A construction of a Fréchet-Urysohn space, and some convergence concepts

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Abstract. Some strong versions of the Fréchet-Urysohn property are introduced and studied. We also strengthen the concept of countable tightness and generalize the notions of first-countability and countable base. A construction of a topological space is described which results, in particular, in a Tychonoff countable Fréchet-Urysohn space which is not first-countable at any point. It is shown that this space can be represented as the image of a countable metrizable space under a continuous pseudoopen mapping. On the other hand, if a topological group G is an image of a separable metrizable space under a pseudoopen continuous mapping, then G is metrizable (Theorem 5.6). Several other applications of the techniques developed below to the study of pseudoopen mappings and intersections of topologies are given (see Theorem 5.17).

Keywords: first-countable, Fréchet-Urysohn, countably compact, closure-sensor, topological group, strong FU-sensor, pseudoopen mapping, side-base, ω-Fréchet-Urysohn space

Classification: 54D20, 54G20, 54J99

1. Introduction

Many convergence properties closely related to first countability have been studied extensively in General Topology and have been found quite useful in its applications. One of those is the Fréchet-Urysohn property. It is very easy to construct a Tychonoff countable non-metrizable Fréchet-Urysohn space — the well known countable Fréchet-Urysohn fan $V(\omega)$ is a standard example. However, the space $V(\omega)$ has only one non-isolated point. Thus, at all other points the space $V(\omega)$ is locally metrizable. It is more tricky to present a transparent example of a Tychonoff countable Fréchet-Urysohn space that is not first-countable at any point.

In the next section, we describe a general construction which is used in Section 3 to define a countable Fréchet-Urysohn space with a series of interesting properties. In particular, it is countable, not first-countable at any point, and has a countable π-base.

Several general concepts, like the concept of a side-base, are introduced in connection with this construction in Sections 2 and 3. They are studied in Sections 3, 4, and 5. In particular, see Theorems 4.5, 4.6, and Theorem 4.9. It is shown that if a topological group G is an image of a separable metrizable space under a pseudoopen continuous mapping, then G is metrizable (Theorem 5.6). In Section 5 we obtain some results on the behaviour of the properties introduced in the paper under pseudoopen mappings and intersection of topologies.

In terminology and notation, we follow [6]. In particular, ω is the set of all natural numbers. We denote by $\mathrm{int}(P)$ the interior of an arbitrary subset P of a space X. Recall that a *prefilter* on a set Y is a non-empty family η of non-empty subsets of Y such that the intersection of any finite collection of elements of η contains an element of η. Another name for a prefilter is "a base of a filter". A prefilter η is called *free* if $\bigcap \eta = \emptyset$. We recall also that a space X is said to be *strongly Fréchet-Urysohn* if, for any decreasing sequence $\{A_n : n \in \omega\}$ of subsets of X and any point x in the intersection of the closures of the sets A_n, one can select $x_n \in A_n$ for every $n \in \omega$ so that the sequence $\{x_n : n \in \omega\}$ will converge to x.

2. A construction of a topological space

A construction. Suppose that Y is an infinite set. We also fix a non-empty family $\mathcal{F}$ of free prefilters on Y.

For every positive integer n, let Z_n be the set of finite sequences $z = (z_1, z_2, \ldots, z_n)$ of elements of Y. Thus, the number of entries in any $z \in Z_n$ is exactly n, and we denote the k-th element of such z by z_k, for $k = 1, \ldots, n$. We put $Z = \bigcup\{Z_n : n \in \omega\}$. If $z \in Z_n$, then we write $r(z) = n$, and say that the *length* of z is n.

Take any $z, h \in Z$. We put $z \leq h$ if $r(z) \leq r(h)$ and $z_i = h_i$, for each $i = 1, \ldots, r(z)$. If we also have $r(z) < r(h)$, then we write $z < h$.

Let us define a certain neighbourhood-like structure on the set Z which will allow to introduce a natural topology on Z.

Fix $z \in Z$, and let $H_z = \{h \in Z : z \leq h\}$. We also put $W(z, U) = \{h \in Z : z < h, h_{n+1} \in U\}$, where $n = r(z)$, and U is any non-empty subset of Y. For $\eta \in \mathcal{F}$, let $\eta_z = \{W(z, U) : U \in \eta\}$. Further, we put $\mathcal{F}_z = \{\eta_z : \eta \in \mathcal{F}\}$.

In this way, we have defined a family $\mathcal{F}_z$ of families η_z of subsets of the set Z for every $z \in Z$. Clearly, every η_z is a prefilter: the intersection of any two elements of η_z contains an element of η_z, and all elements of η_z are non-empty sets.

Every such assignment of families of subsets to elements of a set generates a topology on this set. In the case we consider, the definition runs as follows.

A subset H of Z will be called *open* if, for every $z \in H$ and every $\eta_z \in \mathcal{F}_z$, there exists $W \in \eta_z$ such that $W \subset H$. The set W in this definition need not contain the point z.

