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Local monotonicity of Hausdorff measures restricted to real analytic curves

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Abstract. We prove that the 1-dimensional Hausdorff measure restricted to a simple real analytic curve $\gamma : \mathbb{R} \to \mathbb{R}^N$, $N \geq 2$, is locally 1-monotone.

Keywords: monotone measure, monotonicity formula

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1. Introduction

Definition 1.1. Let μ be a Radon measure on $\mathbb{R}^N$ and $k \in \mathbb{N}$. We say that μ is *k-monotone* if the function $r \mapsto \frac{\mu B(z,r)}{r^k}$ is nondecreasing on $(0, \infty)$ for every $z \in \mathbb{R}^N$. Instead of 1-monotone, we simply write *monotone*.

The monotonicity plays an important role when studying the existence and the regularity problems concerning minimal surfaces (see e.g. [5]). Even though the definition of the monotonicity looks very brief, checking this property in particular cases usually leads to complicated technical computations. These computations are often difficult even for very small radii, i.e. in the case of the local monotonicity.

Definition 1.2. Let μ be a Radon measure on $\mathbb{R}^N$ and $k \in \mathbb{N}$. We say that μ is *locally k-monotone at $z_0 \in \mathbb{R}^N$* if there is $r_0 > 0$ such that the function $r \mapsto \frac{\mu B(z,r)}{r^k}$ is nondecreasing on $(0, r_0)$ for every $z \in B(z_0, r_0)$.
Instead of locally 1-monotone, we simply write *locally monotone*.

The local monotonicity is an important tool for constructing examples of badly behaved monotone measures by the compensation method of Kolář [3].

The first results concerning local monotonicity were obtained by Kirchheim. In his unpublished work he used the Taylor expansion to study the local monotonicity of the 1-dimensional Hausdorff measure restricted to a symmetrical pair of logarithmic spirals.

In recent papers [1] and [2], there are given sufficient conditions for the 1-dimensional Hausdorff measure restricted to a curve to ensure the local monotonicity. The positive results generalize the observation that the 1-dimensional Hausdorff measure restricted to the graph of the function $f(x) = x^2$ or to a sphere in $\mathbb{R}^2$ is locally monotone on $\mathbb{R}^2$. Let us recall one of the results (see [2, Proposition 3.1. and Lemma 3.3.].

Theorem 1.3. *Let $a < b$ and $\gamma : (a, b) \to \mathbb{R}^N$ be a simple regular C^2-curve. If $t_0 \in (a, b)$ and $\ddot{\gamma}(t_0) \neq 0$ then μ_γ is locally monotone at $\gamma(t_0)$.*

Here and in the sequel we use the notation $\mu_\gamma = \mathcal{H}^1 \llcorner \gamma((a, b))$. When γ is a graph of a function f (i.e. $\gamma(t) = te_1 + f(t)e_2$, $t \in (a, b)$), then we prefer to write μ_f. Recall that a curve γ is regular if it has non-vanishing derivative, and γ is simple if $\gamma(t_1) = \gamma(t_2)$ implies $t_1 = t_2$.

Let us give two typical examples of a measure which is not locally monotone. First, set $g(x) = |x|$ and let us prove that the measure μ_g is not locally monotone at the origin. For any $r > 0$ we have $\mu_g B((0, r), r) = 2\sqrt{2}r$. On the other hand $\lim_{\varrho\to\infty} \frac{\mu_g B((0,r),\varrho)}{\varrho} = 2$. This implies that there is $k > 1$ such that $\mu_g B((0, r), kr) < 2\sqrt{2}kr$ and we are done.

Second, set $h(x) = x^p \sin(\frac{1}{x^q})$, $p, q > 0$. We obviously have $\mu_h B((0, 0), r) > 2r$ for every $r > 0$, but we can find arbitrarily small $x_0 > 0$ such that the tangent to the graph of h at $(x_0, h(x_0))$ is orthogonal to the sphere $S((0, 0), \sqrt{x_0^2 + h^2(x_0)})$, and thus we have $\frac{\partial}{\partial r}\mu_h B((0, 0), r)|_{r=r_0} = 2$. Hence

$$\frac{\partial}{\partial r} \frac{\mu_h(B(0, 0), r)}{r}|_{r=r_0} = \frac{1}{r^2}\Big(r \frac{\partial}{\partial r}\mu_h(B(0, 0), r)|_{r=r_0} - \mu_h(B(0, 0), r_0)\Big) < 0.$$

Thus the measure μ_h is not locally monotone at the origin.

It might seem that the pathological behavior from these examples can be prevented if we consider smooth convex functions only. Nevertheless, even if f is a convex C^∞-function, then μ_f is not necessarily monotone. An example is given in [1].

A natural question immediately arises, whether the real analyticity of the function f implies the local monotonicity of the measure μ_f. Surprisingly, it actually does. Our main result is

Theorem 1.4. *Let $a < b$ and $\gamma : (a, b) \to \mathbb{R}^N$, $N \geq 2$ be a simple regular real analytic curve. Then μ_γ is locally monotone at $\gamma(t_0)$ for every $t_0 \in (a, b)$.*

This result is proven by different methods than those used in [1] and [2] because methods using rough estimates for bad centers and bad radii do not work in our case. Let us give the main idea of the proof: Suppose that $\gamma : (a, b) \mapsto \mathbb{R}^N$ is a regular real analytic curve, $\gamma((a, b))$ is not a line segment, and we study the local monotonicity at $\gamma(t_0)$, $a < t_0 < b$. By Lemma 2.2 we can suppose that $t_0 = 0$ and there is $p \in \mathbb{N}$, $p \geq 2$, such that $\gamma(t)$ is very close to $(t, t^p, 0, \dots, 0)$ (up to a small analytic error function for each coordinate) on some neighborhood of 0. We prove the local monotonicity by contradiction. If the restricted measure is not locally monotone at the origin, then there are centers $z_n \to 0$ and radii $r_n \to 0$ such that $\frac{\partial}{\partial r}\frac{\mu_\gamma B(z_n,r)}{r}|_{r=r_n} < 0$. For each pair z_n, r_n there are $\tau_n < \sigma_n$ such that $\{\gamma(\tau_n), \gamma(\sigma_n)\} = \gamma((a, b)) \cap S(z_n, r_n)$. Obviously $\sigma_n \to 0$ and $\tau_n \to 0$. Passing to a subsequence we can suppose that $\sigma_n > 0$, $\frac{\tau_n}{\sigma_n} \to s \in [-1, 1]$. Using a criterion for the monotonicity based on Lemma 2.3 and a suitable blow-up in the case $s \neq 1$ we obtain

$$\int_s^1 f^2(t)\,dt \geq (1-s)\Big(\frac{1}{2}f^2(1)+\frac{1}{2}f^2(s)-\frac{1}{8}\big(f(1)-f(s)\big)^2\Big),$$

where $f(t)=pt^{p-1}-\frac{1-s^p}{1-s}$. For $s=1$ we obtain

$$\int_{-1}^1 f^2(t)\,dt \geq 2\Big(\frac{1}{2}f^2(1)+\frac{1}{2}f^2(-1)-\frac{1}{8}\big(f(1)-f(-1)\big)^2\Big),$$

where $f(t)=t$. Since for any $p\in\mathbb{N}$, $p\geq 2$, and any $s\in[-1,1]$ above inequalities are not satisfied, we have a contradiction.

In the fourth section, our methods are applied to the graph of the function $f(x)=|x|^p$, $p>0$ (in case $p=0$, it is easy to compute that μ_f is monotone). We obtain that μ_f is locally monotone at the origin if and only if $p>\frac{3}{2}$ (and by Theorem 1.3 the local monotonicity is proven at any point but the origin). This result does not follow from Theorem 1.3 because we have $f''(0)=0$ for $p>2$ and $\lim_{x\to 0} f''(x)=\infty$ for $p\in(0,2)$.

We refer to [4] and [5] for other information concerning the geometry of measures and the Monotonicity Formula.

2. Preliminaries

Notation. The *scalar product* of $x,y\in\mathbb{R}^N$ is denoted by $x\cdot y$ and the Euclidean *norm* of x is denoted by $|x|$. The i-th coordinate of x is x_i and the standard orthonormal basis in $\mathbb{R}^N$ is $\{e_1,\dots,e_N\}$. The origin in $\mathbb{R}^N$ is denoted by 0. When $u,v\in\mathbb{R}^N$ and $u\cdot v=0$, we write $u\perp v$. Further $u^\perp=\{v\in\mathbb{R}^N : v\perp u\}$.

We use the convention that C is a generic positive constant that may change from occurence to occurence, as usual.

The *1-dimensional Hausdorff measure* is denoted by $\mathcal{H}^1$ and $\mathcal{H}^1 \llcorner A$ is its restriction to a Borel set A.

For $z\in\mathbb{R}^N$ and $r>0$, we set

$$B(z,r)=\{x\in\mathbb{R}^N : |x-z|\leq r\} \qquad \text{and} \qquad S(z,r)=\{x\in\mathbb{R}^N : |x-z|=r\}.$$

We need the following property of the scalar product.

