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Trigonometric approximation by Nörlund type means in L^p-norm

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Abstract. We show that the same degree of approximation as in the theorems proved by L. Leindler [*Trigonometric approximation in L^p-norm*, J. Math. Anal. Appl. **302** (2005), 129–136] and P. Chandra [*Trigonometric approximation of functions in L^p-norm*, J. Math. Anal. Appl. **275** (2002), 13–26] is valid for a more general class of lower triangular matrices. We also prove that these theorems are true under weakened assumptions.

Keywords: class $\mathrm{Lip}(\alpha, p)$, trigonometric approximation, L^p-norm

Classification: 42A10, 41A25

1. Introduction

Let f be 2π-periodic and $f \in L^p[0, 2\pi] = L^p$ for $p \geq 1$. Denote by

$$S_n(f) = S_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} U_k(f;x)$$

the partial sum of the first $(n+1)$ terms of the Fourier series of $f \in L^p$ $(p \geq 1)$ at a point x, and by

$$\omega_p(f;\delta) = \sup_{0<|h|\leq\delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p \, dx \right\}^{\frac{1}{p}}$$

the integral modulus of continuity of $f \in L^p$. If, for $\alpha > 0$, $\omega_p(f;\delta) = O(\delta^\alpha)$, then we write $f \in \mathrm{Lip}(\alpha, p)$ $(p \geq 1)$.

Throughout the paper $\|\cdot\|_{L^p}$ will denote L^p-norm, defined by

$$\|f\|_{L^p} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}} \quad (f \in L^p(p \geq 1)).$$

We shall consider the approximation of $f \in L^p$ by trigonometrical polynomials $T_n(f;x)$, where

$$T_n(f;x) = T_n(f, A; x) := \sum_{k=0}^{n} a_{n,n-k} S_k(f;x) \quad (n = 0, 1, 2, \ldots)$$

and $A := (a_{n,k})$ is a lower triangular infinite matrix of real numbers such that:

$$a_{n,k} \geq 0 \text{ for } k \leq n, \ a_{n,k} = 0 \text{ for } k > n \quad (k, n = 0, 1, 2 \dots) \tag{1.1}$$

and

$$\sum_{k=0}^{n} a_{n,k} = 1 \quad (n = 0, 1, 2 \dots). \tag{1.2}$$

If $a_{n,n-k} = \frac{p_{n-k}}{P_n}$, where $P_n = p_0 + p_1 + \dots + p_n \neq 0$ $(n \geq 0)$, then we denote the corresponding trigonometrical polynomials by

$$N_n(f; x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(f; x) \quad (n = 0, 1, 2 \dots).$$

The case $a_{n,k} = \frac{1}{n+1}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$ of $T_n(f; x)$ yields

$$\sigma_n(f; x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f; x) \quad (n = 0, 1, 2 \dots).$$

We shall also use the notations

$$\Delta a_k = a_k - a_{k+1}, \qquad \Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$$

and we shall write $I_1 \ll I_2$ if there exists a positive constant K such that $I_1 \leq K I_2$.

A nonnegative sequence $c := (c_n)$ is called *almost monotone decreasing* (*increasing*) if there exists a constant $K := K(c)$, depending on the sequence c only, such that for all $n \geq m$

$$c_n \leq K c_m \quad (K c_n \geq c_m).$$

For such sequences we shall write $c \in$ AMDS and $c \in$ AMIS, respectively.

When we write that a sequence $(a_{n,k})$ belongs to one of the above classes, it means that it satisfies some of the above conditions with respect to $k = 0, 1, 2, \dots, n$ for all n.

Let $A_{n,k} = \frac{1}{k+1} \sum_{i=n-k}^{n} a_{n,i}$. If $(A_{n,k}) \in$ AMDS ($(A_{n,k}) \in$ AMIS), then we shall say that $(a_{n,k})$ is an *almost monotone decreasing* (*increasing*) *upper mean sequence*, briefly $(a_{n,k}) \in$ AMDUMS ($(a_{n,k}) \in$ AMIUMS).

A sequence $c := (c_n)$ of nonnegative numbers will be called the *head bounded variation sequence*, or briefly $c \in$ HBVS, if it has the property

$$\sum_{k=0}^{m-1} |\Delta c_k| \leq K(c) c_m \tag{1.3}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence c has only finite nonzero terms and the last nonzero term is c_N.

