On some equivalent geometric properties in the Besicovitch-Orlicz space of almost periodic functions with Luxemburg norm

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Abstract. The paper is concerned with the characterization and comparison of some local geometric properties of the Besicovitch-Orlicz space of almost periodic functions. Namely, it is shown that local uniform convexity, H-property and strict convexity are all equivalent. In our approach, we first prove some metric type properties for the modular function associated to our space. These are then used to prove our main equivalence result.

Keywords: locally uniform convexity, strict convexity, H-property, Besicovitch-Orlicz space, almost periodic functions

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1. Preliminaries

We start with some notations and definitions.

In what follows φ is a Young function i.e., an even convex function such that $\varphi(0) = 0$, $\varphi(u) > 0$ iff $u > 0$ and $\lim_{|u|\to\infty} \varphi(u) = +\infty$.

Let $M(\mathbb{R})$ be the space of Lebesgue locally integrable functions on $\mathbb{R}$. The functional

$$\rho_{B^\varphi} : M(\mathbb{R}) \to [0, +\infty],\ \rho_{B^\varphi}(f) = \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{-T}^{+T} \varphi(|f(t)|)\, dt$$

is a convex pseudomodular and the associated modular space

$$\begin{aligned} B^\varphi(\mathbb{R}) &= \left\{ f \in M(\mathbb{R}), \lim_{\alpha\to 0}\ \rho_{B^\varphi}(\alpha f) = 0 \right\} \\ &= \left\{ f \in M(\mathbb{R}), \rho_{B^\varphi}(\lambda f) < +\infty, \quad \text{for some} \ \ \lambda > 0 \right\}, \end{aligned}$$

is called the Besicovitch-Orlicz space.

We endow $B^\varphi(\mathbb{R})$ with the Luxemburg pseudonorm (cf. [5], [12])

$$\|f\|_{B^\varphi} = \inf \left\{ k > 0, \rho_{B^\varphi}\left(\frac{f}{k}\right) \leq 1 \right\}.$$

Beside the norm convergence, we may define in $B^\varphi(\mathbb{R})$ the modular convergence: a sequence $\{f_n\}$ is modular convergent to some f if there exists $k > 0$ such that $\lim_{n\to\infty}\rho_{B^\varphi}(k(f_n-f))=0$. It is well known that norm convergence implies modular convergence and these are equivalent when φ satisfies the Δ_2-condition:

(Δ_2) there exist $K>2$ and $u_0\geq 0$ such that $\varphi(2u)\leq K\varphi(u)$, for all $u\geq u_0$.

Let $C^\circ a.p.$ be the class of Bohr's almost periodic functions. The Besicovitch-Orlicz space of almost periodic functions denoted by $B^\varphi a.p.$ (resp. $\widetilde{B}^\varphi a.p.$) is the closure of $C^\circ a.p.$ in $B^\varphi(\mathbb{R})$ with respect to the pseudonorm $\|.\|_{B^\varphi}$ (resp. to the modular convergence), more exactly:

$$\begin{aligned}
B^\varphi a.p. &= \left\{f\in B^\varphi(\mathbb{R}) : \exists\, (p_n)_{n=1}^\infty \subset C^\circ a.p.,\ \lim_{n\to\infty}\|f-p_n\|_{B^\varphi}=0\right\}\\
&= \left\{f\in B^\varphi(\mathbb{R}) : \exists\, (p_n)_{n=1}^\infty \subset C^\circ a.p.,\ \forall\, k>0,\ \lim_{n\to\infty}\ \rho_{B^\varphi}\left(k\left(f-p_n\right)\right)=0\right\},\\
\widetilde{B}^\varphi a.p. &= \left\{f\in B^\varphi(\mathbb{R}) : \exists\, (p_n)_{n=1}^\infty \subset C^\circ a.p.,\ \exists k>0,\ \lim_{n\to\infty}\ \rho_{B^\varphi}\left(k\left(f-p_n\right)\right)=0\right\}.
\end{aligned}$$

It is clear that

$$B^\varphi_{a.p.}\subseteq \widetilde{B}^\varphi_{a.p.}\subseteq B^\varphi(\mathbb{R}).$$

When $\varphi(x)=|x|$, we denote by $B^1(\mathbb{R})$ and $B^1a.p.(\mathbb{R})$ the respective associated spaces. The notation ρ_{B^1} is used for the corresponding pseudomodular.