Lemma 2.1. *Let $a,b,c\in\mathbb{R}$, $ac>-1$ and let $u,\tilde{u},v_1,v_2\in\mathbb{R}^N$, with $|u|=|\tilde{u}|=|v_1|=|v_2|=1$, $v_1\perp v_2$, $u,\tilde{u}\in v_1^\perp\cap v_2^\perp$. Set $F(t)=(v_1+av_2+b\tilde{u})\cdot\frac{v_1+cv_2+tu}{|v_1+cv_2+tu|}$. Then*

$$\max_{t\in\mathbb{R}} F(t)=\max_{|t|\leq\frac{|b|(1+c^2)}{1+ac}} F(t).$$

Proof: Set $\tilde{b}=b(u\cdot\tilde{u})\in[-|b|,|b|]$. The proof obviously follows from

$$F'(t)=\Big(\frac{1+ac+\tilde{b}t}{\sqrt{1+c^2+t^2}}\Big)'=\frac{\tilde{b}+\tilde{b}c^2-t-act}{(\sqrt{1+c^2+t^2})^3}=\frac{1+ac}{(\sqrt{1+c^2+t^2})^3}\Big(\frac{\tilde{b}(1+c^2)}{1+ac}-t\Big).$$

□

The following lemma tells us that any real analytic curve is locally a graph of an analytic function.

Lemma 2.2. *Let $\gamma:(a,b)\to\mathbb{R}^N$, $N\geq 2$, be a regular real analytic curve such that $\gamma((a,b))$ is not a line segment and let $t_0\in(a,b)$. Then there are $\delta>0$, $\sigma_1,\sigma_2>0$, $m\geq 2$, $m\in\mathbb{N}$, and a real analytic function $\tilde{\gamma}:(-\sigma_1,\sigma_2)\to\mathbb{R}^N$ such that $\tilde{\gamma}(0)=0$, $\tilde{\gamma}_1(s)=s$ for $s\in(-\sigma_1,\sigma_2)$, $\dot{\tilde{\gamma}}(0)=e_1$, $\tilde{\gamma}^{(i)}(0)=0$, $i=2,\dots,m-1$, $\tilde{\gamma}^{(m)}(0)=e_2$ and $\tilde{\gamma}$ parameterizes the set $\{\gamma(t+t_0)-\gamma(t_0):t\in(-\delta,\delta)\}$ after suitable rotation and rescaling of coordinates.*

Proof: We observe that for any vector $v\in\mathbb{R}^N$, $F_v(t)=v\cdot(\gamma(t+t_0)-\gamma(t_0))$ is a real analytic function on (a,b). If $v=\dot{\gamma}(t_0)$, then this function is even invertible on some neighborhood of 0 because $F_v'(0)=|\dot{\gamma}(t_0)|^2\neq 0$. Moreover, this inverse function is again real analytic. Let $m\in\mathbb{N}$ be the minimal number such that $\dot{\gamma}(t_0)$ and $\gamma^{(m)}(t_0)$ are linearly independent. This number actually exists because γ is a real analytic curve and $\gamma((a,b))$ is not a line. Let $w\in\operatorname{Span}(\dot{\gamma}(t_0),\gamma^{(m)}(t_0))$ satisfy $|w|=1$, $w\cdot\dot{\gamma}(t_0)=0$ and $w\cdot\gamma^{(m)}(t_0)>0$. Further, we find $v_j\in\mathbb{R}^N$, $|v_j|=1$, $j=3,\dots,N$, complementing w and $\dot{\gamma}(t_0)$ to an orthogonal basis of $\mathbb{R}^N$. Since the composition of real analytic functions is again real analytic, the parameterization with respect to the new basis

$$\breve{\gamma}(s)=(F_{\dot{\gamma}(t_0)}(F^{-1}_{\dot{\gamma}(t_0)}(s)),F_w(F^{-1}_{\dot{\gamma}(t_0)}(s)),F_{v_3}(F^{-1}_{\dot{\gamma}(t_0)}(s)),\dots,F_{v_N}(F^{-1}_{\dot{\gamma}(t_0)}(s)))$$

satisfies all desired equalities but the last one. We have $\breve{\gamma}^{(m)}(0)=ce_2$, where $c>0$. Therefore we set $\tilde{\gamma}(s)=c^{\frac{1}{m-1}}\breve{\gamma}(c^{-\frac{1}{m-1}}s)$ and we are done. □

Local monotonicity. Let us recall some well known facts concerning the local monotonicity. For more details see for example [1] and [2].

Let $\gamma:[a,b]\mapsto\mathbb{R}^N$ be a simple regular C^1-curve. If we want to prove that $r\mapsto\frac{\mu_\gamma B(z,r)}{r}$ is nondecreasing on $(0,r_0)$ for some $z\in\mathbb{R}^N$, then it is done provided

$$\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{\mu_\gamma B(z,r)}{r}=\frac{1}{r^2}\big(r\,\underline{\mathrm{D}}_{\mathrm{r}}\,\mu_\gamma B(z,r)-\mu_\gamma B(z,r)\big)\tag{1}$$

is nonnegative on $(0,r_0)$. Here we use the notation $\underline{\mathrm{D}}_{\mathrm{r}}\,f(r)=\liminf\limits_{\delta\to 0}\frac{f(r+\delta)-f(r)}{\delta}$.

Condition $\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{\mu_\gamma B(z,r)}{r}\geq 0$ is satisfied when $\mu_\gamma B(z,r)\leq 2r$ and $\gamma(a),\gamma(b)\notin B(z,r)$ (if $\mu_\gamma B(z,r)=0$ then the proof is trivial and if $0<\mu_\gamma B(z,r)\leq 2$, then there are at least two points of intersection $S(z,r)\cap\gamma((a,b))$ and the contribution of each of them to $\underline{\mathrm{D}}_{\mathrm{r}}\,\mu_\gamma B(z,r)$ is at least 1).

If $\gamma : (a,b) \mapsto \mathbb{R}^N$ is a C^1-curve satisfying $\gamma(0) = 0$, $a < 0 < b$, $\dot\gamma(t) = e_1 + \mathrm{o}(1)e_2 + \dots + \mathrm{o}(1)e_N$ then there can be found $r_1 > 0$ such that if $|z| < r_1$, $r \in (0, r_1)$ and $\mu_\gamma B(z,r) > 2r$, then we have:
There are $\tau < \sigma$ such that $S(z,r) \cap \gamma((a,b)) = \{\gamma(\tau), \gamma(\sigma)\}$, we have derivatives instead of lower derivatives in (1) and

$$\frac{\partial}{\partial r}\mu_\gamma B(z,r) = \frac{1}{\cos\varphi_\sigma} + \frac{1}{\cos\varphi_\tau},$$

where φ_τ is the angle between $z - \gamma(\tau)$ and $\dot\gamma(\tau)$, φ_σ is the angle between $\gamma(\sigma) - z$ and $\dot\gamma(\sigma)$. Moreover as the center z lies on the perpendicular bisector of a line segment joining $\gamma(\sigma)$ and $\gamma(\tau)$ we have $r = \frac{|\gamma(\sigma)-\gamma(\tau)|}{2\cos\eta}$, where η is the angle between $\gamma(\sigma) - \gamma(\tau)$ and $z - \gamma(\tau)$ (and also between $\gamma(\sigma) - z$ and $\gamma(\sigma) - \gamma(\tau)$). Therefore (note that $\eta, \phi_\tau, \phi_\sigma$ are very small provided r_1 is small) we obtain

$$r\frac{\partial}{\partial r}\mu_\gamma B(z,r) = \frac{|\gamma(\sigma) - \gamma(\tau)|}{2\cos\eta}\Big(\frac{1}{\cos\varphi_\sigma} + \frac{1}{\cos\varphi_\tau}\Big). \tag{2}$$

Our next aim is to obtain a new criterion for the local monotonicity.

For $\tau, \sigma \in \mathbb{R}$, $\tau < \sigma$, $\varepsilon \geq 0$ and a continuous function h we set

$$\Phi^\varepsilon_{\tau,\sigma}(h) = (\sigma - \tau)\Big(\Big(\frac{1}{2} - \varepsilon\Big)h^2(\sigma) + \Big(\frac{1}{2} - \varepsilon\Big)h^2(\tau) - \frac{1}{8}\big(h(\sigma) - h(\tau)\big)^2\Big) - \int_\tau^\sigma h^2(t)\,dt.$$

We observe that if $c > 0$, $d \in \mathbb{R}$ and $\tilde h(t) = h(ct + d)$ then

$$\Phi^\varepsilon_{\tau,\sigma}(ch) = c^2\Phi^\varepsilon_{\tau,\sigma}(h) \qquad \text{and} \qquad \Phi^\varepsilon_{\tau,\sigma}(\tilde h) = \frac{1}{c}\Phi^\varepsilon_{c\tau+d, c\sigma+d}(h). \tag{3}$$

Further, for given C^1-function f and fixed $\tau < \sigma$ set $\phi(t) = f'(t) - \frac{f(\sigma)-f(\tau)}{\sigma-\tau}$ (when $\tau = \tau_n$, $\sigma = \sigma_n$ we write ϕ_n).

The following lemma tells us that the non-negativity of $\frac{\partial}{\partial r}\frac{\mu_\gamma B(z,r)}{r}$ follows from the non-negativity of $\Phi^\varepsilon_{\tau,\sigma}(\phi)$ for the curve $\gamma(t) = tv_1 + f(t)v_2$. We restrict ourselves to planar curves to deal easily with the angles in the proof of the lemma (see (8)).