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (1.3) for the sequence $\alpha_n := (a_{nk})_{k=0}^{\infty}$. Now we can consider the conditions to be used later on. We assume that for all n and $0 \leq m \leq n$

$$\sum_{k=0}^{m-1} |\Delta_k a_{nk}| \leq K a_{nm}$$

holds if $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to HBVS.

It is clear that

$$\text{NDS} \subset \text{HBVS} \subset \text{AMIS}$$

and

$$\text{NIS} \subset \text{AMDS},$$

where NDS (NIS) is the class of nonnegative and nondecreasing (nonincreasing) sequences.

In the present paper we show some embedding relations between the classes AMDS and AMDUMS and the classes AMIS and AMIUMS.

In 1937 E. Quade [8] proved that, if $f \in \text{Lip}(\alpha, p)$ for $0 < \alpha \leq 1$, then $\|\sigma_n(f) - f\|_{L^p} = O(n^{-a})$ for either $p > 1$ and $0 < \alpha \leq 1$ or $p = 1$ and $0 < \alpha < 1$. He also showed that, if $p = \alpha = 1$, then $\|\sigma_n(f) - f\|_{L^1} = O(n^{-1}\log(n+1))$.

There are several generalizations of the above result for $p > 1$ (see, for example [1], [2], [3], [6] and [7]). In [4], P. Chandra extended the work of E. Quade and proved the following theorems:

Theorem 1. *Let $f \in \text{Lip}(\alpha, p)$ and let (p_n) be positive such that*

$$(n+1)p_n = O(P_n). \tag{1.4}$$

If either

(i) $p > 1$, $0 < \alpha \leq 1$, *and*

(ii) (p_n) *is monotonic, or*

(i) $p = 1$, $0 < \alpha < 1$, *and*

(ii) (p_n) *is nondecreasing, then*

$$\|N_n(f) - f\|_{L^p} = O\left(n^{-\alpha}\right).$$

Theorem 2. *Let $f \in \text{Lip}(1, 1)$ and let (p_n) with (1.4) be positive, and*

$$((n+1)^{\eta} p_n) \in \text{NDS} \quad \textit{for some } \eta > 0.$$

Then

$$\|R_n(f) - f\|_{L^1} = O\left(n^{-1}\right),$$

where $R_n(f;x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k S_k(f;x)$ *and* $P_n = p_0 + p_1 + \cdots + p_n \neq 0$ $(n \geq 0)$.

In [5] L. Leindler obtained the same degree of approximation as in Theorem 1, where the conditions of monotonicity were replaced by weaker assumptions. Namely, he proved the following theorem:

Theorem 3. *Let* $f \in \mathrm{Lip}(\alpha, p)$ *and let* (p_n) *be positive. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ *and* $(p_n) \in$ AMDS,

(ii) $p > 1$, $0 < \alpha < 1$, $(p_n) \in$ AMIS *and* (1.4) *holds,*

(iii) $p > 1$, $\alpha = 1$ *and* $\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$,

(iv) $p > 1$, $\alpha = 1$, $\sum_{k=0}^{n-1} |\Delta p_k| = O(P_n/n)$ *and* (1.4) *holds,*

(v) $p = 1$, $0 < \alpha < 1$ *and* $\sum_{k=-1}^{n-1} |\Delta p_k| = O(P_n/n)$

holds, then

$$\|N_n(f) - f\|_{L^p} = O\left(n^{-\alpha}\right).$$

In this paper we shall show that the same degree of approximation as in the above theorems is valid for a more general class of lower triangular matrices. We shall also prove that the cited theorems are true with weakened assumptions.

2. Statement of the results

Our main results are the following.

Theorem 4. *The following properties hold:*

(i) *if* $(a_{n,k}) \in$ AMDS, *then* $(a_{n,k}) \in$ AMIUMS,

(ii) *if* $(a_{n,k}) \in$ AMIS, *then* $(a_{n,k}) \in$ AMDUMS,

(iii) *if* $\sum_{k=0}^{n-1} |\Delta a_{n,k}| = O(n^{-1})$, *then* $\sum_{k=0}^{n-1} |\Delta A_{n,k}| = O(n^{-1})$,

(iv) *if* $\sum_{k=1}^{n-1} k|\Delta a_{n,k}| = O(1)$, *then* $\sum_{k=0}^{n-2} |\Delta A_{n,k}| = O(n^{-1})$.