From [5] and [12] we know that $\varphi(|f|)\in B^1a.p.$ when $f\in B^\varphi a.p.$ Then by a classical result (cf. [1]) the limit exists (and is finite) in the expression of $\rho_{B^\varphi}(f)$, i.e.:

$$\rho_{B^\varphi}(f)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{+T}\varphi(|f(t)|)\,dt,\ \ f\in B^\varphi a.p. \tag{1.1}$$

This fact is very useful in our computations.

We now recall some convexity notions that will be considered below.

Let $(X,\|.\|)$ be a Banach space and $\delta_X:[0,2]\times S(X)\to[0,1]$ be the function defined by

$$\delta_X(\varepsilon,x)=\inf\left\{1-\left\|\frac{1}{2}(x+y)\right\| : y\in B(X),\|x-y\|\geq\varepsilon\right\},$$

where the notations $S(X)$ and $B(X)$ are used for the unit sphere and unit ball respectively.

Then X is called locally uniformly convex (write *LUC*) if $\delta_X(\varepsilon,x)>0$ whenever $\varepsilon\in]0,2]$ and $x\in S(X)$. The function δ_X is the modulus of local uniform convexity of X.

There are also sequential characterization of LUC (cf. [11]):

The space $(X,\|.\|)$ is LUC if and only if for each $x\in S(X)$ and every sequence (y_n) in $S(X)$ (or $B(X)$) such that $\|\frac{1}{2}(x+y_n)\|\to 1$ it follows that $\|y_n-x\|\to 0$.

We recall that a Banach space $(X, \|.\|)$ is strictly convex (write SC) if for any $x, y \in S(X)$ with $\|x - y\| > 0$, we have $\|x + y\| < 2$. It has the H-property if, whenever a sequence (x_n) is weakly convergent in X to some $x \in X$ (write $x_n \xrightarrow{w} x$) and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$ in norm.

Clearly, every uniformly convex space is LUC. We also know from [8] that LUC spaces are SC and have the H-property (also called the Radon-Riesz property or Kadec Klee property).

These properties among others from the geometry of Banach spaces, have some important applications to approximation and optimization theory.
A full characterization and some comparison results concerning these properties in the case of Orlicz spaces may be found in [3], [4], [6], [7], [9], [10], [16].

In [12], we characterized the strict and uniform convexity of the space $\widetilde{B}^{\varphi}a.p.$, namely it is shown that

(1) the space $\widetilde{B}^{\varphi}a.p.$ is SC if and only if φ is strictly convex and satisfies the Δ_2-condition;
(2) the space $\widetilde{B}^{\varphi}a.p.$ is UC if and only if φ is strictly convex on $\mathbb{R}$, uniformly convex for large u (i.e. for $|u| \geq d$ with some fixed $d \geq 0$) and satisfies the Δ_2-condition.

The papers [13], [14] are respectively concerned with similar characterizations in the case where the space $\widetilde{B}^{\varphi}a.p.$ is endowed with the so called Orlicz norm.

In this work, completing the results in [12], it is stated that SC, LUC and H-property are all equivalent in the Besicovitch-Orlicz space of almost periodic functions $\widetilde{B}^{\varphi}a.p.$

2. Auxiliary results

Let $P(\mathbb{R})$ be the family of all subsets of $\mathbb{R}$ and $\Sigma(\mathbb{R})$ the Σ-algebra of its Lebesgue measurable sets. We define the set function:

$$\overline{\mu}(A) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{+T} \chi_A(t)\, dt = \overline{\lim_{T\to\infty}} \frac{1}{2T} \mu(A \cap [-T, +T])$$

where χ_A denotes the characteristic function of $A \in \Sigma(\mathbb{R})$.