Lemma 2.3. *Let $\varepsilon > 0$, $\Delta > 0$, $v_1, v_2 \in \mathbb{R}^N$, $|v_1| = |v_2| = 1$, $v_1 \perp v_2$ and let f be a C^1-function on $(-\Delta, \Delta)$ such that $f(0) = f'(0) = 0$. Set $\gamma(t) = tv_1 + f(t)v_2$, $t \in (-\Delta, \Delta)$. Then there is $\delta > 0$ with the following property:*
If $z \in B(0,\delta)$, $r \in (0,\delta)$ and $\mu_\gamma B(z,r) > 2$ then there are $\tau < \sigma$ such that $S(z,r) \cap \gamma((a,b)) = \{\gamma(\tau), \gamma(\sigma)\}$ and

$$\frac{\partial}{\partial r}\frac{\mu_\gamma B(z,r)}{r} \geq \frac{1}{r^2}\frac{1}{2}\frac{|\gamma(\sigma) - \gamma(\tau)|}{\sigma - \tau}(1 + \varepsilon)\Phi^\varepsilon_{\tau,\sigma}(\phi). \tag{4}$$

Proof: For fixed $t, \tau, \sigma \in [-\Delta, \Delta]$ let α_t be the angle between $\gamma(\sigma) - \gamma(\tau)$ and $\dot\gamma(t)$. Let β be the angle between v_1 and $\gamma(\sigma) - \gamma(\tau)$. Choose $\delta \in (0, \frac{1}{2}\Delta)$ so small that we can use estimate (2) and for every $t, \tau, \sigma \in (-2\delta, 2\delta)$ the angles β and α_t are small enough so that $|\alpha_t| < \frac{\pi}{4}$, $|\beta| < \frac{\pi}{4}$,

$$\begin{aligned}
\frac{1-\varepsilon}{2}\phi^2(t) &= \frac{1-\varepsilon}{2}(\tan(\alpha_t+\beta)-\tan\beta)^2 \\
&= \frac{1-\varepsilon}{2}\Big(\frac{\tan\alpha_t+\tan\beta}{1-\tan\alpha_t\tan\beta}-\tan\beta\Big)^2 \\
&= \frac{1-\varepsilon}{2}\tan^2\alpha_t\Big(\frac{1-\tan^2\beta}{1-\tan\alpha_t\tan\beta}\Big)^2 \le \frac{1}{2}\alpha_t^2
\end{aligned} \tag{5}$$

and similarly

$$\tan^2\alpha_t \le (1+\varepsilon)\phi^2(t). \tag{6}$$

Fix $z \in B(0,\delta)$ and $r \in (0,\delta)$ such that $\mu_\gamma B(z,r) > 2$. Hence there are $\tau, \sigma \in (-2\delta, 2\delta)$ so that $S(z,r) \cap \gamma((-\Delta,\Delta)) = \{\gamma(\tau), \gamma(\sigma)\}$.

Since $\mu_\gamma B(z,r)$ is obtained integrating $t \mapsto \frac{1}{\cos\alpha_t} = \sqrt{1+\tan^2\alpha_t}$ along the line joining $\gamma(\tau)$ and $\gamma(\sigma)$ we have by (6)

$$\begin{aligned}
\mu_\gamma B(z,r) &= \int_\tau^\sigma \sqrt{1+\tan^2\alpha_t}\frac{|\gamma(\sigma)-\gamma(\tau)|}{\sigma-\tau}\,dt \\
&\le \frac{|\gamma(\sigma)-\gamma(\tau)|}{\sigma-\tau}\int_\tau^\sigma \sqrt{1+(1+\varepsilon)\phi^2(t)}\,dt \\
&\le \frac{|\gamma(\sigma)-\gamma(\tau)|}{\sigma-\tau}\Big(\sigma-\tau+\frac{1}{2}(1+\varepsilon)\int_\tau^\sigma \phi^2(t)\,dt\Big).
\end{aligned} \tag{7}$$

As $\varphi_\tau = \alpha_\tau - \eta$, $\varphi_\sigma = \alpha_\sigma + \eta$ and $\frac{1}{\cos t} \ge 1 + \frac{t^2}{2}$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ (indeed, the function $\psi(t) = \frac{1}{\cos t}$ satisfies $\psi(0) = 1$, $\psi'(0) = 0$ and $\psi''(t) = \frac{1+\sin^2 t}{\cos^3 t} \ge 1$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$), using (2) we have

$$\begin{aligned}
r\frac{\partial}{\partial r}\mu_\gamma B(z,r) &= \frac{|\gamma(\sigma)-\gamma(\tau)|}{2}\frac{1}{\cos\eta}\Big(\frac{1}{\cos\varphi_\sigma}+\frac{1}{\cos\varphi_\tau}\Big) \\
&\ge \frac{|\gamma(\sigma)-\gamma(\tau)|}{2}\Big(1+\frac{1}{2}\eta^2\Big)\Big(1+\frac{1}{2}(\alpha_\sigma-\eta)^2+1+\frac{1}{2}(\alpha_\tau+\eta)^2\Big) \\
&\ge |\gamma(\sigma)-\gamma(\tau)|\Big(1+\frac{1}{4}\alpha_\sigma^2+\frac{1}{4}\alpha_\tau^2+\Big(\eta-\frac{1}{4}(\alpha_\sigma-\alpha_\tau)\Big)^2-\frac{1}{16}(\alpha_\sigma-\alpha_\tau)^2\Big) \\
&\ge |\gamma(\sigma)-\gamma(\tau)|\Big(1+\frac{1}{4}\alpha_\sigma^2+\frac{1}{4}\alpha_\tau^2-\frac{1}{16}(\alpha_\sigma-\alpha_\tau)^2\Big).
\end{aligned} \tag{8}$$

Further from $\tan' t \ge 1$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ we have

$$|\alpha_\sigma - \alpha_\tau| = |(\alpha_\sigma+\beta)-(\alpha_\tau+\beta)| \le |\tan(\alpha_\sigma+\beta)-\tan(\alpha_\tau+\beta)| = |\phi(\sigma)-\phi(\tau)|$$

and thus from (5) and $(1+\varepsilon)(\frac{1}{2}-\varepsilon) \le \frac{1-\varepsilon}{2}$ we obtain

$$\begin{aligned}
&\frac{1}{4}\alpha_\sigma^2 + \frac{1}{4}\alpha_\tau^2 - \frac{1}{16}(\alpha_\sigma - \alpha_\tau)^2 = \frac{1}{2}\Big(\frac{1}{2}\alpha_\sigma^2 + \frac{1}{2}\alpha_\tau^2 - \frac{1}{8}(\alpha_\sigma - \alpha_\tau)^2\Big)\\
&\ge \frac{1}{2}\Big(\frac{1-\varepsilon}{2}\phi^2(\sigma) + \frac{1-\varepsilon}{2}\phi^2(\tau) - \frac{1}{8}\big(\phi(\sigma)-\phi(\tau)\big)^2\Big)\\
&\ge \frac{1}{2}(1+\varepsilon)\Big(\Big(\frac{1}{2}-\varepsilon\Big)\phi^2(\sigma) + \Big(\frac{1}{2}-\varepsilon\Big)\phi^2(\tau) - \frac{1}{8}\big(\phi(\sigma)-\phi(\tau)\big)\Big).
\end{aligned} \tag{9}$$

Finally, (1), (7), (8) and (9) imply (4). □

3. Local monotonicity

We want to prove the local monotonicity of $\mathcal{H}^1$ restricted to a real analytic curve and to the graph of the function $|x|^p$. The main step of the proof is the following proposition, where we consider the curves satisfying $\gamma(0) = 0$ and either

$$\begin{gathered}
\dot\gamma(t) = (1, p\,\mathrm{sgn}(t)|t|^{p-1} + \eta_2(t)|t|^{p-1}, \eta_3(t)|t|^{p-1}, \dots, \eta_N(t)|t|^{p-1})\\
\text{for some} \quad p > \frac{3}{2}
\end{gathered} \tag{10}$$

or

$$\begin{gathered}
\dot\gamma(t) = (1, p|t|^{p-1} + \eta_2(t)|t|^{p-1}, \eta_3(t)|t|^{p-1}, \dots, \eta_N(t)|t|^{p-1})\\
\text{for some} \quad p \ge p_0 = \frac{5}{4} + \sqrt{\frac{43}{48}}
\end{gathered} \tag{11}$$

on $(-\Delta, \Delta)$ for some $\Delta > 0$, with $\eta_i \in C^1(-\Delta,\Delta)$, $\eta_i(0) = 0$ for $i = 2, \dots, N$.