Theorem 5. *Let* $f \in \mathrm{Lip}(\alpha, p)$ *and* (1.1), (1.2) *hold. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ *and* $(a_{n,k}) \in$ AMDUMS,

(ii) $p > 1$, $0 < \alpha < 1$, $(a_{n,k}) \in$ AMIUMS *and* $(n+1)a_{n,n} = O(1)$,

(iii) $p > 1$, $\alpha = 1$ *and* $\sum_{k=0}^{n-2} |\Delta_k A_{n,k}| = O(n^{-1})$,

(iv) $p = 1$, $0 < \alpha < 1$ *and* $\sum_{k=-1}^{n-1} |\Delta a_{n,k}| = O(n^{-1})$,

(v) $p = \alpha = 1$, $((k+1)^{-\beta} a_{n,n-k}) \in$ HBVS *for some* $\beta > 0$ *and* $(n+1)a_{n,0} = O(1)$

holds, then

$$\|T_n(f) - f\|_{L^p} = O\left(n^{-\alpha}\right). \tag{2.1}$$

In the particular case $a_{n,k} = \frac{p_k}{P_n}$, where $P_n = p_0 + p_1 + \cdots + p_n$ and $P_{n,k} = \frac{1}{k+1}\sum_{i=n-k}^{n} p_i$, we can derive from Theorem 5 the following corollary:

Corollary 1. *Let* $f \in \mathrm{Lip}(\alpha, p)$ *and let* (p_k) *be positive. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ *and* $(p_k) \in$ AMDUMS,

(ii) $p > 1$, $0 < \alpha < 1$, $(p_k) \in$ AMIUMS *and* $(n+1)p_n = O(P_n)$,

(iii) $p > 1$, $\alpha = 1$ *and* $\sum_{k=0}^{n-2} |\Delta_k P_{n,k}| = O(P_n/n)$,

(iv) $p = 1$, $0 < \alpha < 1$ *and* $\sum_{k=-1}^{n-1} |\Delta p_k| = O(P_n/n)$,

(v) $p = \alpha = 1$, $((k+1)^{-\beta} p_{n-k}) \in$ HBVS *for some* $\beta > 0$ *and* $(n+1)p_0 = O(P_n)$

holds, then

$$\|N_n(f) - f\|_{L^p} = O\left(n^{-\alpha}\right).$$

Remark 1. By Theorem 4 we can observe that Theorem 3 and consequently Theorem 1 follow from Corollary 1(i)–(iv). Moreover, since NDC $\subset$ HBVS we can derive from Corollary 1(v) an analogous estimate as in Theorem 2 for the deviation $N_n(f) - f$ in L^p-norm.

3. Auxiliary results

We shall use the following lemmas in proofs of Theorems 4 and 5.

Lemma 1 ([8, Theorem 4])**.** *If* $f \in \mathrm{Lip}(\alpha, p)$, $p \geq 1$, $0 < \alpha \leq 1$, *then, for any positive integer* n, f *may be approximated in* L^p*-space by a trigonometrical polynomial* t_n *of order* n *such that*

$$\|f - t_n\|_{L^p} = O\left(n^{-\alpha}\right).$$

Lemma 2 ([8, Theorem 5(i)])**.** *If* $f \in \mathrm{Lip}(\alpha, 1)$, $0 < \alpha < 1$, *then*

$$\|\sigma_n(f) - f\|_{L^1} = O\left(n^{-\alpha}\right).$$

Lemma 3 ([8, p. 541, last line])**.** *If* $f \in \mathrm{Lip}(1, p)$ $(p > 1)$, *then*

$$\|\sigma_n(f) - S_n(f)\|_{L^p} = O\left(n^{-1}\right).$$

Lemma 4 ([8, Theorem 6(i), p. 541])**.** *If* $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha \leq 1$, $p > 1$, *then*

$$\|S_n(f) - f\|_{L^p} = O\left(n^{-\alpha}\right).$$

Lemma 5. *Let* (1.1) *and* (1.2) *hold and let one of the following assumptions*

(i) $(a_{n,k}) \in$ AMDUMS,

(ii) $(a_{n,k}) \in$ AMIUMS *and* $(n+1)a_{n,n} = O(1)$

be satisfied. Then

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,n-k} = O\left((n+1)^{-\alpha}\right)$$

holds for all $0 < \alpha < 1$.