Clearly, $\overline{\mu}$ is null on sets with finite measure μ and is not additive.
As usual, a sequence of Σ-measurable functions $\{f_k\}_{k\geq 1}$ is called $\overline{\mu}$-convergent to f when, for all $\varepsilon > 0$

$$\lim_{k\to\infty} \overline{\mu}\, \{t \in \mathbb{R}, |f_k\,(t) - f\,(t)| \geq \varepsilon\} = 0.$$

Let now $\{A_i\}_{i\geq 1}, A_i \in \Sigma$ be such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i\geq 1} A_i \subset [0, \alpha]$, $\alpha < 1$. Put $f = \sum_{i\geq 1} a_i \chi_{A_i}$ with $\sum_{i\geq 1} \varphi(a_i)\mu(A_i) < +\infty$ and let $\widetilde{f}$ be the periodic extension of f to the whole $\mathbb{R}$ (with period $\tau = 1$). Then for all $\varepsilon > 0$ there exists a Bohr almost periodic function P_ε such that

$$\rho_{B^\phi}\left(\frac{\widetilde{f} - P_\varepsilon}{2}\right) \leq \varepsilon \quad \text{(see [12])}. \tag{2.1}$$

The following technical results are of importance in the proof of our main theorem.

Lemma 1. *Let $f \in B^\varphi a.p.$ Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for each $Q \in \Sigma$ with $\overline{\mu}(Q) \leq \delta$ we have $\rho_{B^\varphi}(f\chi_Q) \leq \varepsilon$.*

Proof: We first prove the result in the case when $f \in C^\circ a.p.$

Let $M = \sup_{x\in\mathbb{R}} \varphi(|f(x)|)$ and $\delta = \frac{\varepsilon}{M}$. If $Q \in \Sigma$ is such that $\overline{\mu}(Q) \leq \delta$ we will have

$$\rho_{B^\varphi}\left(f\chi_Q\right) = \rho_{B^1}\left(\varphi(|f|)\chi_Q\right) \leq \rho_{B^1}\left(M\chi_Q\right) = M\overline{\mu}(Q) \leq \varepsilon.$$

Now, if $f \in B^\varphi a.p.$ we take $P \in C^\circ a.p.$ such that $\rho_{B^\varphi}(2(f - P)) \leq \varepsilon$ and the result follows directly from the inequality

$$\rho_{B^\varphi}\left(f\chi_Q\right) \leq \frac{1}{2}\rho_{B^\varphi}\left(2\left(f - P\right)\chi_Q\right) + \frac{1}{2}\rho_{B^\varphi}\left(2P\chi_Q\right).$$

□

Lemma 2. *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^\varphi(\mathbb{R})$. Then:*

(i) *if $\{f_n\}_{n\geq 1}$ is modular convergent to $f \in B^\varphi(\mathbb{R})$ it is also $\overline{\mu}$-convergent to f.*

(ii) *If $\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to $f \in B^\varphi(\mathbb{R})$ and there exists $g \in B^\varphi a.p.$ satisfying $\max(|f_n|, |f|) \leq g$. Then*

$$\lim_{n\to\infty} \rho_{B^\varphi}\left(f_n\right) = \rho_{B^\varphi}(f).$$

(iii) *If $\{f_n\}_{n\geq 1}$ is modular convergent to $f \in B^\varphi a.p.$, we have*

$$\underline{\lim}_{n\to+\infty} \rho_{B^\varphi}(f_n) \geq \rho_{B^\varphi}(f).$$

(iv) *If $f \in B^\varphi a.p.$ and P_n is its sequence of Bochner-Fejèr's polynomials we have $\lim_{n\to\infty} \rho_{B^\varphi}(P_n) = \rho_{B^\varphi}(f)$.*

Proof: These properties are proved in [15] (see Lemma 1, Lemma 4, Proposition 6 and Corollary 7 respectively of this reference). □

Remark 1. Using Lemma 1, we may prove the following property which is in some sense the converse to (i):

(i)′ *If $\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to $f \in B^\varphi(\mathbb{R})$ and there exists $g \in B^\varphi a.p.$ such that $\max(|f_n|, |f|) \leq g$, then $\{f_n\}_{n\geq 1}$ is also modular convergent to f.*