Proposition 3.1. *Let γ satisfy $\gamma(0) = 0$ and either (10) or (11). Then μ_γ is locally monotone at the origin.*

Let us write $f(t) = \gamma_2(t)$ and $\eta(t) = \eta_2(t)$. Hence $f(0) = 0$ and from (10) we have

$$f'(t) = p\,\mathrm{sgn}(t)|t|^{p-1} + \eta(t)|t|^{p-1}, \tag{12}$$

while from (11) we have

$$f'(t) = p|t|^{p-1} + \eta(t)|t|^{p-1}. \tag{13}$$

For fixed $s \in [-1, 1)$, the following functions are important in the sequel:

$$\begin{aligned}
g_{\mathrm{abs}}(t) &= p\,\mathrm{sgn}(t)|t|^{p-1} - \frac{1-|s|^p}{1-s},\\
g_{\mathrm{sgn}}(t) &= p|t|^{p-1} - \frac{1-\mathrm{sgn}(s)|s|^p}{1-s}.
\end{aligned}$$

Lemma 3.2. *The function $g(t) = t$ satisfies $\Phi^0_{-1,1}(g) > 0$.*

Proof: The proof is trivial because

$$2\Big(\frac{1}{2}g^2(1)+\frac{1}{2}g^2(-1)-\frac{1}{8}\big(g(1)-g(-1)\big)^2\Big)-\int_{-1}^{1}g^2(t)\,dt=2-\frac{2}{3}>0.$$

□

Lemma 3.3. *Let $s\in[-1,1)$. If $p>\frac{3}{2}$ then $\Phi^0_{s,1}(g_{\mathrm{abs}})>0$. If $p\geq p_0$ then $\Phi^0_{s,1}(g_{\mathrm{sgn}})>0$.*

The proof of Lemma 3.3 is also straightforward but very long and technical. Therefore it is postponed to the last section.

Lemma 3.4. *Assume that $f(0)=0$ and either (12) or (13) is satisfied. Suppose that the sequences $\{\tau_n\},\{\sigma_n\}$ satisfy $\tau_n<\sigma_n$, $\sigma_n>0$, $\sigma_n\to 0$ and $s_n:=\frac{\tau_n}{\sigma_n}\to s\in[-1,1]$. Then there is $\varepsilon>0$ and $n_0\in\mathbb{N}$ such that*

$$\Phi^{\varepsilon}_{\tau_n,\sigma_n}(\phi_n)\geq\varepsilon\sigma_n^{2p-4}(\sigma_n-\tau_n)^3\qquad\text{for every }\ n>n_0. \tag{14}$$

Proof: We distinguish two cases. First suppose that $s\in[-1,1)$. Assume (12) (for (13) the proof is similar). Set $\psi_n(t)=\frac{1}{\sigma_n^{p-1}}\phi_n(\sigma_n t)$. As $f(0)=0$, from (12) we obtain $f(t)=|t|^p+\tilde\eta(t)|t|^p$, where $\lim_{t\to 0}\tilde\eta(t)=\tilde\eta(0)=0$. Hence

$$\begin{aligned}
\psi_n(t)&=\frac{1}{\sigma_n^{p-1}}\Big(f'(\sigma_n t)-\frac{f(\sigma_n)-f(\tau_n)}{\sigma_n-\tau_n}\Big)\\
&=\frac{1}{\sigma_n^{p-1}}\Big(p\sigma_n^{p-1}\operatorname{sgn}(t)|t|^{p-1}+\eta(\sigma_n t)\sigma_n^{p-1}|t|^{p-1}\\
&\quad-\frac{\sigma_n^p+\tilde\eta(\sigma_n)\sigma_n^p-|\tau_n|^p-\tilde\eta(\tau_n)|\tau_n|^p}{\sigma_n(1-s_n)}\Big)\\
&=p\operatorname{sgn}(t)|t|^{p-1}+\eta(\sigma_n t)|t|^{p-1}-\frac{1+\tilde\eta(\sigma_n)-\frac{|\tau_n|^p}{\sigma_n^p}-\tilde\eta(\tau_n)\frac{|\tau_n|^p}{\sigma_n^p}}{1-s_n}\\
&\to p\operatorname{sgn}(t)|t|^{p-1}-\frac{1-|s|^p}{1-s}=g_{\mathrm{abs}}(t)
\end{aligned}$$

uniformly on $[-2,2]$. By Lemma 3.3 we have $\Phi^0_{s,1}(g_{\mathrm{abs}})>0$. Hence there is $\varepsilon>0$ such that $\Phi^{\varepsilon}_{s,1}(g_{\mathrm{abs}})\geq 2\varepsilon(1-s)^3$. Since $\psi_n\to g_{\mathrm{abs}}$ uniformly on $[-2,2]$, we have $\Phi^{\varepsilon}_{s_n,1}(\psi_n)\to\Phi^{\varepsilon}_{s,1}(g_{\mathrm{abs}})$ and thus there is $n_0\in\mathbb{N}$ such that $\Phi^{\varepsilon}_{s_n,1}(\psi_n)\geq\varepsilon(1-s_n)^3$ for $n>n_0$. Hence (3) implies for all $n>n_0$

$$\Phi^{\varepsilon}_{\tau_n,\sigma_n}(\phi_n)=\sigma_n^{2p-1}\Phi^{\varepsilon}_{s_n,1}(\psi_n)\geq\sigma_n^{2p-1}\varepsilon(1-s_n)^3=\varepsilon\sigma_n^{2p-4}(\sigma_n-\tau_n)^3.$$

Now assume that $s=1$. In this case (12) and (13) coincide. Set $h_n=\frac{\sigma_n-\tau_n}{\sigma_n}$ (hence $\tau_n=\sigma_n(1-h_n)$), $c_p=\frac{2}{p(p-1)}$ and

$$\begin{aligned}\psi_n(t)&=\frac{c_p}{\sigma_n^{p-1}h_n}\phi_n\Big(\sigma_n\Big(1-\frac{1}{2}h_n+\frac{1}{2}h_nt\Big)\Big)\\ &=\frac{c_p}{\sigma_n^{p-1}h_n}\Big(f'\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)\Big)-\frac{f(\sigma_n)-f(\sigma_n(1-h_n))}{\sigma_nh_n}\Big).\end{aligned}\tag{15}$$

Let us show that $\psi_n(t)\to t$ uniformly on $[-1,1]$. From (12) we obtain

$$\begin{aligned}&f'\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)\Big)\\ &=p\sigma_n^{p-1}\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)^{p-1}+\eta\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)\Big)\sigma_n^{p-1}\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)^{p-1},\end{aligned}$$

hence

$$\begin{aligned}&\frac{c_p}{\sigma_n^{p-1}h_n}f'\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)\Big)\\ &=\frac{pc_p}{h_n}\Big(1-\frac{p-1}{2}h_n+\frac{p-1}{2}h_nt\Big)+\frac{c_p}{h_n}\eta\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)\Big)+\theta_n(t),\end{aligned}\tag{16}$$

where $\theta_n(t)\to0$ uniformly on $[-1,1]$. Let us investigate $\frac{c_p}{\sigma^{p-1}h_n}\big(\frac{f(\sigma_n)-f(\sigma_n(1-h_n))}{\sigma_nh_n}\big)$. Set $f_1(t)=t^p$ and $f_2=f-f_1$, hence $f_2'(t)=\eta(t)t^{p-1}$. We have

$$\begin{aligned}\frac{c_p}{\sigma_n^{p-1}h_n}\frac{f_1(\sigma_n)-f_1(\sigma_n(1-h_n))}{\sigma_nh_n}&=\frac{c_p}{h_n^2}\big(1-(1-h_n)^p\big)\\ &=\frac{c_p}{h_n}\Big(p-\frac{p(p-1)}{2}h_n\Big)+\zeta_n,\end{aligned}\tag{17}$$

where $\zeta_n\to0$. Further $f_2'(t)=\eta(t)t^{p-1}$, $\eta\in C^1(-\Delta,\Delta)$ and $\eta(0)=0$ imply

$$\begin{aligned}\frac{c_p}{\sigma_n^{p-1}h_n}\frac{f_2(\sigma_n)-f_2(\sigma_n(1-h_n))}{\sigma_nh_n}&=\frac{c_p}{\sigma_n^ph_n^2}f_2'\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}\xi_n\Big)\Big)h_n\sigma_n\\ &=\frac{c_p}{h_n}\eta\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}\xi_n\Big)\Big)\Big(1-\frac{h_n}{2}+\frac{h_n}{2}\xi_n\Big)^{p-1}\\ &=\frac{c_p}{h_n}\eta\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}\xi_n\Big)\Big)+\tilde\zeta_n,\end{aligned}\tag{18}$$

where $\xi_n\in(-1,1)$ and $\tilde\zeta_n\to0$. Finally there is $K>0$ and a neighborhood of the origin such that the C^1-function η is K-Lipschitz there and thus for n large

$$\Big|\eta\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}t\Big)\Big)-\eta\Big(\sigma_n\Big(1-\frac{h_n}{2}+\frac{h_n}{2}\xi_n\Big)\Big)\Big|\le K\sigma_n\frac{h_n}{2}|t-\xi_n|.\tag{19}$$

From (15), (16), (17), (18) and (19) we obtain $\psi_n(t)\to t$ uniformly on $[-1,1]$. As Lemma 3.2 implies $\Phi^0_{-1,1}(g)>0$, there is $\varepsilon>0$ such that $\Phi^\varepsilon_{-1,1}(g)\ge3\varepsilon c_p^2$. Since

$\psi_n \to g$ uniformly on $[-1,1]$, we have $\Phi^\varepsilon_{-1,1}(\psi_n) \to \Phi^\varepsilon_{-1,1}(g)$ and thus there is $n_0 \in \mathbb{N}$ such that $\Phi^\varepsilon_{-1,1}(\psi_n) \ge 2\varepsilon c_p^2$ for $n > n_0$. Hence using (3) we conclude

$$\Phi^\varepsilon_{\tau_n,\sigma_n}(\phi_n) = \frac{1}{\frac{c_p^2}{\sigma_n^{2p-2}h_n^2}}\frac{1}{2}\sigma_n h_n \Phi^\varepsilon_{-1,1}(\psi_n) \ge \varepsilon\sigma_n^{2p-1}h_n^3 = \varepsilon\sigma_n^{2p-4}(\sigma_n-\tau_n)^3.$$

□

Proof of Proposition 3.1. Suppose that γ satisfies (10) (for (11) the proof is similar). If μ_γ is not locally monotone at the origin, then there are $z_n \in \mathbb{R}^N$, $|z_n| \to 0$, and $r_n > 0$, $r_n \to 0$, such that $\underline{\mathrm{D}}_{\mathrm{r}} \frac{\mu_\gamma B(z_n,r)}{r}|_{r=r_n} < 0$. Hence (see Preliminaries) $\mu_\gamma B(z_n,r_n) > 2r_n$ for all $n \in \mathbb{N}$ and we can suppose that there are $\sigma_n \to 0$ and $\tau_n \to 0$, $\tau_n < \sigma_n$ such that $S(z_n,r_n) \cap \gamma((a,b)) = \{\gamma(\tau_n),\gamma(\sigma_n)\}$, the derivatives in (1) exist and we have (2). Passing to a subsequence and to the curve $\tilde\gamma(t) = \gamma(-t)$ if necessary we can suppose that $\sigma_n > 0$ and $\frac{\tau_n}{\sigma_n} \to s \in [-1,1]$. Set $h_n = \frac{\sigma_n-\tau_n}{\sigma_n}$.