PROOF: Let $r=\left[\frac{n}{2}\right]$. Then, if (1.1) and (1.2) hold,

$$\begin{aligned}\sum_{k=0}^{n}(k+1)^{-\alpha}a_{n,n-k} &\leq \sum_{k=0}^{r}(k+1)^{-\alpha}a_{n,n-k}+(r+1)^{-\alpha}\sum_{k=r+1}^{n}a_{n,n-k}\\ &\leq \sum_{k=0}^{r}(k+1)^{-\alpha}a_{n,n-k}+(r+1)^{-\alpha}.\end{aligned}$$

By Abel's transformation and (1.2), we get

$$\begin{aligned}\sum_{k=0}^{n}(k+1)^{-\alpha}a_{n,n-k} &\leq \sum_{k=0}^{r-1}\left\{(k+1)^{-\alpha}-(k+2)^{-\alpha}\right\}\sum_{i=0}^{k}a_{n,n-i}\\ &\quad+(r+1)^{-\alpha}\sum_{k=0}^{r}a_{n,n-k}+(r+1)^{-\alpha}\\ &\leq \sum_{k=0}^{r-1}\left\{(k+1)^{-\alpha}-(k+2)^{-\alpha}\right\}\sum_{i=n-k}^{n}a_{n,i}+2\left(r+1\right)^{-\alpha}\\ &\leq \sum_{k=0}^{r-1}\frac{(k+2)^{\alpha}-(k+1)^{\alpha}}{(k+1)^{\alpha-1}(k+2)^{\alpha}}A_{n,k}+2(r+1)^{-\alpha}.\end{aligned}$$

Using Lagrange's mean value theorem for the function $f(x)=x^{\alpha}$ $(0<\alpha<1)$ on the interval $(k+1,k+2)$ we obtain

$$\sum_{k=0}^{n}\left(k+1\right)^{-\alpha}a_{n,n-k}\leq \sum_{k=0}^{r-1}\frac{\alpha}{\left(k+2\right)^{\alpha}}A_{n,k}+2\left(r+1\right)^{-\alpha}.$$

If $(a_{n,k})\in$ AMDUMS, then

$$\begin{aligned}\sum_{k=0}^{n}\left(k+1\right)^{-\alpha}a_{n,n-k} &\ll A_{n,r}\sum_{k=0}^{r-1}\frac{1}{(k+2)^{\alpha}}+(r+1)^{-\alpha}\\ &\ll \frac{1}{r+1}\sum_{k=n-r}^{n}a_{n,k}\sum_{k=0}^{r-1}\frac{1}{\left(k+2\right)^{\alpha}}+\left(r+1\right)^{-\alpha}\\ &\ll \left(r+1\right)^{-\alpha}\ll\left(n+1\right)^{-\alpha}.\end{aligned}$$

When $(a_{n,k})\in$ AMIUMS and $(n+1)a_{n,n}=O(1)$ we get

$$\begin{aligned}\sum_{k=0}^{n}\left(k+1\right)^{-\alpha}a_{n,n-k} &\ll A_{n,0}\sum_{k=0}^{r-1}\frac{1}{\left(k+2\right)^{\alpha}}+\left(r+1\right)^{-\alpha}\\ &\ll a_{n,n}\sum_{k=0}^{r-1}\frac{1}{\left(k+2\right)^{\alpha}}+\left(r+1\right)^{-\alpha}\end{aligned}$$

$$\ll a_{n,n}(r+1)^{1-\alpha}+(r+1)^{-\alpha}\ll(n+1)^{-\alpha}.$$

This ends the proof. $\square$

4. Proofs of the results

4.1 Proof of Theorem 4. (i) If $(a_{n,k})\in$ AMDS, then $Ka_{n,m}\geq a_{n,l}$ for $m\leq l$ and all n. We prove that $A_{n,m}\leq\max\{1,K^2\}A_{n,l}$ for $m\leq l$ and all n. For $m=l$ this is true. Let $m<l$. Then

$$\begin{aligned}(l+1)\sum_{i=n-m}^{n}a_{n,i}&=(m+1)\sum_{i=n-m}^{n}a_{n,i}+(l-m)\sum_{i=n-m}^{n}a_{n,i}\\&\leq(m+1)\left\{\sum_{i=n-m}^{n}a_{n,i}+K\left(l-m\right)a_{n,n-m}\right\}\\&\leq(m+1)\left\{\sum_{i=n-m}^{n}a_{n,i}+K^2\sum_{i=n-l}^{n-m-1}a_{n,i}\right\}\\&\leq\max\left\{1,K^2\right\}(m+1)\sum_{i=n-l}^{n}a_{n,i}.\end{aligned}$$

Thus

$$\frac{1}{m+1}\sum_{i=n-m}^{n}a_{n,i}\leq\max\left\{1,K^2\right\}\frac{1}{l+1}\sum_{i=n-l}^{n}a_{n,i}$$

and $(a_{n,k})\in$ AMIUMS.