Indeed, let $\varepsilon > 0$ and $E_n = \{t \in \mathbb{R} : |f_n(t) - f(t)| \geq \varepsilon\}$. Then in view of Lemma 1, there exists $\delta > 0$ such that:

$$\overline{\mu}\left(E_n\right) \leq \delta \Longrightarrow \max\left(\rho_{B^{\varphi}}\left(2f_n\chi_{E_n}\right), \rho_{B^{\varphi}}\left(2f\chi_{E_n}\right)\right) \leq \rho_{B^{\varphi}}\left(2g\chi_{E_n}\right) \leq \varphi(\varepsilon).$$

Now, since $\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to f, we have $\overline{\mu}(E_n) \leq \delta$, $\forall n \geq n_0$ (for some fixed n_0). It follows that

$$\begin{aligned}\rho_{B^{\varphi}}\left(f_n - f\right) &\leq \rho_{B^{\varphi}}\left(\left(f_n - f\right)\chi_{E_n}\right) + \rho_{B^{\varphi}}\left(\left(f_n - f\right)\chi_{E_n^c}\right)\\ &\leq \frac{1}{2}\left[\rho_{B^{\varphi}}\left(2f_n\chi_{E_n}\right) + \rho_{B^{\varphi}}\left(2f\chi_{E_n}\right)\right] + \varphi(\varepsilon)\overline{\mu}\left(E_n^c\right)\\ &\leq \rho_{B^{\varphi}}\left(2g\chi_{E_n}\right) + \varphi(\varepsilon)\\ &\leq 2\varphi(\varepsilon).\end{aligned}$$

Then, since $\varepsilon > 0$ is arbitrary we get $\lim_{n\to\infty} \rho_{B^{\varphi}}(f_n - f) = 0$.

Lemma 3. *For $f \in B^{\varphi}a.p.$, the following equivalences hold:*

(i) $\|f\|_{B^{\varphi}} \leq 1$ *if and only if* $\rho_{B^{\varphi}}(f) \leq 1$.
(ii) $\|f\|_{B^{\varphi}} = 1$ *if and only if* $\rho_{B^{\varphi}}(f) = 1$.

Proof: This lemma was initially proved in [12, Corollary 2], under the additional condition $\varphi \in \Delta_2$. The proof of the present statement may be found in [13, Lemma 4.2]. □

Lemma 4. *Let $\{f_n\}$, $\{g_n\}$ be sequences in $B^{\varphi}a.p.$ such that $\rho_{B^{\varphi}}(f_n) \leq 1$, $\rho_{B^{\varphi}}(g_n) \leq 1$ and $\lim_{n\to\infty} \rho_{B^{\varphi}}\left(\frac{1}{2}(f_n + g_n)\right) = 1$. Suppose that φ is strictly convex. Then the sequence $\{f_n - g_n\}_n$ is $\overline{\mu}$-convergent to zero.*

Proof: Suppose the assertion is false. Then, there exist $\varepsilon > 0$, $\sigma > 0$ and a sequence $(n_k)_k$ increasing to infinity such that $\overline{\mu}(E_k) > \varepsilon$, where $E_k = \{t \in \mathbb{R} : |f_{n_k}(t) - g_{n_k}(t)| \geq \sigma\}$.

Let $k_\varepsilon > 1$ be such that

$$\overline{\mu}(E) \geq \frac{\varepsilon}{4} \Rightarrow \rho_{B^{\varphi}}\left(\chi_E\right) > \frac{1}{k_\varepsilon}.$$

Then putting

$$\begin{aligned}A_k &= \left\{t \in \mathbb{R} : |f_{n_k}(t)| > k_\varepsilon\right\},\\ B_k &= \left\{t \in \mathbb{R} : |g_{n_k}(t)| > k_\varepsilon\right\},\end{aligned}$$

we obtain

$$1 \geq \rho_{B^{\varphi}}\left(f_{n_k}\right) \geq \rho_{B^{\varphi}}\left(f_{n_k}\chi_{A_k}\right) \geq k_\varepsilon \rho_{B^{\varphi}}\left(\chi_{A_k}\right).$$