The set of all open subsets of Z so defined will be denoted by $\mathcal{T}_{\mathcal{F}}$. Clearly, $\mathcal{T}_{\mathcal{F}}$ is a topology on Z, since every η_z is a prefilter on Z. We say that this topology is generated on Z by the family $\mathcal{F}$.

Proposition 2.1. *For each $z \in Z$, the set H_z is open.*

Proof: Indeed, for any $p \in H_z$, any $\eta \in \mathcal{F}$ and any $U \in \eta$, we obviously have $W(p,U) \subset H_z$. Therefore, the set H_z is open. $\square$

Proposition 2.2. *For each $z \in Z$ and each $U \in \eta$, the set $W(z,U)$ is open. In other words, every element of the family η_z is open in the space Z.*

Proof: Put $n = r(z)$, and take any $p \in W(z,U)$. Then $z < p$ and $p_{n+1} \in U$. It follows that similar conditions are satisfied by every element of H_p. Hence, $H_p \subset W(z,U)$. Since $p \in H_p$, and H_p is open in Z, by Proposition 2.1, it follows that $W(z,U)$ is open. $\square$

Let us assume that the family $\mathcal{F}$ is indexed: $\mathcal{F} = \{\eta_\alpha : \alpha \in A\}$.

Suppose that $\xi = \{U_\alpha : \alpha \in A\}$ is a family of subsets of X such that $U_\alpha \in \eta_\alpha$, for every $\alpha \in A$. Then we say that ξ is a *choice family* (on $\mathcal{F}$).

Proposition 2.3. *Suppose that $\xi = \{U_\alpha : \alpha \in A\}$ is an arbitrary choice family on $\mathcal{F}$. Then the set $V_\xi(z) = (\bigcup\{W(z,U_\alpha) : \alpha \in A\}) \cup \{z\}$ is an open subset of Z, for any $z \in Z$.*

Proof: Indeed, the definition of an open set is, clearly, satisfied at the point z. It is also satisfied at all other points of $V_\xi(z)$, by Proposition 2.2 and by the definition of $V_\xi(z)$. Hence, $V_\xi(z)$ is open. $\square$

The sets $V_\xi(z)$, such as in the last statement, will be called *special open sets*. Proposition 2.3 and the definition of an open set in Z immediately lead to the next statement:

Proposition 2.4. *The family $\mathcal{B}$ of all special open sets is a base of the space Z.*

Proposition 2.5. *Let z and p be any two elements of Z such that $H_z \cap H_p$ is not empty. Then either $z \le p$ or $p \le z$.*

Proof: Put $n = r(z)$ and $m = r(p)$. We may assume that $n \le m$. Let us show that $z \le p$.

Assume the contrary. Then $z_i \ne p_i$, for some $i \in \{1, \dots, n\}$. Then $h_i \ne p_i$, for any $h \in H_z$. Therefore, any such h is not in H_p, a contradiction. $\square$

Proposition 2.6. *Every special open set $V_\xi(z)$ is closed in the space Z.*

Proof: Take any $p \in Z \setminus V_\xi(z)$, and let $n = r(z)$, $m = r(p)$.

Case 1: $m < n$. Put $y = z_{m+1}$. Clearly, we can find a choice family $\kappa = \{U_\alpha : \alpha \in A\}$ on $\mathcal{F}$ such that $h_{m+1} \ne z_{m+1}$, for any $h \in V_\kappa(p)$. Hence, $H_z \cap V_\kappa(p) = \emptyset$. The conclusion follows, since $V_\kappa(p)$ is open and $V_\xi(z) \subset H_z$.

Case 2: $m = n$. Then $H_z \cap H_p = \emptyset$. The rest is obvious.

Case 3: $n < m$. If it is not true that $z < p$, then $H_p \cap H_z = \emptyset$, and we complete the argument as in Case 2. Assume now that $z < p$. Since p does not belong to $V_\xi(z)$, we have $p_{n+1} \notin \bigcup\{U_\alpha : \alpha \in A\}$. Again, we conclude that H_p and $V_\xi(z)$ do not meet. Since H_p is an open neighbourhood of p, the argument is complete. $\square$

Theorem 2.7. *The space Z is a zero-dimensional T_1-space. Hence, Z is Tychonoff.*

Proof: This statement immediately follows from Propositions 2.4 and 2.6. We also use the assumption that every prefilter in $\mathcal{F}$ is free. $\square$

Observe that the space Z does not have isolated points.

3. On ω-first-countable and related spaces

In this section, we present an example of a non-empty Tychonoff countable Fŕechet-Urysohn space that is not first-countable at any point. We also introduce some related general concepts and study them.

A prefilter η on a topological space X is called *open* if all elements of η are open subsets of X.

A family $\mathcal{P}$ of prefilters on a topological space X will be called *closure-generating* at a point $x \in X$ if, for every subset A of X, the next condition is satisfied:

$x \in \overline{A}$ if and only if there exists $\eta \in \mathcal{P}$ such that every element of η meets A.

A family $\mathcal{P}$ of prefilters on a topological space X will be called *base-generating* at a point $x \in X$ if all $\eta \in \mathcal{P}$ are open prefilters on X and the family $\mathcal{P}$ is closure-generating at x.