(ii) Let $(a_{n,k})\in$ AMIS. We get that $a_{n,m}\leq Ka_{n,l}$ for $m\leq l$ and all n. We show that $\frac{1}{\min\{1,\frac{1}{K^2}\}}A_{n,m}\geq A_{n,l}$ for $m\leq l$ and all n. If $m=l$, then it is true. Suppose $m<l$, then we have

$$\begin{aligned}(l+1)\sum_{i=n-m}^{n}a_{n,i}&=(m+1)\sum_{i=n-m}^{n}a_{n,i}+(l-m)\sum_{i=n-m}^{n}a_{n,i}\\&\geq(m+1)\left\{\sum_{i=n-m}^{n}a_{n,i}+\frac{1}{K}\left(l-m\right)a_{n,n-m}\right\}\\&\geq(m+1)\left\{\sum_{i=n-m}^{n}a_{n,i}+\frac{1}{K^2}\sum_{i=n-l}^{n-m-1}a_{n,i}\right\}\\&\geq\min\left\{1,\frac{1}{K^2}\right\}(m+1)\sum_{i=n-l}^{n}a_{n,i}.\end{aligned}$$

Hence

$$\frac{1}{\min\left\{1,\frac{1}{K^2}\right\}}\frac{1}{m+1}\sum_{i=n-m}^{n}a_{n,i}\geq\frac{1}{l+1}\sum_{i=n-l}^{n}a_{n,i}$$

and $(a_{n,k})\in$ AMDUMS.

(iii) An elementary calculation yields that

$$\begin{aligned}\sum_{k=0}^{n-1}\left|\Delta A_{n,k}\right| &= \sum_{k=0}^{n-1}\frac{1}{(k+1)(k+2)}\\ &\quad\times\left|(k+2)\sum_{i=n-k}^{n}a_{n,i}-(k+1)\sum_{i=n-k-1}^{n}a_{n,i}\right| \qquad (4.1)\\ &= \sum_{k=0}^{n-1}\frac{1}{(k+1)(k+2)}\left|\sum_{i=n-k}^{n}a_{n,i}-(k+1)a_{n,n-k-1}\right|.\end{aligned}$$

Clearly,

$$\sum_{i=n-k}^{n}a_{n,i}-(k+1)a_{n,n-k-1}=\sum_{i=0}^{k}(i+1)\left(a_{n,n-i}-a_{n,n-i-1}\right) \qquad (4.2)$$

holds for any $0\leq k\leq n$.

Using (4.2) and (4.1) we have

$$\begin{aligned}\sum_{k=0}^{n-1}\left|\Delta A_{n,k}\right| &= \sum_{k=0}^{n-1}\frac{1}{(k+1)(k+2)}\left|\sum_{i=0}^{k}(i+1)\left(a_{n,n-i}-a_{n,n-i-1}\right)\right|\\ &\leq \sum_{k=0}^{n-1}\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(i+1)\left|a_{n,n-i}-a_{n,n-i-1}\right|\\ &\leq \sum_{k=0}^{n-1}(k+1)\left|a_{n,n-k}-a_{n,n-k-1}\right|\sum_{i=k}^{\infty}\frac{1}{(i+1)(i+2)}\\ &\ll \sum_{k=0}^{n-1}\left|a_{n,n-k}-a_{n,n-k-1}\right|=\sum_{k=0}^{n-1}\left|a_{n,k}-a_{n,k+1}\right|.\end{aligned}$$

If $\sum_{k=0}^{n-1}\left|\Delta a_{n,k}\right|=O(n^{-1})$, then we obtain that $\sum_{k=0}^{n-1}\left|\Delta A_{n,k}\right|=O(n^{-1})$, too.

