Two notes on eventually differentiable families of operators

Tomáš Bárta

Abstract. In the first note we show for a strongly continuous family of operators $(T(t))_{t\geq 0}$ that if every orbit $t \mapsto T(t)x$ is differentiable for $t > t_x$, then all orbits are differentiable for $t > t_0$ with t_0 independent of x. In the second note we give an example of an eventually differentiable semigroup which is not differentiable on the same interval in the operator norm topology.

Keywords: eventually differentiable semigroups, operator families

Classification: 47D06

A strongly continuous semigroup is said to be *eventually differentiable* if the mapping $t \mapsto T(t)x$ is differentiable for all $x \in X$ and all $t > t_0$. Batty asked in [2], whether this property is equivalent to

$$\forall\, x \;\exists\, t_x > 0;\; T(\cdot)x \;\text{ is differentiable }\; \forall\, t > t_x. \tag{1}$$

This question was answered positively by Iley in [3]. However, the proof of Iley depends heavily on the semigroup property. In the present result we show that the semigroup property is not needed in the proof and that equivalence of differentiability and (1) holds for every strongly continuous family of operators.

For C_0-semigroups, eventual differentiability is equivalent to differentiability of the mapping $t \mapsto T(t)$ in the operator norm topology. More precisely, if $t \mapsto T(t)$ is differentiable on $(t_0, +\infty)$ then obviously $t \mapsto T(t)x$ is differentiable on the same interval for every $x \in X$. If $t \mapsto T(t)x$ is differentiable on $(t_0, +\infty)$ for every x then $t \mapsto T(t)$ is differentiable on $(2t_0, +\infty)$ (see e.g. [4]). It was not known, whether $2t_0$ can be replaced by t_0 in the last assertion. In the second part of this paper we give an example, which shows it is not possible. It is also an example of a translation semigroup, which is eventually differentiable and neither immediately differentiable nor nilpotent (an example of an immediately differentiable translation semigroup that is not analytic was given in [1]).

Theorem 1. *Let X be a Banach space and $S : \mathbb{R}_+ \to B(X)$ be a strongly continuous family of bounded linear operators with the following property. For every $x \in X$ there exists t_x such that $t \mapsto S(t)x$ is differentiable on $(t_x, +\infty)$.*

This work is a part of the project MSM 0021620839 and was partly supported by Nečas center for mathematical modelling LC 06052 (MŠMT ČR).

Then there exists $t_0 \in \mathbb{R}_+$ such that $t \mapsto S(t)x$ is differentiable on $(t_0, +\infty)$ for every $x \in X$.

Proof: We prove Theorem 1 by contradiction. Assume that there exists a sequence $t_n \to +\infty$ (satisfying $t_{n+1} \geq t_n + 1$) and $x_n \in X$ such that $u_n : t \mapsto S(t)x_n$ is not differentiable at t_n and it is differentiable on $(t_n + \frac{1}{2}, +\infty)$. We will find $y \in X$ such that $v(t) := S(t)y$ will be differentiable at none of t_n, $n \in \mathbb{N}$.

This y will be in the form

$$y := \sum_{n=1}^{\infty} c_n x_n. \tag{2}$$

Observe that we can assume that $\|x_n\| \leq 1$. If $|c_n| \leq 2^{-n}$ then the series in (2) converges and y is well defined. We will find the numbers c_n inductively, such that the orbit v_n, defined by

$$v_n := S(t)y_n, \quad y_n := \sum_{k=1}^{n} c_k x_k, \tag{3}$$

is not differentiable at t_k for all $k \leq n$.

Before we start constructing the sequence c_n, let us mention that nondifferentiability of v_m at t_k $(k \leq m)$ is equivalent to existence of $\varepsilon(k,m) > 0$ and two sequences $t(k,m) = (t(k,m)_i)_{i=1}^{\infty}$, $s(k,m) = (s(k,m)_i)_{i=1}^{\infty}$ converging to t_k such that

$$\left\| \frac{v_m(t(k,m)_i) - v_m(t_k)}{t(k,m)_i - t_k} - \frac{v_m(s(k,m)_i) - v_m(t_k)}{s(k,m)_i - t_k} \right\| \geq \varepsilon(k,m) \qquad \text{for all } \ i \in \mathbb{N}. \tag{4}$$

Simultaneously with c_n we will construct sequences $t(k,n)$, $s(k,n)$ and numbers $\varepsilon(k,n)$. We will construct them in such a way that $t(k,n+1)$ will be a subsequence of $t(k,n)$ and $s(k,n+1)$ will be the corresponding subsequence of $s(k,n)$ and

$$t(k,n+1)_i = t(k,n)_i, \quad s(k,n+1)_i = s(k,n)_i \quad \text{for} \quad i \leq n. \tag{5}$$

Moreover, we will take

$$\varepsilon(k,n) := \varepsilon(k)\left(1 + \frac{1}{n}\right) \tag{6}$$

for some $\varepsilon(k) > 0$. Let us conclude our requirements in the following claim.