The following elementary statement is easy to prove.

Proposition 3.1. *Suppose that x is a point of a space X, and $\mathcal{P} = \{\eta_\alpha : \alpha \in M\}$ is a closure-generating at x family of prefilters on X. Then:*

1) *for any open neighbourhood $O(x)$ of x, one can select $P_\alpha \in \eta_\alpha$, for each $\alpha \in M$, so that the set $B = \{x\} \cup \{P_\alpha : \alpha \in M\}$ is contained in $O(x)$;*
2) *the interior of the set $B = \{x\} \cup \{P_\alpha : \alpha \in M\}$ contains x, for any choice of $P_\alpha \in \eta_\alpha$ for every $\alpha \in M$.*

A topological space X is said to be *side-first-countable* at a point $x \in X$ if there exists a base-generating at x family $\mathcal{P}$ of countable prefilters on X. If, in addition, this family $\mathcal{P}$ can be chosen to be countable, then we call X *almost first-countable* at x (or *ω-first-countable* at x).

If a space is ω-first-countable at every point, we will say that this space is ω-first-countable (or almost first-countable).

A *side-base* of a space X is a family S of open subsets of X such that, for each $x \in X$, there exists a family $\mathcal{P}$ of prefilters such that every $\eta \in \mathcal{P}$ is contained in S and $\mathcal{P}$ is closure-generating at x. If, in addition, for each $x \in X$ the family $\mathcal{P}$ can be selected to be countable, we say that S is an *ω-side-base* of X. Clearly, every base of a space X is an ω-side-base of X.

The minimum of cardinalities of side-bases (of ω-side-bases) of a space X will be called *side-weight* (*ω-side-weight*, respectively) of X.

The concepts introduced above can be illustrated using objects in the construction presented in the preceding section. Indeed, we, obviously, have:

Proposition 3.2. *Let Y, Z, $\mathcal{F}$, and other notation, be as in the Construction. Then, for every $z \in Z$, the family $\mathcal{F}_z$ is base-generating, and the family $S = \bigcup\{\mathcal{F}_z : z \in Z\}$ is a side-base of Z.*

Proof: Let $B \subset Z$, $z \in Z$, and $\eta \in \mathcal{F}_z$ be such that every element of η meets B. Since any open neighbourhood $O(z)$ of z contains some element of η, by the definition of the topology of Z, it follows that $O(z) \cap B \neq \emptyset$. Hence, $z \in \overline{B}$.

Conversely, assume that $z \in \overline{B}$. Then the set $C = \overline{B} \setminus \{z\}$ is not closed. Therefore, the set $V = \{z\} \cup (Z \setminus \overline{B})$ is not open. Since $Z \setminus \overline{B}$ is open, from the definition of open sets in Z it follows that for the set V this definition is violated precisely at the point z. Hence, there exists $\eta \in \mathcal{F}_z$ such that every element of η meets $\overline{B}$. Since every element of η is open in Z, it follows that every element of η meets B. □

When we impose strong assumptions on Y and $\mathcal{F}$, the properties of the space Z in the Construction may become rather interesting. In particular, we have the following statements.

Proposition 3.3. *If $\mathcal{P}$ is a closure-generating family of countable prefilters on X at some $x \in X$, then the space X is Fréchet-Urysohn at x, that is, whenever x is in the closure of a subset B of X, then some sequence of points of B converges to x.*

Proof: Since every prefilter in $\mathcal{P}$ is countable, this follows immediately from the definitions of a closure-generating family and of a prefilter. □

Theorem 3.4. *Suppose that, in the construction, Y is countable, $\mathcal{F}$ is countable, and every $\eta \in \mathcal{F}$ is countable. Then the space Z is countable and has a countable ω-side-base. Thus, the ω-side-weight of Z is countable, and Z is an almost first-countable Frećhet-Urysohn space.*

Proof: The first part of the statement follows from Proposition 3.2 and from the definitions of $\mathcal{F}_z$ and ω-side-base. The second conclusion follows from Proposition 3.3. □

Example 3.5. Let $Y = \omega \times \omega$. Put $P_m = \{j \in \omega : m \leq j\}$ for $m \in \omega$, and $\eta_i = \{P_m \times \{i\} : m \in \omega\}$ for $i \in \omega$. Now let us apply the construction using this Y and the family $\mathcal{F} = \{\eta_i : i \in \omega\}$ of free prefilters on Y. We denote by Z_ω the space Z defined in this way. The space Z_ω is a countable normal Fŕechet-Urysohn space with a countable side-base. This follows immediately from Theorem 3.4. Thus, Z_ω is almost first-countable and has a countable π-base. However, there is no point in Z_ω at which the space Z_ω is first-countable.

To prove this, it is enough to show that, for every $z \in Z_\omega$, the space Z_ω contains a topological copy $V_z(\omega)$ of the countable Fŕechet-Urysohn fan $V(\omega)$ such that z is the only non-isolated point of $V_z(\omega)$. The last conclusion easily follows from the definition of the side-base of Z_ω at z. Since the space $V_z(\omega)$ is not first-countable at z, it follows that Z_ω is not first-countable at z. This also means that no point of Z_ω is isolated.