It follows that $\rho_{B^{\varphi}}\left(\chi_{A_k}\right) \leq \frac{1}{k_\varepsilon}$ and then

$$\overline{\mu}\left(A_k\right) \leq \frac{\varepsilon}{4}.$$

In the same way we may show that

$$\overline{\mu}\,(B_k) \le \frac{\varepsilon}{4}\,.$$

Now, define the set

$$Q = \{(u,v) \in \mathbb{R}^2 : |u| \le k_\varepsilon, |v| \le k_\varepsilon, |u-v| \ge \sigma\},$$

and consider the function

$$F(u,v) = \frac{2\varphi\left(\frac{u+v}{2}\right)}{\varphi\,(u) + \varphi\,(v)}\,.$$

Since φ is strictly convex we have $F(u,v) < 1$ for all $(u,v) \in Q$. Then using the continuity of φ on Q (where Q is a compact set of $\mathbb{R}^2$), it follows that

$$\sup_Q F(u,v) = 1-\delta \quad \text{for some} \quad \delta \in \left]0,1\right[\,.$$

More precisely, we have

$$\varphi\left(\frac{u+v}{2}\right) \le (1-\delta)\,\frac{\varphi(u)+\varphi(v)}{2}, \forall\,(u,v) \in Q.$$

Let now $t \in E_k \backslash (A_k \cup B_k)$. Then $f_{n_k}(t), g_{n_k}(t) \in Q$ and consequently

$$\varphi\left(\frac{|f_{n_k}(t)+g_{n_k}(t)|}{2}\right) \le (1-\delta)\,\frac{\varphi\,(|f_{n_k}(t)|) + \varphi\,(|g_{n_k}(t)|)}{2}\,. \tag{2.2}$$

It follows that

$$\begin{aligned}
&1-\rho_{B^\varphi}\left(\frac{f_{n_k}+g_{n_k}}{2}\right)\\
&\ge \frac{\rho_{B^\varphi}\,(f_{n_k}) + \rho_{B^\varphi}\,(g_{n_k})}{2} - \rho_{B^\varphi}\left(\frac{f_{n_k}+g_{n_k}}{2}\right)\\
&\ge \overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{[E_k\backslash(A_k\cup B_k)]\cap[-T,+T]}\\
&\quad \left[\frac{\varphi\,(|f_{n_k}(t)|) + \varphi\,(|g_{n_k}(t)|)}{2} - \varphi\left(\frac{|f_{n_k}(t)+g_{n_k}(t)|}{2}\right)\right] dt\\
&\ge \frac{\delta}{2}\,\overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{[E_k\backslash(A_k\cup B_k)]\cap[-T,+T]} \left[\varphi\,(|f_{n_k}(t)|) + \varphi\,(|g_{n_k}(t)|)\right] dt\\
&\ge \delta\,\overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{[E_k\backslash(A_k\cup B_k)]\cap[-T,+T]} \varphi\left(\frac{|f_{n_k}(t)-g_{n_k}(t)|}{2}\right) dt\\
&\ge \delta\varphi\left(\frac{\sigma}{2}\right)\left(\varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4}\right) = \delta\frac{\varepsilon}{2}\varphi\left(\frac{\sigma}{2}\right),
\end{aligned}$$

a contradiction with the hypothesis $\lim_{n\to+\infty} \rho_{B^\varphi}(\frac{f_n+g_n}{2}) = 1$. □

Lemma 5. *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}a.p.$ Suppose that $\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to some f in $B^{\varphi}a.p.$ and $\lim_{n\to+\infty}\rho_{B^{\varphi}}(f_n)=\rho_{B^{\varphi}}(f)$. Then*

$$\lim_{n\to+\infty}\rho_{B^{\varphi}}\left(\frac{f_n-f}{2}\right)=0.$$

PROOF: It is clear that $\{\varphi(\frac{|f_n-f|}{2})\}_n$ is $\overline{\mu}$-convergent to 0 and that the sequence

$$g_n=\frac{\varphi\left(|f_n|\right)+\varphi\left(|f|\right)}{2}-\varphi\left(\frac{|f_n-f|}{2}\right)$$

is $\overline{\mu}$-convergent to $g=\varphi(|f|)$.