Further, by Lemma 3.4 there are $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that we have (14).

Now, for fixed $n \in \mathbb{N}$ (i.e. for τ_n,σ_n fixed) we construct a suitable orthonormal basis in $\mathbb{R}^N$. Pick $\xi_n \in (\tau_n,\sigma_n)$ such that $\gamma(\sigma_n) - \gamma(\tau_n) = \dot\gamma(\xi_n)(\sigma_n-\tau_n)$. Set $\tilde v_1 = \dot\gamma(\xi_n) - (\dot\gamma(\xi_n)\cdot e_2)e_2$, $\tilde v_2 = e_2$, further $\tilde v_i = e_i - \sum_{j=1}^{i-1}(e_i\cdot\tilde v_j)\frac{\tilde v_j}{|\tilde v_j|^2}$, $i = 3,\dots,N$, and finally $v_i = \frac{\tilde v_i}{|\tilde v_i|}$, $i = 1,\dots,N$. We see that $(v_1)_1 = 1 - C_n$, where $C_n \to 0$ as $n \to \infty$.

By (10), the fact that η_i are $\mathcal{C}^1$-functions and $\eta_i(0) = 0$ we have for $i = 3,\dots,N$

$$\begin{aligned}|\gamma_i'(t) - \gamma_i'(\xi_n)| &\le |\eta_i(t)||t^{p-1} - \xi_n^{p-1}| + |\eta_i(t) - \eta_i(\xi_n)||\xi_n^{p-1}| \\ &\le C\sigma_n|t-\xi_n|\sigma_n^{p-2} + C|t-\xi_n|\sigma_n^{p-1} = C\sigma_n^{p-1}|t-\xi_n|.\end{aligned}$$

Hence we obtain from $|v_i| = 1$

$$\dot\gamma(t)\cdot v_i = (\dot\gamma(t) - \dot\gamma(\xi_n))\cdot v_i = \breve\eta_i(t)\sigma_n^{p-1}(t-\xi_n) \qquad \text{for } \ i = 3,\dots,N, \tag{20}$$

where $\breve\eta_i$ are bounded functions and the bound is independent of n. Further we have $\gamma(t)\cdot v_1 = (1-C_n)t$ and $\gamma(t)\cdot v_2 = f(t)$.

Let $\tilde\gamma$ be a projection of γ to $\mathrm{Span}(v_1,v_2)$, i.e. $\tilde\gamma(t) = (1-C_n)tv_1 + f(t)v_2$. We also define $\tilde z_n = (z_n\cdot v_1)v_1 + (z_n\cdot v_2)v_2$ and $\tilde r_n = |\tilde\gamma(\tau_n) - \tilde z_n| = |\tilde\gamma(\sigma_n) - \tilde z_n|$. Let $\tilde\varphi_{\tau_n}$ be the angle between $\tilde z_n - \tilde\gamma(\tau_n)$ and $\dot{\tilde\gamma}(\tau_n)$ and let $\tilde\varphi_{\sigma_n}$ be the angle between $\tilde\gamma(\sigma_n) - \tilde z_n$ and $\dot{\tilde\gamma}(\sigma_n)$.

Let us define $\breve\gamma(t) = \tilde\gamma((1-C_n)t)$. Then Lemma 2.3 applied to $\breve\gamma$ at $\breve\tau_n = \frac{\tau_n}{1-C_n}$ and $\breve\sigma_n = \frac{\sigma_n}{1-C_n}$ gives $n_1 > n_0$ such that for $n > n_1$ we have

$$\begin{aligned}\frac{\partial}{\partial r}\frac{\mu_{\tilde\gamma}B(\tilde z_n,r)}{r}\Big|_{r=\tilde r_n} &= \frac{\partial}{\partial r}\frac{\mu_{\breve\gamma}B(\tilde z_n,r)}{r}\Big|_{r=\tilde r_n} \\ &\ge \frac{1}{\tilde r_n^2}\frac{1}{2}\frac{|\breve\gamma(\breve\sigma_n) - \breve\gamma(\breve\tau_n)|}{|\breve\sigma_n - \breve\tau_n|}(1+\varepsilon)\Phi^\varepsilon_{\breve\tau_n,\breve\sigma_n}(\breve\phi_n),\end{aligned}$$

where

$$\breve{\phi}_n(t) = \big(f((1-C_n)t)\big)' - \frac{f((1-C_n)\breve{\sigma}_n) - f((1-C_n)\breve{\tau}_n)}{\breve{\sigma}_n - \breve{\tau}_n}$$
$$= (1-C_n)f'((1-C_n)t) - \frac{f(\sigma_n) - f(\tau_n)}{\frac{\sigma_n}{1-C_n} - \frac{\tau_n}{1-C_n}} = (1-C_n)\phi_n((1-C_n)t).$$

Hence (3) and (14) imply that for $n > n_1$ we have

$$\frac{\partial}{\partial r}\frac{\mu_{\tilde{\gamma}}B(z_n,r)}{r}\Big|_{r=\tilde{r}_n} \geq \frac{1}{\tilde{r}_n^2}\frac{1}{2}(1-C_n)\varepsilon\sigma_n^{2p-1}h_n^3. \tag{21}$$

As $|\dot{\gamma}| \geq 1$, by (20) there is $n_2 > n_1$ such that for $n > n_2$ we have

$$\begin{aligned}
&\mu_\gamma B(z_n,r_n) - \mu_{\tilde{\gamma}}B(\tilde{z}_n,\tilde{r}_n)\\
&= \int_{\tau_n}^{\sigma_n} \sqrt{\sum_{i=1}^{N}(\dot{\gamma}(t)\cdot v_i)^2} - \sqrt{\sum_{i=1}^{2}(\dot{\gamma}(t)\cdot v_i)^2}\,dt\\
&\leq \int_{\tau_n}^{\sigma_n}\sum_{i=3}^{N}(\dot{\gamma}(t)\cdot v_i)^2\,dt \leq C(\sigma_n-\tau_n)(C\sigma_n^{p-1}(\sigma_n-\tau_n))^2 \leq C\sigma_n^{2p+1}h_n^3.
\end{aligned} \tag{22}$$

We want to estimate $\frac{\partial}{\partial r}\mu_\gamma B(z_n,r)|_{r=r_n} - \frac{\partial}{\partial r}\mu_{\tilde{\gamma}}B(\tilde{z}_n,r)|_{r=\tilde{r}_n}$. Let us start by proving that there is $n_3 > n_2$ such that for $n > n_3$ we have

$$\Big|z_n - \frac{\gamma(\tau_n)+\gamma(\sigma_n)}{2}\Big| \leq \frac{1}{4}\sigma_n h_n = \frac{1}{4}(\sigma_n - \tau_n). \tag{23}$$

Suppose that (23) is not satisfied. From (10) for n large enough we have $|\dot{\gamma}(t)| < \sqrt{\frac{5}{4}}$ for $t \in [\tau_n, \sigma_n]$ and thus

$$\begin{aligned}
\mu_\gamma B(z_n,r_n) &= \int_{\tau_n}^{\sigma_n}|\dot{\gamma}(t)|\,dt \leq \sqrt{\frac{5}{4}}(\sigma_n-\tau_n)\\
&= 2\sqrt{\Big(\frac{\sigma_n-\tau_n}{2}\Big)^2 + \frac{1}{16}(\sigma_n-\tau_n)^2}\\
&\leq 2\sqrt{\Big(\frac{|\gamma(\tau_n)-\gamma(\sigma_n)|}{2}\Big)^2 + \Big|z_n - \frac{\gamma(\tau_n)+\gamma(\sigma_n)}{2}\Big|^2} = 2r_n.
\end{aligned}$$

Hence $\frac{\partial}{\partial r}\frac{\mu_\gamma B(z_n,r)}{r}|_{r=r_n} \geq 0$ for n large which contradicts the choice of $B(z_n,r_n)$.

Set $w = z_n - \gamma(\tau_n)$. We distinguish two cases.