(iv) Let $r=\left[\frac{n}{2}\right]$. Using (4.2) we get

$$\begin{aligned}\sum_{k=0}^{n-2}\left|\Delta A_{n,k}\right| &\leq \sum_{k=0}^{n-2}\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(i+1)\left|a_{n,n-i}-a_{n,n-i-1}\right|\\ &= \sum_{k=0}^{r-1}\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(i+1)\left|a_{n,n-i}-a_{n,n-i-1}\right|\end{aligned}$$

$$+\sum_{k=r}^{n-2}\frac{1}{(k+1)(k+2)}\sum_{i=0}^{k}(i+1)\left|a_{n,n-i}-a_{n,n-i-1}\right| := I_1+I_2.$$

An elementary calculation yields that

$$\begin{aligned}
I_1 &\leq \sum_{k=0}^{r-1}(k+1)\left|a_{n,n-k}-a_{n,n-k-1}\right|\sum_{i=k}^{\infty}\frac{1}{(i+1)(i+2)}\\
&\ll \sum_{k=0}^{r-1}\left|a_{n,n-k}-a_{n,n-k-1}\right| = \sum_{k=n-r}^{n-1}\left|a_{n,k}-a_{n,k+1}\right|\\
&\ll \frac{1}{n}\sum_{k=n-r}^{n-1}k\left|\Delta a_{n,k}\right|.
\end{aligned}$$

If

$$\sum_{k=1}^{n-1}k\left|\Delta a_{n,k}\right| = O(1), \tag{4.3}$$

then $I_1 = O(n^{-1})$.

On the other hand,

$$\begin{aligned}
I_2 = \sum_{k=r}^{n-2}\frac{1}{(k+1)\,(k+2)}&\left\{\sum_{i=0}^{r-1}(i+1)\left|a_{n,n-i}-a_{n,n-i-1}\right|\right.\\
&\left.+\sum_{i=r}^{k}(i+1)\left|a_{n,n-i}-a_{n,n-i-1}\right|\right\} := I_{21}+I_{22}.
\end{aligned}$$

Furthermore, using again (4.3), we get

$$\begin{aligned}
I_{21} &\leq \sum_{k=r}^{n-2}\frac{1}{(k+1)}\sum_{i=0}^{r-1}\left|a_{n,n-i}-a_{n,n-i-1}\right| \leq \frac{n-r}{r+1}\sum_{k=n-r}^{n-1}\left|a_{n,k}-a_{n,k+1}\right|\\
&\ll \frac{1}{n}\sum_{k=n-r}^{n-1}k\left|\Delta a_{n,k}\right| = O\left(n^{-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
I_{22} &\leq \sum_{k=r}^{n-2}\frac{1}{(k+1)}\sum_{i=r}^{k}\left|a_{n,n-i}-a_{n,n-i-1}\right| \leq \frac{1}{r+1}\sum_{k=r}^{n-2}\sum_{i=r}^{k}\left|a_{n,n-i}-a_{n,n-i-1}\right|\\
&\ll \frac{1}{n}\sum_{k=1}^{n-r}k\left|a_{n,k}-a_{n,k+1}\right| = 0\left(n^{-1}\right).
\end{aligned}$$

Summing up our partial result, we obtain that $\sum_{k=0}^{n-2}|\Delta A_{n,k}| = O(n^{-1})$. The proof is complete. □

4.2 Proof of Theorem 5. We prove the cases (i) and (ii) together using Lemmas 4 and 5. Since

$$T_n(f;x) - f(x) = \sum_{k=0}^{n} a_{n,n-k}\left(S_k(f;x) - f(x)\right),$$

we have

$$\|T_n(f) - f\|_{L^p} \leq \sum_{k=0}^{n} a_{n,n-k}\|S_k(f) - f\|_{L^p} \ll \sum_{k=0}^{n}(k+1)^{-\alpha} a_{n,n-k} = O\left(n^{-\alpha}\right)$$

and this is (2.1).

Next we consider the case (iii).

Using two times Abel's transformation and (1.2) we get

$$\begin{aligned}
T_n(f;x) - f(x) &= \sum_{k=0}^{n} a_{n,n-k}\left(S_k(f;x) - f(x)\right) \\
&= \sum_{k=0}^{n-1}\left(S_k(f;x) - S_{k+1}(f;x)\right)\sum_{i=0}^{k} a_{n,n-i} + S_n(f;x) - f(x) \\
&= \sum_{k=0}^{n-1}\left(S_k(f;x) - S_{k+1}(f;x)\right)\sum_{i=n-k}^{n} a_{n,i} + S_n(f;x) - f(x) \\
&= S_n(f;x) - f(x) - \sum_{k=0}^{n-1}(k+1)\,U_{k+1}(f;x)\,A_{n,k} \\
&= S_n(f;x) - f(x) - \sum_{k=0}^{n-2}\left(A_{n,k} - A_{n,k+1}\right)\sum_{i=0}^{k}(i+1)U_{i+1}(f;x) \\
&\quad - A_{n,n-1}\sum_{k=0}^{n-1}(k+1)\,U_{k+1}(f;x) \\
&= S_n(f;x) - f(x) - \sum_{k=0}^{n-2}\left(A_{n,k} - A_{n,k+1}\right)\sum_{i=0}^{k}(i+1)U_{i+1}(f;x) \\
&\quad - \frac{1}{n}\sum_{i=1}^{n} a_{n,i}\sum_{k=0}^{n-1}(k+1)\,U_{k+1}(f;x).
\end{aligned}$$

