means that $\overline{U_n}$ is one of the sets $A_{m,j}$. Let $V \supseteq \overline{U_n}$ be open such that $V \cap \overline{U_i} = \emptyset$ for all $i \neq n$ and $V \cap \overline{W_n} = \emptyset$. Let $W = V \cup U_{m,j}$.

Claim: $\emptyset \neq \mathcal{L} \cap \mathcal{K}(W) \subseteq K^*$.

It is clear that $\mathcal{L}_{m,j} \cap \mathcal{K}(W) \neq \emptyset$. Let $L \in \mathcal{K}(W) \cap \mathcal{L}$. For any $i \notin \{n, m\}$, $L \cap U_i = \emptyset$. Also, $L \cap W_n = \emptyset$ and $L \cap U_{m,j'} = \emptyset$ for all $j' \neq j$. So the only possibility is that $L \in \mathcal{L}_{m,j}$, i.e., there exists a set $F \in \mathcal{K}_m$ such that $F \cap A_{m,j} = F \cap \overline{U_n} \subseteq L$. Since $F \cap \overline{U_n} \cap K \neq \emptyset$, we have $L \cap K \neq \emptyset$.

Case 2: There exists j such that $K_j \notin I$. Fix m such that $\mathcal{K}_m \subseteq K_j^*$. Fix $\delta > 0$ such that $K_j \cap \bigcup_{i<m} \overline{(U_i + \delta)} = \emptyset$ and let $k \in \mathbb{N}$ such that $k \geq m$ and $1/k < \delta$. Let $W = (W_m \setminus \bigcup_{i<m} \overline{U_i}) \cup U_{m,k}$.

Claim: $\emptyset \neq \mathcal{L} \cap \mathcal{K}(W) \subseteq K^*$.

It is clear that $\mathcal{K}(W) \cap \mathcal{L}_{m,k} \neq \emptyset$. (To get something in this set, we can simply take any $F \in \mathcal{K}_m$ and join some piece of $U_{m,k}$ to $F \cap A_{m,k}$.) So $\mathcal{K}(W) \cap \mathcal{L} \neq \emptyset$.

Now let $L \in \mathcal{K}(W) \cap \mathcal{L}$. As before, the only possibility is that $L \in \mathcal{L}_{m,k}$, i.e., there exists a set $F \in \mathcal{K}_m$ such that $F \cap A_{m,k} = F \setminus \bigcup_{i<m} (U_i + 1/k) \subseteq L$. Since $F \in \mathcal{K}_m$, $F \cap K_j \neq \emptyset$. Let $x \in F \cap K_j$. Since $1/k < \delta$, we have $x \in L$. Therefore $L \in K_j^* \subseteq K^*$.

So in both cases, K^* contains a non-empty relatively open subset of $\mathcal{L}$. Finally, set $\mathcal{F}' = \overline{\mathcal{L}}$.

To deal with the case where I has no infinite set, we note that in this situation I is of the form $\mathcal{K}(A)$, where A is a countable G_δ set. (In fact, A is just $\bigcup I$, which is G_δ since I is G_δ.) In this case, we let C_n, $n \in \mathbb{N}$, be closed subsets of E such that $E \setminus A = \bigcup_i C_i$, and set $\mathcal{K}_n = \{C_n\}$. The sets $\mathcal{K}_n$ satisfy the conditions (1) and (2). Now let $x \in A$. (If no such x exists then $I = \{\emptyset\}$; for this ideal we may simply set $\mathcal{F}' = \{E\}$.) Since $\{x\}$ is in I, it is not open and we may find a sequence of distinct points x_i in the dense set $E \setminus A$, converging to x. For any n, C_n does not contain x. So by replacing $\{x_i\}$ with a suitable subsequence, we may assume that C_n is disjoint from $\{x\} \cup \{x_i : i \geq n\}$. We may now let U_i' be open neighbourhoods of x_i with disjoint closures, and exactly as in the case where I had an infinite set, proceed to define sets (U_i, F_i, W_i) satisfying all the listed properties. The construction of these sets succeeds because it remains true that if $n_i \geq i$, then $\mathcal{K}_i \setminus \{x, x_{n_i}\}^* \neq \emptyset$.

At this point we deal with two subcases. Suppose first that the sequence $\{x_n\}$ contains infinitely many non-isolated points. In this case we assume that in fact each x_n is non-isolated; this allows us to construct the sets $U_{n,j}$ and carry out the rest of the proof exactly as before.

Now consider the alternative: all but finitely many x_n are isolated. In this case we assume that every x_n is isolated. For $n \in \mathbb{N}$, define

$$\mathcal{L}_n = \{F \in \mathcal{K}(E) : C_n \setminus \{x_0, \dots, x_{n-1}\} \subseteq F \ \text{ and } \ x_n \in F\},$$

and set $\mathcal{L} = \bigcup_n \mathcal{L}_n$, which is obviously closed upwards. Now for any $K \in \mathcal{K}(E)$, K^* is nowhere dense in $\mathcal{L}$ if and only if $K \in I$. To see this, let $K \in I$. K consists of finitely many points of A, which are all non-isolated. So if $F \in \mathcal{L}_n$ we may remove a small open superset of K from F without removing x_n or any point of C_n, resulting in a set in $\mathcal{L}_n \setminus K^*$ that is close to F. (Recall that $x_n \notin A$.)

Conversely, if $K \notin I$, pick $y \in K \setminus A$. If $y = x_n$ for some n, then $\{y\}$ is open, and $\{y\}^* \cap \mathcal{L}$ is a non-empty open subset of $\mathcal{L}$, which is all we need. If on the other hand $y \in E \setminus \{x_n : n \in \mathbb{N}\}$, fix m such that $y \in C_m$. Consider the open set $V = W_m \setminus \{x_i : 0 \leq i < m\} \cup \{x_m\}$; it is immediate that $\emptyset \neq \mathcal{K}(V) \cap \mathcal{L} \subseteq \{y\}^*$. The set $\mathcal{L}$ is thus as required, and we may set $\mathcal{F}' = \overline{\mathcal{L}}$. □

Corollary 3. *Let $I \subseteq \mathcal{K}(E)$ be a coanalytic ideal with property $(*)$ containing no non-meager sets. Then there exists a closed set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that $\mathcal{F}$ is closed upwards and for any $K \in \mathcal{K}(E)$,*

$$K \in I \iff K^* \cap \mathcal{F} \ \text{ is meager in } \ \mathcal{F}.$$

Proof: An immediate consequence of Theorem 1 and Theorem 2. □

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Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, IL 61801, USA

Email: msaran@math.uiuc.edu

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