Claim. *For all k, $n \in \mathbb{N}$, $k \leq n$, there exist numbers c_n $(0 < c_n \leq 2^{-n})$, $\varepsilon(k) > 0$ and sequences $t(k,n)$, $s(k,n)$ converging to t_k, such that $t(k,n+1)$ is a subsequence of $t(k,n)$ satisfying* (5)*, $s(k,n+1)$ is the corresponding subsequence of $s(k,n)$, and* (4) *holds for $\varepsilon(k,n)$ defined by* (6) *and v_n defined by* (3).

Assume the Claim is true and fix $k \in \mathbb{N}$. Condition (5) guarantees that the diagonal sequence $(t(k))_{i=k}^{\infty}$ defined by $t(k)_i := t(k,i)_i$, $i \geq k$, is a subsequence of $t(k,n)$ for all $n \geq k$ (and similarly for $s(k)$). Therefore, we have for every $n \geq k$

$$\left\| \frac{v_n(t(k)_i) - v_n(t_k)}{t(k)_i - t_k} - \frac{v_n(s(k)_i) - v_n(t_k)}{s(k)_i - t_k} \right\| \geq \varepsilon(k,n) = \varepsilon(k)\left(1 + \frac{1}{n}\right) \geq \varepsilon(k)$$

for all $i \geq k$. Hence, the function $v(t) = \lim_{n\to\infty} v_n(t)$ satisfies

$$\left\| \frac{v(t(k)_i) - v(t_k)}{t(k)_i - t_k} - \frac{v(s(k)_i) - v(t_k)}{s(k)_i - t_k} \right\| \geq \varepsilon(k), \quad i \geq k.$$

It follows that v is not differentiable at t_k. Since k was arbitrary, this is the desired contradiction (we have found an orbit that is not eventually differentiable).

It remains to prove the Claim. We will construct the numbers and sequences inductively. Set $c_1 := \frac{1}{2}$. Since $v_1 = \frac{1}{2}S(t)x_1$ is not differentiable at t_1, there exist $\varepsilon(1,1) > 0$ and two sequences $t(1,1) = (t(1,1)_i)_{i=1}^{\infty}$, $s(1,1) = (s(1,1)_i)_{i=1}^{\infty}$ converging to t_1 and satisfying (4) for $m = k = 1$. Take $\varepsilon(1) := \varepsilon(1,1)/2$ and define $\varepsilon(1,n)$ by (6) for $n \geq 2$ (for $n = 1$ identity (6) also holds).

Let us fix $n \in \mathbb{N}$ and assume that we already have c_m, $\varepsilon(m)$ and sequences $t(k,m)$, $s(k,m)$ satisfying all conditions of the Claim for all $1 \leq k \leq m \leq n$. We will find c_{n+1}, $\varepsilon(n+1)$ and sequences $t(k,n+1)$, $s(k,n+1)$ for all $1 \leq k \leq n+1$ such that they satisfy the conditions of the Claim.

Let $v_{n+1} := v_n + c'_{n+1}u_n$ where c'_{n+1} satisfies $0 < c'_{n+1} < 2^{-n-1}$ and its exact value will be specified later. It holds that

$$\left\| \frac{v_{n+1}(t(k,n)_i) - v_{n+1}(t_k)}{t(k,n)_i - t_k} - \frac{v_{n+1}(s(k,n)_i) - v_{n+1}(t_k)}{s(k,n)_i - t_k} \right\| \tag{7}$$

$$\geq \left\| \frac{v_n(t(k,n)_i) - v_n(t_k)}{t(k,n)_i - t_k} - \frac{v_n(s(k,n)_i) - v_n(t_k)}{s(k,n)_i - t_k} \right\| \tag{8}$$

$$- c'_{n+1} \left\| \frac{u_{n+1}(t(k,n)_i) - u_{n+1}(t_k)}{t(k,n)_i - t_k} - \frac{u_{n+1}(s(k,n)_i) - u_{n+1}(t_k)}{s(k,n)_i - t_k} \right\|. \tag{9}$$

Define

$$t(k,n+1)_i := t(k,n)_i, \quad s(k,n+1)_i := s(k,n)_i \quad \text{for} \quad i,k \leq n \tag{10}$$

(so, (5) will be satisfied for these sequences). If c'_{n+1} is small enough then (9) is less than $\varepsilon(k)\frac{1}{n(n+1)}$ for all i, $k \leq n$. It follows that (4) holds (with $m = n+1$) for i, $k \leq n$. In fact, (7) is larger than

$$\varepsilon(k,n) - \varepsilon(k)\frac{1}{n(n+1)} = \varepsilon(k)\left(1 + \frac{1}{n} - \frac{1}{n(n+1)}\right) = \varepsilon(k)\frac{n+2}{n+1} = \varepsilon(k,n+1).$$

Let us point out that further reduction of $c'_{n+1} > 0$ will not destroy the inequality (4) for i, $k \leq n$.