In particular, Z_ω is not metrizable and does not have a countable base.

It is not difficult to show that the space Z_ω is homogeneous. However, Z_ω is not homeomorphic to a topological group. Indeed, otherwise, Z_ω would have been metrizable, as any topological group with a countable π-base.

4. Some applications to topological groups and to compact spaces

In this section, we prove a few results, involving several concepts introduced above, for compact spaces and for topological groups.

The following convergence concept resembles the notion of a closure-generating family of prefilters.

Let X be a T_1-space, and x be a point of X. A family $\mathcal{S}$ of subsets of X will be called *closure-sensitive at x* (or just *sensitive at x*) if, for each open neighbourhood $O(x)$ of x and for each subset A of $X \setminus \{x\}$ such that $x \in \overline{A}$, there exists $P \in \mathcal{S}$ satisfying the following conditions: $P \subset O(x)$ and $P \cap A$ is infinite.

Obviously, if $\mathcal{B}$ is a base of X at a point x, and X is a T_1-space, then $\mathcal{B}$ is closure-sensitive at x. Every sensitive at x family of sets is a π-network at x, but its elements need not be open. They also need not contain the point x.

The proof of the next statement is omitted, since it is easy and standard.

Proposition 4.1. *If $\mathcal{F}$ is a closure-generating family of prefilters at a point x of a T_1-space X, then $\bigcup \mathcal{F}$ is a closure-sensitive at x family of subsets of X.*

The minimum of cardinalities of sensitive at x families of subsets of X is called the *sensitivity* of X at x and is denoted by $\mathrm{sn}(x, X)$. Clearly, $\mathrm{sn}(x, X) \leq \chi(x, X)$, for any T_1-space X, where $\chi(x, X)$ is the character of X at x. The countable Fréchet-Urysohn fan is not first-countable, but its sensitivity is countable at every point. Such spaces we will call *countably sensitive*.

A space X is *ω-Fréchet-Urysohn at a point $x \in X$* if there exists a countable family $\mathcal{F}$ of countable prefilters on X such that $\mathcal{F}$ is closure-generating at x. Clearly, every ω-first-countable at x space X is ω-Fréchet-Urysohn at x, and if X is ω-Fréchet-Urysohn at x, then X is Fréchet-Urysohn at x. The next statement is obvious in view of Proposition 4.1:

Proposition 4.2. *If a space X is ω-Fréchet-Urysohn at $x \in X$, then X is countably sensitive at x.*

Let us show that the converse to the last statement does not hold.

Example 4.3. Put $X = (\omega \times \omega) \cup \{\theta\}$, where θ is some object not in $\omega \times \omega$. Let us define a topology on X. Every point of $\omega \times \omega$ we declare to be isolated in X. Put $Y_i = \omega \times \{i\}$, for each $i \in \omega$. For any subset W of X we denote by K_W the set of all $i \in \omega$ such that $Y_i \setminus W$ is infinite. A subset W of X containing θ is declared to be open if the set K_W is finite. Let η_i be the set of all subsets P of Y_i such that $Y_i \setminus P$ is finite. Then $\eta = \bigcup \{\eta_i : i \in \omega\}$ is a closure-sensitive at θ countable family of subsets of X. However, the space X is easily seen not to be Fréchet-Urysohn at θ. Hence, X is not ω-Fréchet-Urysohn at θ.

One of main results of this section is Theorem 4.5 below. To prove it, we need the following elementary fact:

Proposition 4.4. *If a T_1-space X is countably sensitive at a point x, then the tightness of X at x is countable.*

PROOF: Let us fix a countable family $\mathcal{S}$ of subsets of X which is closure-sensitive at x, and take any $A \subset X$ such that $x \in \overline{A \setminus \{x\}}$. Fix a point $c(P) \in A \cap P$, for each $P \in \mathcal{S}$ such that $A \cap P \neq \emptyset$. Then $C = \{c(P) : P \in \mathcal{S}, A \cap P \neq \emptyset\}$ is a countable subset of A such that $x \in \overline{C}$, since $\mathcal{S}$ is closure-sensitive at x. Therefore, the tightness of X at x is countable. □

Theorem 4.5. *If a countably compact regular T_1-space X is countably sensitive at a point $x \in X$, then X is first-countable at x.*

PROOF: By Proposition 4.4, the tightness of X is countable. Since X is a countably compact regular T_1-space, it follows that the fan-tightness $\mathrm{vet}(X)$ is countable [3]. Let us fix a countable family $\mathcal{S}$ of subsets of X which is closure-sensitive at x. We may assume that x belongs to every element of $\mathcal{S}$.

Let $O(x)$ be any open neighbourhood of x, and let $\gamma = \{P \in \mathcal{S} : P \subset O(x)\}$. Then γ is countable, since $\mathcal{S}$ is countable. Thus, $\gamma = \{P_n : n \in \omega\}$. Put $B_n = \bigcup\{P_i : i \leq n\}$, for $n \in \omega$.