Now, we shall prove that

$$\varliminf_{n\to+\infty}\rho_{B^1}(g_n)\geq\rho_{B^1}(g)=\rho_{B^{\varphi}}(f).$$

For, let

$$h_n(t)=\begin{cases}g(t) & \text{if } |g_n(t)|>|g(t)|,\\ g_n(t) & \text{if } |g_n(t)|\leq|g(t)|.\end{cases}$$

Then,

$$|h_n(t)-g(t)|=\begin{cases}0 & \text{if } |g_n(t)|>|g(t)|,\\ |g_n(t)-g(t)| & \text{if } |g_n(t)|\leq|g(t)|.\end{cases}$$

It follows that $|h_n(t)-g(t)|\leq|g_n(t)-g(t)|$ and, consequently, the sequence $\{h_n\}_n$ is $\overline{\mu}$-convergent to g.

Now, since $|h_n(t)|\leq|g(t)|$ and $g\in B^1a.p.$, using Lemma 2 we deduce that $\lim_{n\to+\infty}\rho_{B^1}(h_n)=\rho_{B^1}(g)$. Hence,

$$\rho_{B^{\varphi}}\left(f\right)=\rho_{B^1}(g)=\lim_{n\to+\infty}\rho_{B^1}(h_n)\leq\varliminf_{n\to+\infty}\rho_{B^1}(g_n).$$

Then

$$\begin{aligned}\rho_{B^{\varphi}}(f)&=\rho_{B^1}(\varphi(|f|))\\ &\leq\varliminf_{n\to+\infty}\rho_{B^1}\left(\frac{\varphi\left(|f_n|\right)+\varphi\left(|f|\right)}{2}-\varphi\left(\frac{|f_n-f|}{2}\right)\right)\\ &\leq\varliminf_{n\to+\infty}\left\{\frac{1}{2}\rho_{B^{\varphi}}\left(f_n\right)+\frac{1}{2}\rho_{B^{\varphi}}\left(f\right)-\rho_{B^{\varphi}}\left(\frac{f_n-f}{2}\right)\right\}\\ &\leq\rho_{B^{\varphi}}(f)-\varlimsup_{n\to+\infty}\rho_{B^{\varphi}}\left(\frac{f_n-f}{2}\right).\end{aligned}$$

Thus $\lim_{n\to+\infty}\rho_{B^{\varphi}}(\frac{f_n-f}{2})=0$. $\square$

Let $L^{\varphi}([0,1])$ be the usual Orlicz space of functions defined on $[0,1]$. We denote by ρ_{φ} and $\|.\|_{\varphi}$ the respective associated modular and Luxemburg norm. We will show that $L^{\varphi}([0,1])$ is isometrically embedded in the Besicovitch-Orlicz space of almost periodic functions $\widetilde{B}^{\varphi}a.p.$

Proposition 1. *Let $f \in L^{\varphi}([0,1])$. Then*

(1) *if $\widetilde{f}$ is the periodic extension of f to the whole $\mathbb{R}$ (with period $\tau = 1$), we have $\widetilde{f} \in \widetilde{B}^{\varphi}a.p.$,*
(2) *the injection map $i : L^{\varphi}([0,1]) \hookrightarrow \widetilde{B}^{\varphi}a.p.$, $i(f) = \widetilde{f}$ is an isometry with respect to the modulars and for the respective Luxemburg norms.*

Proof: Let $f \in L^{\varphi}([0,1])$ and $E_N = \{t \in [0,1] : |f(t)| \geq N\}$. It is known that $f \in L^1([0,1])$ (see [3]) and consequently $\lim_{N\to+\infty} \mu(E_N) = 0$.