We will construct the rest ($i > n$) of the sequences $t(k, n+1)$, $s(k, n+1)$, $k \le n$. Let us fix $k \le n$ and define $v^k_{n+1} := v_n + d(k)u_{n+1}$ where $d(k) > 0$ is arbitrary. Then exactly one of the following is true: (A) for every $d(k) > 0$ there exists a subsequence $t(k, n+1)$ of $t(k, n)$ such that (4) holds for v^k_{n+1} and $m = n+1$, or (B) there exists $d(k) > 0$ such that

$$\left\| \frac{v^k_{n+1}(t(k,n)_i) - v^k_{n+1}(t_k)}{t(k,n)_i - t_k} - \frac{v^k_{n+1}(s(k,n)_i) - v^k_{n+1}(t_k)}{s(k,n)_i - t_k} \right\| < \varepsilon(k, n+1) \tag{11}$$

for all $i > n$ with at most finitely many exceptions.

If (A) is true, set $\tilde{d}(k) := +\infty$. If (B) is true, take d such that for $d(k) = d$ inequality (11) holds for all $i > n$ with at most finitely many exceptions. Set

$$\tilde{d}(k) = d \frac{\varepsilon(k,n) - \varepsilon(k,n+1)}{\varepsilon(k,n) + \varepsilon(k,n+1)}.$$

Denote the expression in the norm in (8) by O_i and the expression in the norm in (9) by P_i. Then (11) is equivalent to

$$d(k)P_i \in B_i := B(O_i, \varepsilon(k, n+1)),$$

where $B(O, r)$ denotes the ball of radius r centered at O. Since the ball B_i does not contain zero ($\|O_i\| \ge \varepsilon(k,n) > \varepsilon(k,n+1)$), for $d_i > 0$ small enough we have $d_i B_i \cap B_i = \emptyset$. In fact, d_i must be smaller than

$$\frac{\|O_i\| - \varepsilon(k,n+1)}{\|O_i\| + \varepsilon(k,n+1)}.$$

Since the function $z \mapsto \frac{z-c}{z+c}$ is increasing on $(0, +\infty)$ if $c > 0$, we have

$$\frac{\|O_i\| - \varepsilon(k,n+1)}{\|O_i\| + \varepsilon(k,n+1)} \ge \frac{\varepsilon(k,n) - \varepsilon(k,n+1)}{\varepsilon(k,n) + \varepsilon(k,n+1)}$$

for all $i \in \mathbb{N}$. Hence, if we take an arbitrary $d(k)$ satisfying

$$d(k) \le d \frac{\varepsilon(k,n) - \varepsilon(k,n+1)}{\varepsilon(k,n) + \varepsilon(k,n+1)},$$

then we have

$$dP_i \in B_i \quad \Rightarrow \quad d(k)P_i \notin B_i. \tag{12}$$

Hence, for all $d(k) \le \tilde{d}(k)$, there exists a subsequence $t(k, n+1)$ of $t(k, n)$ (and corresponding subsequence $s(k, n+1)$) such that (4) holds with v^k_{n+1} and $m = n+1$. The last sentence is true in both cases (A) and (B). Hence, $c_{n+1} := \min\{c'_{n+1},\ d(k); k \le n\}$ is the desired number, for which the beginning ($i \le n$) and also the rest ($i > n$) of the subsequences $t(k, n+1)$, $s(k, n+1)$ satisfy (4).

It remains to note that u_k, $k \leq n$ are differentiable at t_{n+1} and u_{n+1} is not. Hence v_{n+1} is not differentiable at t_{n+1} ($c_{n+1} > 0$) and we can find $\varepsilon(n+1, n+1)$ and sequences $t(n+1, n+1)$, $s(n+1, n+1) \to t_{n+1}$ such that (4) holds. Take $\varepsilon(n+1) := \frac{n+1}{n+2} \cdot \varepsilon(n+1, n+1)$. By defining $\varepsilon(n+1, l)$ for $l > n+1$ by (6) we finish the inductive step and thus conclude the proof of the Claim. □

Now we give an example of a semigroup where intervals of "strong differentiability" and "operator norm differentiability" do not coincide.