If $|w\cdot v_i| \leq \sigma_n^p h_n^2$ for all $i = 3,\dots,N$, then (20) implies

$$|\dot{\gamma}(\tau_n)\cdot(z_n-\gamma(\tau_n)) - \dot{\tilde{\gamma}}(\tau_n)\cdot(\tilde{z}_n - \tilde{\gamma}(\tau_n))| \leq \sum_{i=3}^{N}|\dot{\gamma}(\tau_n)\cdot v_i||w\cdot v_i| \leq C\sigma_n^{2p}h_n^3. \tag{24}$$

Further there is $n_4 > n_3$ such that for $n > n_4$ we have $|\dot{\tilde{\gamma}}(\tau_n)| \leq |\dot{\gamma}(\tau_n)| < 2$, $\tilde{r}_n \leq r_n$, $\dot{\gamma}(\tau_n) \cdot (z_n - \gamma(\tau_n)) \geq \frac{\tilde{r}_n}{2}$ and $\dot{\tilde{\gamma}}(\tau_n) \cdot (\tilde{z}_n - \tilde{\gamma}(\tau_n)) \geq \frac{\tilde{r}_n}{2}$ (the last two estimates follow from (23)). Hence from (24) we obtain for $n > n_4$

$$\begin{aligned}\frac{r_n}{\cos\varphi_{\tau_n}} - \frac{\tilde{r}_n}{\cos\tilde{\varphi}_{\tau_n}} &= \frac{|\dot{\gamma}(\tau_n)| r_n^2}{\dot{\gamma}(\tau_n) \cdot (z_n - \gamma(\tau_n))} - \frac{|\dot{\tilde{\gamma}}(\tau_n)| \tilde{r}_n^2}{\dot{\tilde{\gamma}}(\tau_n) \cdot (\tilde{z}_n - \tilde{\gamma}(\tau_n))} \\ &\geq -|\dot{\tilde{\gamma}}(\tau_n)| \tilde{r}_n^2 \frac{|\dot{\tilde{\gamma}}(\tau_n) \cdot (\tilde{z}_n - \tilde{\gamma}(\tau_n)) - \dot{\gamma}(\tau_n) \cdot (z_n - \gamma(\tau_n))|}{(\dot{\gamma}(\tau_n) \cdot (z_n - \gamma(\tau_n)))(\dot{\tilde{\gamma}}(\tau_n) \cdot (\tilde{z}_n - \tilde{\gamma}(\tau_n)))} \qquad (25) \\ &\geq -2\tilde{r}_n^2 \frac{C\sigma_n^{2p} h_n^3}{\frac{1}{2}\tilde{r}_n \, \frac{1}{2}\tilde{r}_n} \geq -C\sigma_n^{2p} h_n^3.\end{aligned}$$

Conversely, if there is $i_0 \in \{3, \dots, N\}$ such that $|w \cdot v_{i_0}| > \sigma_n^p h_n^2$, set

$$\begin{aligned}\tilde{w} &= w - (w \cdot v_1)v_1 - (w \cdot v_2)v_2, \qquad u = \frac{\tilde{w}}{|\tilde{w}|}, \\ \breve{w}(t) &= (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + t(w \cdot v_1)u \qquad \text{for } \ t \in \mathbb{R}.\end{aligned}$$

We observe that $w = \breve{w}(\frac{|\tilde{w}|}{w \cdot v_1})$. Finally set

$$\begin{aligned}G(t) &= \dot{\gamma}(\tau_n) \cdot \frac{\breve{w}(t)}{|\breve{w}(t)|} \\ &= (\dot{\gamma}(\tau_n) \cdot v_1)\Big[\Big(v_1 + \frac{\dot{\gamma}(\tau_n) \cdot v_2}{\dot{\gamma}(\tau_n) \cdot v_1} v_2 + \sum_{i=3}^N \frac{\dot{\gamma}(\tau_n) \cdot v_i}{\dot{\gamma}(\tau_n) \cdot v_1} v_i\Big) \cdot \frac{v_1 + \frac{w \cdot v_2}{w \cdot v_1} v_2 + tu}{|v_1 + \frac{w \cdot v_2}{w \cdot v_1} v_2 + tu|}\Big].\end{aligned}$$

By (23) we observe that there is $n_5 > n_4$ such that $w \cdot v_1 \in [\frac{\tilde{r}_n}{2}, \tilde{r}_n]$ and $|w \cdot v_i| \leq w \cdot v_1$, for all $i = 2, \dots, N$ provided $n > n_5$. Hence using Lemma 2.1 (with $a = \frac{\dot{\gamma}(\tau_n) \cdot v_2}{\dot{\gamma}(\tau_n) \cdot v_1}$, $b\tilde{u} = \sum_{i=3}^N \frac{\dot{\gamma}(\tau_n) \cdot v_i}{\dot{\gamma}(\tau_n) \cdot v_1} v_i$ and $c = \frac{w \cdot v_2}{w \cdot v_1}$) and (20) we obtain that there is $t_0 \in \mathbb{R}$ such that

$$\dot{\gamma}(\tau_n) \cdot \frac{w}{|w|} = G\Big(\frac{|\tilde{w}|}{w \cdot v_1}\Big) \leq G(t_0) \tag{26}$$

and

$$|t_0| \leq \frac{C \max_{i \in \{3, \dots, N\}} \frac{|\dot{\gamma}(\tau_n) \cdot v_i|}{\dot{\gamma}(\tau_n) \cdot v_1} (1 + (\frac{w \cdot v_2}{w \cdot v_1})^2)}{1 + \frac{\dot{\gamma}(\tau_n) \cdot v_2}{\dot{\gamma}(\tau_n) \cdot v_1} \frac{w \cdot v_2}{w \cdot v_1}} \leq \frac{C\sigma_n^p h_n (1+1)}{1 - C\sigma_n^p h_n \cdot 1} \leq C\sigma_n^p h_n.$$

Therefore there is $n_6 > n_5$ such that $\breve{w}(t_0)$ satisfies for $n > n_6$ and all $i = 3, \dots, N$

$$|\breve{w}(t_0) \cdot v_i| \leq |t_0| |w \cdot v_1| |u| \leq C\sigma_n^p h_n \, C\sigma_n h_n \, 1 \leq C\sigma_n^{p+1} h_n^2 < \sigma_n^p h_n^2. \tag{27}$$

Thus for the auxiliary center $z_0 = \gamma(\tau_n) + w(t_0)$, $r_0 = |w(t_0)|$ and φ_0 being the angle between $z_0 - \gamma(\tau_n)$ and $\dot\gamma(\tau_n)$ we can use (25). Therefore

$$\frac{r_0|\dot\gamma(\tau_n)|}{G(t_0)} - \frac{\tilde r_n}{\cos\tilde\varphi_{\tau_n}} = \frac{r_0}{\cos\varphi_0} - \frac{\tilde r_n}{\cos\tilde\varphi_{\tau_n}} \geq -C\sigma_n^{2p}h_n^3. \tag{28}$$

Moreover (27) and assumption $|w\cdot v_{i_0}| > \sigma_n^p h_n^2$ imply that $r_n = |w| > |\breve w(t_0)| = r_0$. Therefore we obtain from (26) and (28) for $n > n_6$

$$\frac{r_n}{\cos\varphi_{\tau_n}} - \frac{\tilde r_n}{\cos\tilde\varphi_{\tau_n}} = \frac{r_n|\dot\gamma(\tau_n)|}{G(\frac{|\breve w|}{w\cdot v_1})} - \frac{r_n|\dot\gamma(\tau_n)|}{G(t_0)} + \frac{r_n|\dot\gamma(\tau_n)|}{G(t_0)} - \frac{\tilde r_n}{\cos\tilde\varphi_{\tau_n}} \geq 0 - C\sigma_n^{2p}h_n^3.$$

Similarly there is $n_7 > n_6$ such that $\frac{r_n}{\cos\varphi_{\sigma_n}} - \frac{\tilde r_n}{\cos\tilde\varphi_{\sigma_n}} \geq C\sigma_n^{2p}h_n^3$ whenever $n > n_7$. And thus we have

$$r_n\frac{\partial}{\partial r}\mu_\gamma B(z_n,r)|_{r=r_n} - \tilde r_n\frac{\partial}{\partial r}\mu_{\tilde\gamma} B(\tilde z_n,r)|_{r=\tilde r_n} \geq -C\sigma_n^{2p}h_n^3. \tag{29}$$

Therefore by (21), (22) and (29) there is $n_8 > n_7$ so that for $n > n_8$ we obtain

$$\begin{aligned}\frac{\partial}{\partial r}\frac{\mu_\gamma B(z_n,r)}{r}|_{r=r_n} &= \frac{\tilde r_n^2}{r_n^2}\frac{\partial}{\partial r}\frac{\mu_{\tilde\gamma} B(\tilde z_n,r)}{r}|_{r=\tilde r_n} \\ &\quad + \frac{1}{r_n^2}\Big(r_n\frac{\partial}{\partial r}\mu_\gamma B(z_n,r)|_{r=r_n} - \tilde r_n\frac{\partial}{\partial r}\mu_{\tilde\gamma} B(\tilde z_n,r)|_{r=\tilde r_n} \\ &\quad - \mu_\gamma B(z_n,r_n) + \mu_{\tilde\gamma} B(\tilde z_n,\tilde r_n)\Big) \\ &\geq \frac{1}{r_n^2}\frac{\varepsilon}{2}(1-C_n)\sigma_n^{2p-1}h_n^3 + \frac{1}{r_n^2}\Big(-C\sigma_n^{2p}h_n^3 - C\sigma_n^{2p+1}h_n^3\Big) > 0.\end{aligned}$$

Hence we have a contradiction with the choice of the balls $B(z_n, r_n)$. □

Proof of Theorem 1.4: Fix $t_0 \in (a,b)$. If $\gamma((a,b))$ is a line segment, then it is well known and easy to compute that μ_γ is locally monotone at $\gamma(t_0)$. Otherwise we apply Lemma 2.2 and we obtain that γ can be parameterized as a graph of an analytic function which satisfies either (10) for $p \geq 2$ even or (11) for $p \geq 3$ odd. Finally Proposition 3.1 concludes the proof. □

4. Graph of $|x|^p$

Papers [1] and [2] were motivated by the fact that μ_f with $f(x) = x^2$ is locally monotone at every $z_0 \in \mathbb{R}^2$. As there is a convex C^∞-function such that $\mathcal{H}^1$ restricted to its graph is not locally monotone at the origin, it might be interesting to study the local monotonicity at the origin of μ_f with $f(x) = |x|^p$.