Claim 1: $x \in \mathrm{int}(B_n)$, for some $n \in \omega$.

Assume the contrary. Then, for each $n \in \omega$, we put $A_n = X \setminus B_n$ and observe that $x \in \overline{A_n}$, for each n in ω. Since the tightness of X is countable, we can find a countable subset C_n of A_n such that $x \in \overline{C_n}$, for each $n \in \omega$. The next claim is obvious:

Claim 2: $P_i \cap C_n = \emptyset$, for each $i \leq n$.

Since $\mathrm{vet}(x, X) \leq \omega$, we can choose a finite subset $K_n \subset C_n$ for every $n \in \omega$ so that x is in the closure of the set $M = \bigcup\{K_n : n \in \omega\}$. Clearly, x is not in M.

Fix $k \in \omega$. It follows from Claim 2 that $P_k \cap K_n$ is empty for every $n > k$. Therefore, $M \cap P_k$ if finite. Thus, the following claim holds:

Claim 3: There exists $M \subset X \setminus \{x\}$ such that $x \in \overline{M}$ and $M \cap P$ is finite, for every $P \in \gamma$.

Now we can produce a contradiction completing the proof of Claim 1. Indeed, since $\mathcal{S}$ is closure-sensitive at x, we can find $P \in \mathcal{S}$ such that $P \subset O(x)$ and $M \cap P$ is infinite. Then $P \in \gamma$, by the definition of γ, and we have a contradiction with Claim 3. Claim 1 is established.

Observe that $B_n \subset O(x)$, for each $n \in \omega$. This observation and Claim 1 imply that the family $\mathcal{E}$ of all subsets V of X such that $V = \mathrm{int}(\bigcup \xi)$, for some finite subfamily ξ of $\mathcal{S}$, and $x \in V$, is a base of X at x. Clearly, $\mathcal{E}$ is countable, since $\mathcal{S}$ is countable. Hence, X is first-countable at x. □

It follows that the Σ-product of uncountably many copies of the closed unit interval $[0,1]$ is not countably sensitive at any point. Notice that this space is countably compact and Fréchet-Urysohn (and enjoys many other properties).

The proof of Theorem 4.5 shows that the following statement is also true:

Theorem 4.6. *A regular T_1-space X is first-countable at a point $x \in X$ if and only if X is countably sensitive at x and the fan-tightness* $\mathrm{vet}(x, X)$ *of X at x is countable.*

Theorem 4.5 can be considerably generalized using the next obvious statement:

Proposition 4.7. *If a space X is countably sensitive at a point $x \in X$, then every subspace Y of X containing x is countably sensitive at x.*

Recall that a point x of a space X is called a *q-point* if there exists a sequence $\{V_i : i \in \omega\}$ of open neighbourhoods of x such that, for any choice of x_i in V_i, the resulting sequence has an accumulation point in X (see [7]).

Theorem 4.8. *If a regular T_1-space X is countably sensitive at a q-point $x \in X$, then X is first-countable at x.*

This theorem follows from Theorem 4.5 and Proposition 4.7 by a standard argument which is omitted.

There is a version of Theorem 4.5 that concerns topological groups.

Theorem 4.9. *Suppose that G is a topological group, and that the space G is Fréchet-Urysohn and countably sensitive at some point. Then G is metrizable.*

PROOF: By a well-known result of P.J. Nyikos [8], the space G is strongly Fréchet-Urysohn. Therefore, the fan-tightness of G is countable. It follows from Theorem 4.6 that G is first-countable. Since every first-countable topological group is metrizable, the conclusion follows. $\square$

We know that a Fréchet-Urysohn topological group need not be metrizable: the Σ-product of uncountably many copies of the space $\mathbb{R}$ of reals can serve as an example. It is natural to ask whether every countably-sensitive topological group G is metrizable. I do not know the answer to this question at this time, though I strongly suspect that it is in the negative.

Here is a useful general statement.

Proposition 4.10. *Let X be a T_1-space, $x \in X$, and $X = \bigcup\{X_n : n \in \omega\}$, where $x \in X_n$ for each $n \in \omega$, and each X_n is first-countable at x. Suppose further that if A is any subset of X such that $x \in \overline{A}$, then $x \in \overline{A \cap X_n}$, for some $n \in \omega$.*

Then X is countably sensitive at x.

PROOF: Fix a countable base γ_n of the subspace X_n at x, for every $n \in \omega$. We claim that the countable family $\mathcal{S} = \bigcup\{\gamma_n : n \in \omega\}$ is closure-sensitive in X at x.