Example 1. Denote $\mathbb{N} := \{0, 1, 2, \dots\}$. Consider the set

$$X := \{f \in \mathrm{BUC}([0, +\infty)) : \exists B > 0, \forall n \in \mathbb{N}$$
$$f^{(n)} \in \mathrm{BUC}((2\pi n, +\infty)) \ \&\ |f^{(n)}(t)| \leq B \ \forall\ t \in (2\pi n, +\infty) \ \&\ f \text{ satisfies (U)}\},$$

where (U) is the following property

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall n \in \mathbb{N}, \ s, t \in (2\pi n, +\infty), \ |s - t| < \delta \Rightarrow |f^{(n)}(s) - f^{(n)}(t)| < \varepsilon,$$

i.e., uniform continuity of the derivatives is uniform with respect to n. We define a norm on X by

$$\|f\|_X := \sup_{n \in \mathbb{N}} \sup\{|f^{(n)}(t)| : \ t \in (2\pi n, +\infty)\}.$$

Then $(X, \|\cdot\|_X)$ is a Banach space. In fact, let f_k be a Cauchy sequence. Then clearly f_k converge to a function f in sup-norm with all the derivatives on appropriate intervals. The derivatives converge uniformly since the Cauchy estimates hold uniformly, hence we have convergence in the norm $\|\cdot\|_X$. The limit function satisfies the estimate $|f^{(n)}(t)| \leq B$ for some $B > 0$ since the functions f_k satisfy such an estimate and $\|f_k - f\|_X$ is small. Property (U) follows by the same argument as closedness of $\mathrm{BUC}([0, +\infty))$.

Define for $t \geq 0$ the mapping $T(t) : X \to X$ by

$$(T(t)f)(s) := f(t + s), \quad s \geq 0.$$

Then $(T(t))_{t \geq 0}$ is a strongly continuous semigroup on X (this is due to property (U)) which is eventually differentiable on $(2\pi, +\infty)$. In fact, for $n \in \mathbb{N}$, $t > 2\pi n$, $s > 2\pi$, $h > 0$ and $k \leq n$ we have

$$\left| \frac{1}{h}(f^{(k)}(t + s + h) - f^{(k)}(t + s)) - f^{(k+1)}(t + s) \right| = |f^{(k+1)}(t + \xi) - f^{(k+1)}(t + s)|$$

for some $\xi \in (s, s + h)$. If $h < \delta$ then the last term is less than ε (for all s and k) by property (U). This yields convergence of $\frac{1}{h}(T(s+h)f - T(s)f)$ in the norm $\|\cdot\|_X$.

We show that the semigroup is not differentiable in the operator norm topology on $(2\pi, 4\pi)$. The idea is similar to the proof that the translation semigroup is not norm continuous on BUC$([0,+\infty))$. Let $t\in(2\pi,4\pi)$. Take (for $k\in\mathbb{N}$, $k\geq 1$)

$$f_k(x) := \begin{cases} \sin x, & x\in[0,2\pi]\cup[4\pi,+\infty),\\ \frac{1}{k}\sin(kx), & x\in(2\pi,4\pi). \end{cases}$$

We will show that

$$\left\|\frac{1}{h}(T(t+h)-T(t))-AT(t)\right\|_{L(X)}\not\to 0.$$

In particular, we show that there exists $\varepsilon>0$ such that for every fixed $h>0$ $(h<4\pi-t)$ there exist $k\in\mathbb{N}$ and $s\in(0,4\pi-t-h)$ such that

$$\left|\frac{1}{h}(f_k(t+h+s)-f_k(t+s))-f_k'(t+s)\right|>\varepsilon.$$

Take $s:=2\pi/k$ and k so large, such that $s\in(0,4\pi-t-h)$. Then we have

$$f_k'(t+s)=\cos(k(t+s))=\cos(kt)$$

and

$$|f_k(t+h+s)|\leq\frac{1}{k},\quad |f_k(t+s)|\leq\frac{1}{k}\,.$$

An easy computation yields

$$\frac{1}{h}(f_k(t+h+s)-f_k(t+s))-f_k'(t+s)=\sin(kt)\frac{\cos(kh)-1}{kh}+\cos(kt)\left(\frac{\sin(kh)}{kh}-1\right).$$

If we choose k large enough and such that $\cos(kt)>1/2$, then

$$\left|\frac{1}{h}(f_k(t+h+s)-f_k(t+s))-f_k'(t+s)\right|>\frac{1}{4}\,.$$

References

[1] Bárta T., *On smooth solutions of Volterra equations via semigroups*, Bull. Austral. Math. Soc. **78** (2008), 249–260.

[2] Batty C.J.K., *Differentiability of perturbed semigroups and delay semigroups*, Perspectives in Operator Theory, 39–53, Banach Center Publ., 75, Polish Acad. Sci., Warsaw, 2007.

[3] Iley P., *Perturbations of differentiable semigroups*, J. Evol. Equ. **7** (2007), no. 4, 765–781.

[4] Pazy A., *Semigroups of Linear operators and Applications to Partial Differential Equations*, Springer, Berlin, 1983.

Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Prague 8, Czech Republic

(*Received* October 12, 2009, *revised* January 7, 2010)