It follows that

$$\lim_{N\to+\infty} \int_{E_N} \varphi(\lambda\|f(t)\|)\,dt = 0 \text{ for some } \lambda > 0.$$

Let $f_N = f\chi_{E_N^c}$, where E_N^c denotes the complement to E_N. Then, for a given $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbb{N}$ such that

$$\int_0^1 \varphi\left(\lambda\,|f(t) - f_{N_\varepsilon}(t)|\right)\,dt \leq \int_{E_{N\varepsilon}} \varphi\left(\lambda\,|f(t)|\right)\,dt \leq \varepsilon.$$

Now the function f_{N_ε} being bounded, there exists a sequence of simple functions $(S_{N_\varepsilon})_n$ uniformly convergent to f_{N_ε}. In particular, there exists a simple function S_{N_ε} such that $\sup_{t\in[0,1]} |\lambda(f_{N_\varepsilon}(t) - S_{N_\varepsilon}(t))| \leq \varepsilon$. It follows

$$\begin{aligned}
&\int_0^1 \varphi\left(\frac{\lambda}{2}\,|f(t) - S_{N_\varepsilon}(t)|\right)\,dt \\
&\leq \frac{1}{2}\int_0^1 \varphi\left(\lambda\,|f(t) - f_{N_\varepsilon}(t)|\right)\,dt + \frac{1}{2}\int_0^1 \varphi\left(\lambda\,|f_{N_\varepsilon}(t) - S_{N_\varepsilon}(t)|\right)\,dt \leq \varepsilon.
\end{aligned}$$

We denote by $\widetilde{f}$, $\widetilde{f}_{N_\varepsilon}$ and $\widetilde{S}_{N_\varepsilon}$ the respective periodic extensions (with period $\tau = 1$) of the functions f, f_{N_ε} and S_{N_ε}. We have from the periodicity properties of $\widetilde{f}$, $\widetilde{f}_{N_\varepsilon}$ and $\widetilde{S}_{N_\varepsilon}$:

$$\begin{aligned}
\rho_{B^{\varphi}}\left(\frac{\lambda}{2}\left(\widetilde{f} - \widetilde{S}_{N_\varepsilon}\right)\right) &= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{+T} \varphi\left(\frac{\lambda}{2}\left|\widetilde{f}(t) - \widetilde{S}_{N_\varepsilon}(t)\right|\right)\,dt \\
&= \int_0^1 \varphi\left(\frac{\lambda}{2}\,|f(t) - S_{N_\varepsilon}(t)|\right)\,dt \leq \varepsilon.
\end{aligned}$$

Moreover from (2.1) there exists $P_\varepsilon \in C^{\circ}a.p.$ for which

$$\rho_{B^{\varphi}}\left(\frac{1}{2}\left(\widetilde{S}_{N_\varepsilon} - P_\varepsilon\right)\right) \leq \varepsilon.$$

Finally, putting $\alpha = \min(\frac{\lambda}{2}, \frac{1}{4})$ we get:

$$\rho_{B^{\varphi}}\left(\frac{\alpha}{2}\left(\widetilde{f} - P_{\varepsilon}\right)\right) \leq \frac{1}{2}\left\{\rho_{B^{\varphi}}\left(\frac{\lambda}{2}\left(\widetilde{f} - \widetilde{S}_{N_{\varepsilon}}\right)\right) + \rho_{B^{\varphi}}\left(\frac{1}{4}\left(\widetilde{S}_{N_{\varepsilon}} - P_{\varepsilon}\right)\right)\right\} \leq \varepsilon.$$

This means that $\widetilde{f} \in \widetilde{B}^{\varphi}a.p.$

It remains to show that i is an isometry. Indeed, for $f \in B^{\varphi}a.p.$ we have in view of the periodicity of $\widetilde{f}$ (with period $T = 1$)

$$\rho_{B^{\varphi}}\left(\widetilde{f}\right) = \lim_{T \to +\infty} \frac{1}{2T}\int_{-T}^{+T} \varphi\left(\left|\widetilde{f}(t)\right|\right)\, dt = \rho_{\varphi}(f).$$

In the same way, for each $\lambda > 0$, we have:

$$\rho_{B^{\varphi}}\left(\frac{\widetilde{f}}{\lambda}\right) = \rho_{\varphi}\left(\frac{f}{\lambda}\right),$$

and then

$$\left\|\widetilde{f}\right\|_{B^{\varphi}} = \inf\left\{\lambda > 0, \rho_{B^{\varphi}}\left(\frac{\widetilde{f}}{\lambda}\right) \leq 1\right\} = \inf\left\{\lambda > 0, \rho_{\varphi}\left(\frac{f}{\lambda}\right) \leq 1\right\} = \|f\|_{\varphi}.$$

Hence, i is an isometry for the respective modulars and also for the respective norms. □

3. Main result

We can now state our main result.