Indeed, take any $A \subset X \setminus \{x\}$ such that $x \in \overline{A}$. Take also an arbitrary open neighbourhood $O(x)$ of x in X. By the assumption in the Theorem, $x \in \overline{A \cap X_k}$, for some $k \in \omega$. Since γ_k is a base of X_k at x, there exists $V \in \gamma_k$ such that $x \in V \subset O(x) \cap X_k \subset O(x)$. Since V is an open neighbourhood of x in X_k, it follows that $x \in \overline{V \cap (A \cap X_k)} \subset \overline{V \cap A}$. Taking into account that X is a T_1-space and that x is not in $V \cap A$, we conclude that the set $V \cap A$ is infinite. $\square$

Theorem 4.11. *Let G be a topological group, $x \in G$, and $G = \bigcup\{X_n : n \in \omega\}$, where $x \in X_n$ for each $n \in \omega$, and each X_n is first-countable at x. Suppose further that if A is any subset of G such that $x \in \overline{A}$, then $x \in \overline{A \cap X_n}$, for some $n \in \omega$.*

Then G is metrizable.

Proof: It follows from the second condition in the theorem that the space G is Fréchet-Urysohn. Now Proposition 4.10 and Theorem 4.9 imply that G is metrizable. □

We present now a result that easily follows from a well-known theorem on the cardinality of first-countable compacta (see [1]) and Theorem 4.5:

Theorem 4.12. *If a compact Hausdorff space X is countably sensitive at each point, then the cardinality of X does not exceed 2^ω.*

The conclusion can be considerably strengthened if the assumptions in the last theorem are replaced with their global version.

A *closure-sensor* of a space X (or in X) is a family $\mathcal{S}$ of subsets of X which is closure-sensitive at each point of X. Clearly, every base of a space X is a closure-sensor of X. On the other hand, a network of a space X need not be a closure-sensor of X.

Here is one of the main results in this section.

Theorem 4.13. *If a regular countably compact T_1-space X has a countable closure-sensor $\mathcal{S}$, then X has a countable base (and hence, X is a metrizable compactum).*

Proof: We just have to add a few observations to those used in the proof of Theorem 4.5. Indeed, let Y be the set of all non-isolated points of X. Clearly, Y is closed in X, and therefore the subspace Y is countably compact.

Arguing as in the proof of Theorem 4.5, we see that the family $\mathcal{E}$ of all subsets V of X such that $V = \operatorname{int}(\bigcup \xi)$, for some finite subfamily ξ of $\mathcal{S}$, satisfies the following condition:

(s) For every $x \in Y$ and every open neighbourhood $O(x)$ of x in X, there exists $V \in \mathcal{E}$ such that $V \subset O(x)$ and $V \cup \{x\}$ is open in X.

Observe that if x, V, and $O(x)$ are such as in condition (s), then $x \in \overline{V} \subset \overline{O(x)}$. Hence, $x \in V \cup \{x\} \subset \operatorname{int}(\overline{V}) \subset \overline{O(x)}$. Since X is regular, it follows that the next claim holds:

Claim 1: The family $\mathcal{H} = \{\operatorname{int}(\overline{V}) : V \in \mathcal{E}\}$ contains a base of X at x, for each $x \in Y$.

In particular, the space Y has a countable base. Since Y is also countably compact, we conclude that Y is compact.

Observe that the family $\mathcal{H}$ in Claim 1 is countable, since $\mathcal{E}$ is countable. Therefore, the family of all finite subfamilies of $\mathcal{H}$ is also countable. It follows now from compactness of Y and Claim 1 that Y is a G_δ-subset of X. However, every closed subset of X contained in $X \setminus Y$ is finite, since X is countably compact and each point of $X \setminus Y$ is isolated. Therefore, the set $X \setminus Y$ is countable. Hence, the family $\mathcal{B} = \mathcal{H} \cup \{\{x\} : x \in X \setminus Y\}$ is a countable base of X. □

The space Z_ω in Example 3.5 is a countable Tychonoff space with a countable side-base. Clearly, any side-base of an arbitrary T_1-space is a closure-sensor of the space. Thus, Z_ω has a countable open closure-sensor. However, Z is not first-countable.

5. Some applications to pseudoopen mappings

Some of the notions introduced above naturally appear in the study of mappings. This opens an opportunity to apply results obtained in this paper to investigate connections between topological spaces established by means of mappings. We present a few such applications in this section.

Let f be a continuous mapping of a space X onto a space Y, and y be a point of Y. Recall that f is said to be *pseudoopen at y* if for every open subset U of X containing $f^{-1}(y)$ we have: $y \in \operatorname{int}(f(U))$. We say that f is a *strict S-mapping at y* if there exists a countable family γ of open subsets of X such that, for each $x \in f^{-1}(y)$, the family γ contains a base of X at x. If f is a pseudoopen mapping (a strict S-mapping) at each $y \in Y$, we call f *pseudoopen* (a *strict S-mapping*, respectively).

Proposition 5.1. *Suppose that f is a continuous mapping of a space X onto a T_1-space Y, and let y be a point of Y such that f is a pseudoopen strict S-mapping at y. Then Y is countably sensitive at y.*

PROOF: Fix a countable family γ of open subsets of X such that, for each $x \in f^{-1}(y)$, γ contains a base of X at x. Put $\mathcal{P} = \{f(U) : U \in \gamma\}$. We claim that the countable family $\mathcal{P}$ of subsets of Y is closure-sensitive at y. Let us check this.