Proposition 4.1. *The 1-dimensional Hausdorff measure restricted to the graph of the function $f(x) = |x|^p$, $p > 0$, is locally monotone at the origin if and only if $p > \frac{3}{2}$.*

Proof: If $p > \frac{3}{2}$ then the proof follows from Proposition 3.1.

Suppose $p \le \frac{3}{2}$. A necessary condition for the local monotonicity at the origin for even C^1-functions given in [1] is that there is $\delta > 0$ such that

$$\frac{2x\sqrt{1+f'^2(x)}}{1+\sqrt{1+f'^2(x)}} \ge \int_0^x \sqrt{1+f'^2(t)}\,dt$$

for all $x \in (0,\delta)$. But the Taylor expansion gives us

$$\begin{aligned}\frac{2x\sqrt{1+f'^2(x)}}{1+\sqrt{1+f'^2(x)}} &= x\Big(1+\frac{1}{4}f'^2(x)-\frac{1}{8}f'^4(x)+\mathrm{O}(f'^6(x))\Big)\\ &= x+\frac{p^2}{4}x^{2p-1}-\frac{p^4}{8}x^{4p-3}+\mathrm{O}(x^{6p-5})\end{aligned}$$

and

$$\begin{aligned}\int_0^x \sqrt{1+f'^2(t)}\,dt &= \int_0^x 1+\frac{1}{2}f'^2(t)-\frac{1}{8}f'^4(t)+\mathrm{O}(f'^6(t))\,dt\\ &= x+\frac{p^2}{2(2p-1)}x^{2p-1}-\frac{p^4}{8(4p-3)}x^{4p-3}x^{4p-3}+\mathrm{O}(x^{6p-5}).\end{aligned}$$

Now it is enough to compare both expansions. If $p < \frac{3}{2}$ then we have $\frac{p^2}{4} < \frac{p^2}{2(2p-1)}$. If $p = \frac{3}{2}$ then we have $\frac{p^2}{4} = \frac{p^2}{2(2p-1)}$ and $-\frac{p^4}{8} < -\frac{p^4}{8(4p-3)}$. □

Remark 4.2. If z_0 is not on the graph, then the local monotonicity at z_0 is proven trivially and if $z_0 \neq 0$ is on the graph, then the local monotonicity at z_0 follows from Theorem 1.3.

Remark 4.3. By Proposition 3.1 we see that $\mathcal{H}^1$ restricted to the graph of the function $\operatorname{sgn}(x)|x|^p$ is locally monotone provided $p \ge p_0 = \frac{5}{4} + \sqrt{\frac{43}{48}}$. The bound p_0 is given by our weak version of Lemma 5.4 bellow (but for example it can be easily shown that the assertion of the lemma holds for $p = 2$ too). The author was not able to find a method how to get p_0 smaller. Computer approximations indicate that the critical parameter is in the interval $(1.796, 1.797)$.

Remark 4.4. In [2] it is proved that the $(N-1)$-dimensional Hausdorff measure restricted to a sphere in $\mathbb{R}^N$, $N \ge 2$, is locally $(N-1)$-monotone if and only if $N \le 3$. A similar method gives for a graph of $|x|^p$ in $\mathbb{R}^N$, that the restricted measure cannot be locally $(N-1)$-monotone for $p \le \frac{N+1}{2}$. In $\mathbb{R}^3$ it is quite surprising because for the sphere and the graph of $f(x_1,x_2) = x_1^2 + x_2^2$ we have different results.

5. Proof of Lemma 3.3

For $s \in [0,1)$ the definitions of g_{abs} and g_{sgn} coincide with

$$f(t) = pt^{p-1} - \frac{1-s^p}{1-s} = \frac{1}{1-s}(p(1-s)t^{p-1} - (1-s^p)).$$

We have

$$\begin{aligned}
&\int_s^1 f'^2(t)\,dt \\
&\quad = \int_s^1 \frac{1}{(1-s)^2}\Big(p^2(1-s)^2 t^{2p-2} - 2p(1-s)(1-s^p)t^{p-1} + (1-s^p)^2\Big)\,dt \\
&\quad = \frac{1}{(1-s)^2}\Big(\frac{p^2(1-s)^2(1-s^{2p-1})}{2p-1} - (1-s)(1-s^p)^2\Big)
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
&(1-s)\Big(\frac{1}{2}f'^2(1) + \frac{1}{2}f'^2(s) - \frac{1}{8}\big(f'(1)-f'(s)\big)^2\Big) \\
&\qquad = \frac{1}{8(1-s)}\Big(4\big(p(1-s)-(1-s^p)\big)^2 + 4\big(p(1-s)s^{p-1} - (1-s^p)\big)^2 \\
&\qquad\quad - \big(p(1-s)(1-s^{p-1})\big)^2\Big) \\
&\qquad = \frac{1}{8(1-s)}\Big(4p^2(1-s)^2(1+s^{2p-2}) - 8p(1-s)(1-s^p)(1+s^{p-1}) \\
&\qquad\quad + 8(1-s^p)^2 - p^2(1-s)^2(1-s^{p-1})^2\Big).
\end{aligned} \tag{31}$$

We define $F(s) := 8(1-s)(2p-1)\Phi^0_{s,1}(f)$. Therefore $F(s)$ has the same sign as $\Phi^0_{s,1}(g_{\mathrm{abs}})$ and $\Phi^0_{s,1}(g_{\mathrm{sgn}})$ for $s \in [0,1)$. Equations (30) and (31) imply

$$\begin{aligned}
F(x) = {} & (6p^3 - 27p^2 + 40p - 16)x^{2p} - 2p(6p^2 - 15p + 4)x^{2p-1} \\
& + 3p^2(2p-1)x^{2p-2} + 2p(p-4)(2p-1)x^{p+1} + 4(1-2p)(p^2-4p+8)x^p \\
& + 2p(p-4)(2p-1)x^{p-1} + 3p^2(2p-1)x^2 - 2p(6p^2-15p+4)x \\
& + (6p^3 - 27p^2 + 40p - 16).
\end{aligned}$$

For $s \in [-1,0)$ we consider each function g_{abs} and g_{sgn} separately. Set $x = -s$ (hence $x \in (0,1]$). The same way as above we obtain that the sign of $\Phi^0_{s,1}(g_{\mathrm{abs}})$

and $\Phi^0_{s,1}(g_{\mathrm{sgn}})$ is the same as the sign of

$$\begin{aligned}F_a(x) &= (6p^3-27p^2+40p-16)x^{2p}+2p(6p^2-15p+4)x^{2p-1}\\ &\quad+3p^2(2p-1)x^{2p-2}-2p(p-4)(2p-1)x^{p+1}+4(1-2p)(p^2-4p+8)x^p\\ &\quad-2p(p-4)(2p-1)x^{p-1}+3p^2(2p-1)x^2+2p(6p^2-15p+4)x\\ &\quad+(6p^3-27p^2+40p-16)\end{aligned}$$

and

$$\begin{aligned}F_s(x) &= (6p^3-27p^2+40p-16)x^{2p}+2p(6p^2-15p+4)x^{2p-1}\\ &\quad+3p^2(2p-1)x^{2p-2}+2p(p-4)(2p-1)x^{p+1}-4(1-2p)(p^2-4p+8)x^p\\ &\quad+2p(p-4)(2p-1)x^{p-1}+3p^2(2p-1)x^2+2p(6p^2-15p+4)x\\ &\quad+(6p^3-27p^2+40p-16),\end{aligned}$$

respectively. In the sequel, we need the following auxiliary lemma.

Lemma 5.1. *The following polynomials are positive on* $[\frac{3}{2},\infty)$*:*

$$\begin{aligned}P_1(t) &= 6t^3-27t^2+40t-16,\\ P_2(t) &= 3t^2-5t+2,\\ P_3(t) &= 23t^2-74t+68,\\ P_4(t) &= 15t^2-54t+56,\\ P_5(t) &= 5t^2-16t+14,\\ P_6(t) &= 11t^2-42t+42,\\ P_7(t) &= 12t^3-49t^2+57t-9,\\ P_8(t) &= 12t^4-60t^3+103t^2-67t+18,\\ P_9(t) &= 24t^4-144t^3+321t^2-310t+116,\\ P_{10}(t) &= 24t^4-114t^3+175t^2-70t-8.\end{aligned}$$

Proof: The assertion trivially holds for $P_3(t)$, $P_4(t)$, $P_5(t)$ and $P_6(t)$ because of the negative discriminant.