Theorem 1. *The following properties are equivalent:*

(1) $\widetilde{B}^{\varphi}a.p.$ *is LUC;*
(2) $\widetilde{B}^{\varphi}a.p.$ *has the H-property;*
(3) $\widetilde{B}^{\varphi}a.p.$ *is SC;*
(4) φ *is strictly convex and* φ *satisfies the* Δ_2*-condition.*

PROOF: It is known from [12] that the space $\widetilde{B}^{\varphi}a.p.$ is strictly convex if and only if φ is strictly convex and satisfies the Δ_2-condition. We will show that these properties are equivalent to the local uniform convexity and the H-property of the space.

The implication (1) $\Rightarrow$ (2) holds in general Banach spaces.

(2) $\Rightarrow$ (3): Suppose that $\widetilde{B}^{\varphi}a.p.$ has the H-property. We prove first that the Orlicz space $L^{\varphi}([0,1])$ has the H-property as well.

For, let $\{f_n\}$ be a sequence in $L^{\varphi}([0,1])$ such that:

- $\{f_n\}$ converge weakly to some f in $L^{\varphi}([0,1])$,
- $\|f_n\|_{\varphi} \longrightarrow \|f\|_{\varphi}$.

Then, for each G in the dual space $(\widetilde{B}^{\varphi}a.p.)^*$, we have $G \circ i \in (L^{\varphi}([0,1]))^*$. Moreover, since $f_n \longrightarrow f$ weakly in $L^{\varphi}([0,1])$, we get

$$G \circ i\,(f_n) \longrightarrow G \circ i\,(f)$$

or equivalently $G(\widetilde{f_n}) \longrightarrow G(\widetilde{f})$. Thus $\widetilde{f_n} \longrightarrow \widetilde{f}$ weakly in $\widetilde{B}^{\varphi}a.p.$

It is clear that $\|\widetilde{f_n}\|_{B^{\varphi}} \longrightarrow \|\widetilde{f}\|_{B^{\varphi}}$ and since $\widetilde{B}^{\varphi}a.p.$ has the H-property, we can write $\|\widetilde{f_n - f}\|_{B^{\varphi}} \longrightarrow 0$ and finally $\|f_n - f\|_{\varphi} \longrightarrow 0$. This means that $L^{\varphi}([0,1])$ has the H-property.

It follows from [16] that φ is strictly convex and satisfies the Δ_2-condition. Thus using [12] we conclude that $\widetilde{B}^{\varphi}a.p.$ is strictly convex.

Let us show finally that (4) $\Rightarrow$ (1): For, let f_n, f be in $\widetilde{B}^{\varphi}a.p.$ with

$$\|f_n\|_{B^{\varphi}} = \|f\|_{B^{\varphi}} = 1 \quad \text{and} \quad \left\|\frac{f+f_n}{2}\right\|_{B^{\varphi}} \to 1 \quad \text{as} \quad n \to +\infty.$$

Recall that since φ satisfies the Δ_2-condition, we have $B^{\varphi}a.p. = \widetilde{B}^{\varphi}a.p.$ From Lemma 3, we have also

$$\rho_{B^{\varphi}}(f_n) = \rho_{B^{\varphi}}(f) = 1 \quad \text{and} \quad \rho_{B^{\varphi}}\left(\frac{f+f_n}{2}\right) \to 1 \quad \text{as} \quad n \to +\infty.$$

In view of Lemma 4, it follows that the sequence $\{f_n\}_n$ is $\overline{\mu}$-convergent to f. Then using Lemma 5 and the Δ_2-condition on φ, we conclude that

$$\|f_n - f\|_{B^{\varphi}} \to 0 \quad \text{as} \quad n \to +\infty.$$

$\square$

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