Take any $A \subset Y \setminus \{y\}$ such that y is in the closure of A, and put $B = f^{-1}(A)$. Clearly, the sets $f^{-1}(y)$ and B are disjoint. However, since f is pseudoopen at y, there exists $x \in f^{-1}(y)$ such that x is in the closure of B.

Now take an arbitrary open neighbourhood $O(y)$ of y. Since γ contains a base of X at x, and f is continuous, there exists $U \in \gamma$ such that $x \in U$ and $f(U) \subset O(y)$.

We have $x \in \overline{U \cap B}$, since U is an open neighbourhood of x. It follows that $f(U \cap B)$ is infinite. Indeed, otherwise we can find $z \in f(U \cap B)$ such that x is in the closure of $f^{-1}(z)$. Then, by continuity of f, $f(x) = z$, since Y is a T_1-space. However, $f(x) = y$, and $y \neq z$, since $z \in f(B) = A$ and y is not in A. This contradiction shows that $f(U \cap B)$ is infinite. Since $f(U \cap B) \subset f(U) \cap f(B) \subset f(U) \cap A$, it follows that $f(U) \cap A$ is infinite. Taking into account that $f(U) \in \mathcal{P}$, we conclude that $\mathcal{P}$ is closure-sensitive at y. □

Theorem 5.2. *Suppose that f is a continuous pseudoopen strict S-mapping of a space X onto a T_1-space Y. Then Y is Fréchet-Urysohn and countably sensitive at each point.*

Proof: Clearly, X is first-countable. Since f is pseudoopen, it follows that Y is Fréchet-Urysohn (see [6]). The remaining part of the statement immediately follows from Proposition 5.1. □

Theorem 5.3. *Suppose that f is a continuous pseudoopen strict S-mapping of a space X onto a regular T_1-space Y which is a q-space. Then Y is first-countable.*

Proof: This statement follows from Theorems 5.2 and 4.8. □

Corollary 5.4. *Suppose that f is a continuous pseudoopen strict S-mapping of a space X onto a regular countably compact T_1-space Y. Then Y is first-countable.*

Theorem 5.5. *Suppose that f is a continuous pseudoopen strict S-mapping of a space X onto a regular T_1-space Y, and that the fan-tightness of Y is countable. Then Y is first-countable.*

Proof: This statement follows from Theorems 5.2 and 4.6. □

Theorem 5.6. *Suppose that f is a continuous pseudoopen strict S-mapping of a space X onto a T_1-space Y which is a topological group. Then Y is metrizable.*

Proof: This statement follows from Theorems 5.2 and 4.9. □

The countable Fréchet-Urysohn fan $V(\omega)$ can be naturally represented as an image of a separable metrizable space under a continuous pseudoopen strict S-mapping. However, $V(\omega)$ is not metrizable.

We have already mentioned that, for every strict S-mapping of a space X onto a space Y, the space X is first-countable. The next obvious statement goes in the opposite direction.

Proposition 5.7. *Every countable-to-one mapping of a first-countable space X to a space Y is a strict S-mapping.*

Thus, Proposition 5.1 is applicable to pseudoopen countable-to-one mappings of first-countable spaces.

Corollary 5.8. *Suppose that f is a continuous mapping of a first-countable space X onto a T_1-space Y, and let y be a point of Y such that f is pseudoopen and countable-to-one at y. Then Y is countably sensitive at y and Fréchet-Urysohn at y.*

Can we say more? The proof of the next statement should be obvious by now and is omitted.

Proposition 5.9. *For every countable-to-one pseudoopen continuous mapping of a first-countable space X onto a space Y, the space Y is ω-Fréchet-Urysohn.*

Let us now consider a continuous pseudoopen mapping of a space with a countable base onto an arbitrary space Y. What can we say about the properties of Y? Observe that any such mapping is a strict S-mapping. To formulate the results in the strongest form, we need the following version of the concept of a sensor.

A family $\mathcal{S}$ of subsets of a space X will be called a *Fréchet-Urysohn sensor* on X (or a *FU-sensor* on X) if, for each $A \subset X$ and each $x \in \overline{A} \setminus A$, there exists a countable prefilter $\eta \subset \mathcal{S}$ converging to x such that $P \cap A \neq \emptyset$ for every $P \in \eta$.

A family $\mathcal{S}$ of subsets of a space X will be called a *strong Fréchet-Urysohn sensor* on X (or a *strong FU-sensor* on X) if, for each $x \in X$, there exists a countable family $\mathcal{F}$ of countable prefilters on X converging to x and contained in $\mathcal{S}$ and satisfying the following condition:

For each $A \subset X$ such that $x \in \overline{A}$, there exists $\eta \in \mathcal{F}$ such that $P \cap A \neq \emptyset$ for every $P \in \eta$.

Proposition 5.10. *If f is a continuous pseudoopen mapping of a space X with a countable base $\mathcal{B}$ onto a space Y, then Y has a countable FU-sensor.*

Proof: It is easy to verify that the family $\mathcal{S} = \{f(U) : U \in \mathcal{B}\}$ is a countable FU-sensor on Y. □

Similarly, the next statement is established.