We have $P_1(1)=3$, $P_1(\frac{5}{3})=\frac{31}{9}$ and

$$P_1'(t)=18t^2-54t+40=2(3t-4)(3t-5),$$

hence $P_1(t)\geq 3$ on $[1,\infty)$.

Further, $P_2(t)=(t-1)(3t-2)>0$ on $[\frac{3}{2},\infty)$.

Now, we set

$$\varphi_7(t)=12t^3-49t^2+52t-9.$$

We obtain $\varphi_7(1) = 6$, $\varphi_7(2) = -5$ and

$$\varphi_7'(t) = 36t^2 - 98t + 52 = 2(t-2)(18t-13).$$

Hence $\varphi_7(t) \geq -5$ on $[1, \infty)$ and thus $P_7(t) = \varphi_7(t) + 5t > 0$ on $[\frac{3}{2}, \infty)$.

Let

$$\varphi_8(t) = 12t^4 - 60t^3 + 99t^2 - 60t + 18 \qquad \text{and} \qquad \psi_8(t) = 4t^2 - 8t.$$

The first function satisfies $\varphi_8(1) = 9$, $\varphi_8(2) = 6$ and

$$\varphi_8'(t) = 6(8t^3 - 30t^2 + 33t - 10) = 6(t-2)(2t-1)(4t-5).$$

This implies $\varphi_8(t) \geq 6$ on $[1, \infty)$. We observe $\psi_8(1) = -4$ and $\psi_8'(t) = 8t-8$, thus $\psi_8(t) \geq -4$ on $[1, \infty)$ and the assertion follows from $P_8(t) = \varphi_8(t) + \psi_8(t) + t$.

We set

$$\varphi_9(t) = 24t^4 - 144t^3 + 321t^2 - 324t + 116.$$

We obtain $\varphi_9(2) = -16$ and

$$\varphi_9'(x) = 96t^3 - 432t^2 + 642t - 324 = 6(t-2)(16t^2 - 40t + 27) = 6(t-2)((4t-5)^2 + 2).$$

Therefore

$$P_9(t) = \varphi_9(t) + 14t \geq 14t + \varphi_9(2) \geq 14t - 16 > 0 \qquad \text{on } [\tfrac{3}{2}, \infty).$$

Let

$$\varphi_{10}(t) = 24t^4 - 114t^3 + 150t^2 \qquad \text{and} \qquad \psi_{10}(t) = 25t^2 - 70t - 8.$$

We have $\varphi_{10}(2) = 72$, $\varphi_{10}(\frac{3}{2}) = \frac{297}{4} > 72$ and

$$\varphi_{10}'(t) = 96t^3 - 342t^2 + 300t = 6t(t-2)(16t-25),$$

therefore $\varphi_{10}(t) \geq 72$ on $[\frac{3}{2}, \infty)$. Further, we observe $\psi_{10}'(t) = 50t - 70$ and thus $\psi_{10}(t) \geq \psi_{10}(\frac{7}{5}) = -57$ on $\mathbb{R}$. Hence $P_{10}(t) = \varphi_{10}(t) + \psi_{10}(t) > 0$ on $[\frac{3}{2}, \infty)$. $\square$

The proofs of the following Lemma 5.2 and Lemma 5.3 use Lemma 5.1. These technical proofs are based on the idea that if we need to show that a smooth function f is positive on (a, ∞), it is enough to show that $f(a), f'(a), \ldots, f^{(k)}(a) \geq 0$ and $f^{(k+1)}(x) > 0$ on (a, ∞) for some $k \in \mathbb{N}$. To improve the readability of the proofs, there are not given formulae for $F'(x)$, $F''(x)$, etc., but only the properties we actually use.

Lemma 5.2. *Assume $p > \frac{3}{2}$. Then the function $F(x)$ is positive on $[0, 1)$.*

Proof: First, we have

$$F(0) = 6p^3 - 27p^2 + 40p - 16 = P_1(p) > 0.$$

Further, as $F(\frac{1}{x}) = \frac{F(x)}{x^{2p}}$, we restrict ourselves to $[1,\infty)$. The function F satisfies

$$F(1) = F'(1) = F''(1) = 0$$

and

$$F'''(x) = 2p(p-1)(2p-1)x^{p-4}g(x),$$

where

$$\begin{aligned} g(x) = 2(6p^3 - 27p^2 + 40p - 16)x^{p+1} + 2(3-2p)(6p^2 - 15p + 4)x^p \\ + 6p(p-2)(2p-3)x^{p-1} + p(p+1)(p-4)x^2 \\ + 2(2-p)(p^2 - 4p + 8)x + (p-2)(p-3)(p-4). \end{aligned}$$

Therefore it is enough to show $g(x) > 0$ on $(1,\infty)$. The function g satisfies

$$g(1) = 0, \quad g'(1) = 4p(p-1) > 0, \quad g''(1) = 4p(3p^2 - 5p + 2) = 4pP_2(p) > 0$$

and

$$g'''(x) = 2p(p-1)x^{p-4}h(x),$$

where

$$\begin{aligned} h(x) = (p+1)(6p^3 - 27p^2 + 40p - 16)x^2 - (p-2)(2p-3)(6p^2 - 15p + 4)x \\ + 3(p-3)(p-2)^2(2p-3). \end{aligned}$$

Our aim is to prove that $h(x) > 0$ on $[1,\infty)$. We have

$$\begin{aligned} h(1) &= 23p^2 - 74p + 68 = P_3(p) > 0, \\ h'(1) &= (2p-1)(15p^2 - 54p + 56) = (2p-1)P_4(p) > 0, \\ h''(x) &= 2(p+1)(6p^3 - 27p^2 + 40p - 16) = 2(p+1)P_1(p) > 0 \quad \text{on} \quad [1,\infty). \end{aligned}$$

Thus, we are done. □

Lemma 5.3. *Assume $p > \frac{3}{2}$. Then the function $F_a(x)$ is positive on $(0,1]$.*

Proof: As $F_a(\frac{1}{x}) = \frac{F_a(x)}{x^{2p}}$, we restrict ourselves to $[1,\infty)$. The function F_a satisfies

$$F_a(1) = 16p^2(2p-3) > 0, \qquad F_a'(1) = 16p^3(2p-3) > 0,$$

$$F_a''(1) = 8p^2(10p^3 - 37p^2 + 44p - 14) = 8p^2(2p-1)P_5(p) > 0$$

and

$$F_a'''(x) = 2p(p-1)(2p-1)x^{p-4}g(x),$$

where

$$g(x) = 2(6p^3 - 27p^2 + 40p - 16)x^{p+1} + 2(2p-3)(6p^2 - 15p + 4)x^p + 6p(p-2)(2p-3)x^{p-1} + p(p+1)(4-p)x^2 + 2(2-p)(p^2 - 4p + 8)x + (2-p)(p-3)(p-4).$$

It is enough to show that $g(x) > 0$ on $[1, \infty)$. For this function we have

$$g(1) = 4p(11p^2 - 42p + 42) = 4pP_6(p) > 0,$$
$$g'(1) = 4p(12p^3 - 49p^2 + 57p - 9) = 4pP_7(p) > 0,$$
$$g''(1) = 4p(12p^4 - 60p^3 + 103p^2 - 67p + 18) = 4pP_8(p) > 0$$

and

$$g'''(x) = 2p(p-1)x^{p-4}h(x),$$

where

$$h(x) = (p+1)(6p^3 - 27p^2 + 40p - 16)x^2 + (p-2)(2p-3)(6p^2 - 15p + 4)x + 3(p-3)(p-2)^2(2p-3).$$

Proving that $h(x) > 0$ on $[1, \infty)$ we observe

$$h(1) = 24p^4 - 144p^3 + 321p^2 - 310p + 116 = P_9(p) > 0,$$
$$h'(1) = 24p^4 - 114p^3 + 175p^2 - 70p - 8 = P_{10}(p) > 0,$$
$$h''(x) = 2(p+1)(6p^3 - 27p^2 + 40p - 16) = 2(p+1)P_1(p) > 0 \quad \text{on} \quad [1, \infty).$$

□

In the proof of the following lemma we cannot use the same method because we do not have $F_s'''(1) \geq 0$. Therefore we use some rough estimates. In fact, $p = p_0$ is far from the borderline case.

Lemma 5.4. *Let $p \geq p_0 = \frac{5}{4} + \sqrt{\frac{43}{48}}$. Then $F_s(x)$ is positive on $(0, 1]$.*

Proof: We have

$$F_s(x) = F(x) + 2\Big(2p(6p^2 - 15p + 4)x^{2p-1} + 4(2p-1)(p^2 - 4p + 8)x^p + 2p(6p^2 - 15p + 4)x^{2p-1}x\Big).$$

Since $p^2 - 4p + 8 = (p-2)^2 + 4 > 0$ on $\mathbb{R}$ and

$$6p^2 - 15p + 4 \geq 0 \quad \text{for} \quad p \geq \frac{15 + \sqrt{15^2 - 4 \cdot 6 \cdot 4}}{2 \cdot 6} = p_0,$$

the proof follows from Lemma 5.2. □

Now, the proof of Lemma 3.3 follows from Lemmata 5.2, 5.3 and 5.4.

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