Hence

$$\begin{aligned}\|T_n(f)-f\|_{L^p} \le \|S_n(f)-f\|_{L^p} \\ + \sum_{k=0}^{n-2} |A_{n,k}-A_{n,k+1}| \left\|\sum_{i=1}^{k+1} iU_i(f)\right\|_{L^p} \\ + \frac{1}{n}\left\|\sum_{k=1}^{n} kU_k(f;x)\right\|_{L^p}.\end{aligned} \tag{4.4}$$

Since

$$\sigma_n(f;x)-S_n(f;x) = -\frac{1}{n+1}\sum_{k=1}^{n} kU_k(f;x),$$

by Lemma 3 we get

$$\left\|\sum_{k=1}^{n} kU_k(f)\right\|_{L^p} = (n+1)\|\sigma_n(f)-S_n(f)\|_{L^p} = O(1). \tag{4.5}$$

By (4.4), (4.5) and Lemma 4 we have that

$$\|T_n(f)-f\|_{L^p} \ll \frac{1}{n} + \sum_{k=0}^{n-2}|A_{n,k}-A_{n,k+1}|.$$

If $\sum_{k=0}^{n-2}|\Delta_k A_{n,k}| = O(n^{-1})$, then

$$\|T_n(f)-f\|_{L^p} = O\left(n^{-1}\right)$$

and (2.1) holds.

Now, we prove the cases (iv).

By Abel's transformation

$$\begin{aligned}T_n(f;x)-f(x) &= \sum_{k=0}^{n} a_{n,n-k}(S_k(f;x)-f(x)) \\ &= \sum_{k=0}^{n-1}(a_{n,n-k}-a_{n,n-k-1})\sum_{i=0}^{k}(S_i(f;x)-f(x)) \\ &\quad + a_{n,0}\sum_{k=0}^{n}(S_k(f;x)-f(x)) \\ &= \sum_{k=0}^{n-1}(a_{n,n-k}-a_{n,n-k-1})(k+1)(\sigma_k(f;x)-f(x)) \\ &\quad + a_{n,0}(n+1)(\sigma_n(f;x)-f(x)).\end{aligned}$$

Using Lemma 2 we get

$$\begin{aligned}
\|T_n(f)-f\|_{L^1} &\leq \sum_{k=0}^{n-1}|a_{n,n-k}-a_{n,n-k-1}|\,(k+1)\,\|\sigma_k(f)-f\|_{L^1} \\
&\quad + a_{n,0}\,(n+1)\,\|\sigma_n(f)-f\|_{L^1} \\
&\ll \sum_{k=0}^{n-1}|a_{n,n-k}-a_{n,n-k-1}|\,(k+1)^{1-\alpha} \\
&\quad + a_{n,0}(n+1)^{1-\alpha} \\
&\leq (n+1)^{1-\alpha}\left(\sum_{k=0}^{n-1}|a_{n,n-k}-a_{n,n-k-1}|+a_{n,0}\right) \\
&= (n+1)^{1-\alpha}\sum_{k=-1}^{n-1}|\Delta a_{n,k}|,
\end{aligned}$$

where $a_{n,-1}=0$. When the assumptions (iv) hold we get

$$\|T_n(f)-f\|_{L^1}=O\left(n^{-\alpha}\right).$$

This ends the proof of the case (iv).

Finally, we prove the case (v).

Let t_n be a trigonometrical polynomial of Lemma 1 of the present paper. Then for $m\leq n$,

$$S_m(t_n;x)=t_m \quad \text{and} \quad S_m(f;x)-t_m=S_m(f-t_n;x).$$

Thus

$$T_n(f;x)-\sum_{k=0}^{n}a_{n,n-k}t_k(x)=\sum_{k=0}^{n}a_{n,n-k}S_k(f-t_n;x),$$

where

$$S_k(f-t_n;x)=\frac{1}{\pi}\int_0^{2\pi}\{f(x+u)-t_n(x+u)\}\frac{\sin\left(k+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}\,du.$$