Proposition 5.11. *If f is a continuous countable-to-one pseudoopen mapping of a space X with a countable base $\mathcal{B}$ onto a space Y, then Y has a countable strong FU-sensor.*

Theorem 5.12. *A countable space Y can be represented as an image of a countable space with a countable base under a continuous pseudoopen mapping if and only if Y has a countable strong FU-sensor, that is, if and only if Y is ω-Fréchet-Urysohn at each point y of Y.*

Proof: The necessity follows from Proposition 5.11. Let us prove the sufficiency.

Fix a strong FU-sensor on Y. Thus, for each $y \in Y$ and each $i \in \omega$, we have a countable family $\{\eta_{(y,i)} : i \in \omega\}$ of countable prefilters $\eta_{(y,i)}$, each of which is converging to y, such that the following condition (t) is satisfied:

(t) For any subset A of Y such that $y \in \overline{A}$, and for some $i \in \omega$, we have $P \cap A \neq \emptyset$, for every $P \in \eta_{(y,i)}$.

For each pair $(y,i) \in Y \times \omega$, we define a topological space $X_{(y,i)}$ as follows. First, we define a topological space $Y_{(y,i)}$. The set of points of this space is the set Y. The topology $\mathcal{T}_{(y,i)}$ of it consists of all subsets of $Y \setminus \{y\}$, as well as of the subsets W of Y such that $y \in W$ and $P \subset W$, for some $P \in \eta_{(y,i)}$. Now we let $X_{(y,i)} = Y_{(y,i)} \times \{(y,i)\}$, with the product topology.

The space $X_{(y,i)}$ so defined is first-countable, since $\eta_{(y,i)}$ is countable. Clearly, this space has only one non-isolated point. Let X be the free topological sum of the countable family $\{X_{(y,i)} : y \in Y, i \in \omega\}$ of topological spaces defined in this way. Obviously, X has a countable base.

Let f be the standard mapping of X to Y defined by the rule: $f(z,(y,i)) = z$, for $z \in Y$, $y \in Y$, and $i \in \omega$. Then f is continuous, since $\eta_{(y,i)}$ converges to y. The mapping f is also pseudoopen (see Proposition 3.1). And f is, clearly, onto Y. □

Corollary 5.13. *There exists a continuous pseudoopen mapping of a countable metrizable space X onto a Tychonoff space Y such that, for each $y \in Y$, the space Y is not first-countable at y.*

Proof: Suffices to take the space Z_ω described in Example 3.5 and to apply the construction in the proof of Theorem 5.12. □

In connection with the last result, the following questions naturally arise:

Problem 5.14. *Does there exist a continuous pseudoopen mapping of a non-empty locally compact separable metrizable space X onto a Tychonoff space Y such that, for each $y \in Y$, the space Y is not first-countable at y?*

Problem 5.15. *Does there exist a continuous pseudoopen mapping of a non-empty locally compact countable metrizable space X onto a Tychonoff space Y such that, for each $y \in Y$, the space Y is not first-countable at y?*

If in the above questions we replace "pseudoopen" by the weaker requirement that f is quotient, then both questions get positive answers. Indeed, it suffices to take as Y the free topological group of a countable non-discrete compact Hausdorff space and to use a natural mapping.

Problem 5.16. *Suppose that Y is a non-empty (countable) Tychonoff space with a countable strong FU-sensor $\mathcal{S}$ such that each element of $\mathcal{S}$ is compact. Is then Y first-countable at some point?*

Results of this section can be applied to intersections of countable families of first-countable topologies. Let Y be a set, and $\eta = \{\mathcal{T}_i : i \in \omega\}$ be a countable family of first-countable topologies on Y. Then their intersection $\mathcal{T} = \bigcap \eta$ is also a topology on Y. Provided with this topology, Y is a sequential topological space [4]. Besides, Y with this topology is a T_1-space if every $\mathcal{T}_i$ is a T_1-topology. We will now consider an important special case in which the space $(Y, \mathcal{T})$ so obtained is Fréchet-Urysohn and Hausdorff.

Theorem 5.17. *Suppose that G is a Hausdorff Fréchet-Urysohn topological group the topology $\mathcal{T}$ of which is the intersection of a countable family $\eta = \{\mathcal{T}_i : i \in \omega\}$ of first-countable topologies on the set G. Then the space G is metrizable.*

Proof: Let X be the free topological sum of the spaces X_i, where X_i is the set G given the topology $\mathcal{T}_i$. The obvious natural mapping f of the space X onto the space G (with the topology $\mathcal{T}$) is a quotient mapping. However, since G is Fréchet-Urysohn and Hausdorff, it follows that f is pseudoopen [4]. Clearly, f is countable-to-one, onto, and continuous, and X is first-countable. By Proposition 5.7 and Theorem 4.9, G is metrizable. □

Acknowledgment. I am grateful to Professor Miroslav Hušek for his valuable advice while I was working on this paper.

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(*Received* December 10, 2009*, revised* February 8, 2010)