By the general form of Minkowski inequality we get

$$
\begin{aligned}
&\left\| T_n(f) - \sum_{k=0}^{n} a_{n,n-k} t_k \right\|_{L^1} \\
&\leq \frac{1}{2\pi^2} \int_0^{2\pi} |K_n(u)| \, du \int_0^{2\pi} |f(x+u) - t_n(x+u)| \, dx \\
&= \frac{1}{2\pi^2} \int_0^{2\pi} |K_n(u)| \int_0^{2\pi} |f(x) - t_n(x)| \, dx \\
&= \frac{1}{\pi} \|f - t_n\|_{L^1} \int_0^{2\pi} |K_n(u)| \, du \\
&= \frac{2}{\pi} \|f - t_n\|_{L^1} \int_0^{\pi} |K_n(u)| \, du \\
&= \frac{2}{\pi} \|f - t_n\|_{L^1} \left(\int_0^{\pi/n} |K_n(u)| \, du + \int_{\pi/n}^{\pi} |K_n(u)| \, du \right) \\
&= \frac{2}{\pi} \|f - t_n\|_{L^1} (I_1 + I_2),
\end{aligned} \tag{4.6}
$$

where

$$
K_n(u) = \sum_{k=0}^{n} a_{n,n-k} \frac{\sin\left(k + \frac{1}{2}\right) u}{2 \sin \frac{u}{2}}.
$$

Now, we estimate the quantities I_1 and I_2. By (1.2)

$$
I_1 \ll \int_0^{\pi/n} \sum_{k=0}^{n} (k+1) a_{n,n-k} \, du = O(1). \tag{4.7}
$$

If $((k+1)^{-\beta} a_{n,n-k}) \in$ HBVS, then $((k+1)^{-\beta} a_{n,n-k}) \in$ AMIS. Hence, for $0 \leq l \leq m \leq n$,

$$
K a_{n,n-m} \geq a_{n,n-l} \left(\frac{m+1}{l+1} \right)^{\beta} \geq a_{n,n-l}.
$$

Thus $(a_{n,n-k}) \in$ AMIS. Using this and the assumption $(n+1)a_{n,0} = O(1)$ we obtain that

$$
I_2 \ll a_{n,0} \int_{\pi/n}^{\pi} u^{-2} \, du = O(1). \tag{4.8}
$$

Combining (4.6)–(4.8) we have

$$
\left\| T_n(f) - \sum_{k=0}^{n} a_{n,n-k} t_k \right\|_{L^1} \ll \|f - t_n\|_{L^1}. \tag{4.9}
$$

Further, by using (4.9) and Lemma 1 for $p=\alpha=1$, we get

$$\begin{aligned}\left\|T_n\left(f\right)-f\right\|_{L^1} &\leq \left\|T_n(f)-\sum_{k=0}^{n}a_{n,n-k}t_k\right\|_{L^1}+\left\|\sum_{k=0}^{n}a_{n,n-k}t_k-f\right\|_{L^1}\\ &\ll \frac{1}{n}+\left\|\sum_{k=0}^{n}a_{n,n-k}t_k-f\right\|_{L^1}\leq\frac{1}{n}+\sum_{k=0}^{n}a_{n,n-k}\left\|t_k-f\right\|_{L^1}\\ &\ll \frac{1}{n}+\sum_{k=0}^{n}a_{n,n-k}\left(k+1\right)^{-1}.\end{aligned}$$

By Abel's transformation

$$\begin{aligned}\|T_n(f)-f\|_{L^1} &\ll \frac{1}{n}+\sum_{k=0}^{n-1}\left|\frac{a_{n,n-k}}{(k+1)^{\beta}}-\frac{a_{n,n-k-1}}{(k+2)^{\beta}}\right|\sum_{i=0}^{k}(i+1)^{\beta-1}\\ &\quad+\frac{a_{n,0}}{(n+1)^{\beta}}\sum_{k=0}^{n}(k+1)^{\beta-1}\\ &\ll \frac{1}{n}+(n+1)^{\beta}\sum_{k=0}^{n-1}\left|\frac{a_{n,n-k}}{(k+1)^{\beta}}-\frac{a_{n,n-k-1}}{(k+2)^{\beta}}\right|+a_{n,0}.\end{aligned}$$

Since $((k+1)^{-\beta}a_{n,n-k})\in$ HBVS and $(n+1)a_{n,0}=O(1)$, we obtain

$$\|T_n(f)-f\|_{L^1}=O\left(n^{-1}\right)$$

and (2.1) holds. This completes the proof of Theorem 5. $\square$

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