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An algorithm for free algebras

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Abstract. We present an algorithm for constructing the free algebra over a given finite partial algebra in the variety determined by a finite list of equations. The algorithm succeeds whenever the desired free algebra is finite.

Keywords: reflection, free algebra, variety, algorithm

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1. Introduction

Let A be a partial algebra of a signature σ and V be a variety (an equationally defined class) of algebras of the same signature σ. By a *free algebra* in V over the partial algebra A we mean a *reflection* of A in V, i.e., a pair $\langle B, h\rangle$ consisting of an algebra $B \in V$ and a homomorphism h of A such that $h(A)$ is a generating subset of B and for any homomorphism f of A into any algebra $C \in V$ there exists a homomorphism $g : B \to C$ with $f = gh$. As it is well known, the free algebra in V exists over any partial algebra of the given signature and it is unique up to isomorphism. It is also known that there is no algorithm deciding for any finite partial algebra A and any finite set of equations E whether the free algebra over A in the variety determined by E is finite or infinite. The aim of this paper is to present an algorithm trying to construct the free algebra and succeeding each time when the free algebra is finite; if the free algebra is infinite, the algorithm never halts; of course, we cannot predict which case will take place or estimate the number of necessary steps.

The idea is to modify the given partial algebra in a sequence of steps, each of which either extends the partial algebra by adding a new element or factors the partial algebra if an equation requires two different elements to be identified; the algorithm halts if the last partial algebra is complete and satisfies all the equations. The extending steps depend on the choice of an undefined place in the table for a partial operation. We will see, however, that an arbitrary choice, or even one that could seem to be the most natural, may result in an infinite number of steps even if the free algebra is finite. So, we must pay attention to a proper way how the selection should be done.

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The algorithm has been implemented in the computer program Alg which can be found at `www.karlin.mff.cuni.cz/~jezek`.

For notation and terminology not introduced in this paper we refer to the books [1] and [4].

2. Partial algebras

By a *partial operation* of arity n (for $n \geq 0$) on a nonempty set A we mean a mapping of a subset of A^n into A. If the domain is the whole of A^n, we say that the partial operation is complete; complete partial operations are called operations.

Let σ be a fixed finite *signature*, i.e., a finite set of operation symbols each of which is assigned a nonnegative integer, called its arity. By a *partial algebra* (of this fixed signature σ) we mean a nonempty set A together with a mapping assigning to each symbol $F \in \sigma$ a partial operation of the same arity on A; this partial operation will be denoted by F_A. A partial algebra is said to be *complete* if F_A is complete for all $F \in \sigma$. *Algebras* are complete partial algebras.

For a partial algebra A and a nonempty subset S of A we denote by $A \restriction S$ the partial algebra with underlying set S and partial operations defined as follows: for $a_1, \dots, a_n, b \in S$, $F_{A\restriction S}(a_1, \dots, a_n) = b$ if and only if $F_A(a_1, \dots, a_n) = b$.

Let A, B be two partial algebras. By a *homomorphism* of A into B we mean a mapping $f : A \to B$ such that $f(F_A(a_1, \dots, a_n)) = F_B(f(a_1), \dots, f(a_n))$ whenever $F_A(a_1, \dots, a_n)$ is defined.

By a *congruence* of a partial algebra A we mean an equivalence r on A such that $\langle F_A(a_1, \dots, a_n), F_A(b_1, \dots, b_n)\rangle \in r$ whenever these two elements are defined and $\langle a_i, b_i\rangle \in r$ for all i. If r is a congruence of A then the *factor* of A by r is the partial algebra with the underlying set A/r (the set of blocks of r) and operations defined as follows. Let $F \in \sigma$ be of arity n and $a_1/r, \dots, a_n/r$ be elements of A/r. If there exist elements $b_1, \dots, b_n \in A$ such that $\langle a_i, b_i\rangle \in r$ for all i and $F_A(b_1, \dots, b_n)$ is defined, then $F_{A/r}(a_1/r, \dots, a_n/r) = F_A(b_1, \dots, b_n)/r$; otherwise, $F_{A/r}(a_1/r, \dots, a_n/r)$ is not defined. The factor of A by r will be denoted by A/r (as its underlying set).

Let us remark that for a binary operation symbol F (in contrast to the situation for complete algebras), congruences of a partial algebra A do not coincide with the equivalences r on A satisfying the following two weaker conditions: $\langle F_A(a, c), F_A(b, c)\rangle \in r$ whenever $\langle a, b\rangle \in r$ and $F_A(a, c)$, $F_A(b, c)$ are defined; and $\langle F_A(c, a), F_A(c, b)\rangle \in r$ whenever $\langle a, b\rangle \in r$ and $F_A(c, a)$, $F_A(c, b)$ are both defined. For example, let A be the partial groupoid with three elements a, b, c and $aa = a$, $bb = c$ the only products defined in A. The equivalence with blocks $\{a, b\}$ and $\{c\}$ satisfies both weaker conditions but is not a congruence of A.

2.1 Lemma. *Let f be a homomorphism of a partial algebra A into a partial algebra B and let r be a congruence of A such that $r \subseteq \ker(f)$. Then the unique mapping $g : A/r \to B$ satisfying $g(a/r) = f(a)$ for all $a \in A$, is a homomorphism of A/r into B.*

Proof: It is obvious. □

Let $x_1, x_2, \dots$ be countably many variables. Denote by T the algebra of *terms* (of the given signature) over the set of variables. For a positive integer k denote by T_k the subalgebra generated by $\{x_1, \dots, x_k\}$, so that T_k is the algebra of terms over $x_1, \dots, x_k$.

The *length* of a term t is the total number of occurrences of variables and operation symbols in t.

By an *elementary lift* in a partial algebra A we mean a partial mapping of the form $a \mapsto F_A(b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_k)$ for an operation symbol F of an arity k, a number $i \in \{1, \dots, k\}$ and some elements $b_i \in A$. A composition of m elementary lifts is called a *lift* of depth m. For a complete algebra A, all lifts are mappings of A into A.

By the *depth* of a term t we mean the maximal number m such that $t = L(u)$ for a term u and a lift L of depth m in the algebra of terms. Thus variables and constants are terms of depth 0, and all other terms are of positive depth.

A partial algebra A is said to be 0-*complete* if all constants (operation symbols of arity 0) are defined in A. The 0-completion of A is the partial algebra A' with the underlying set $A \cup \{c_0, \dots, c_m\}$ where $c_0, \dots, c_m$ are all the constants of σ that are not defined in A (we assume that the constants do not belong to A); the value of c_i in A' is c_i, and $F_{A'} = F_A$ for all operation symbols F of σ not belonging to $\{c_0, \dots, c_m\}$. It is easy to see that the free algebras over A and over A' in a variety V coincide. Therefore, it will be sufficient to work with 0-complete partial algebras only.

3. The order of an element of a renovation of A

In the sequel let A be a fixed finite 0-complete partial algebra of the signature σ. Put $n = |A|$ and let $a_1, \dots, a_n$ be all elements of A.

By a *renovation* of A we will mean a pair $\langle B, h \rangle$ consisting of a finite partial algebra B and a homomorphism h of A into B such that the range $h(A)$ of h is a generating subset of B.

Let $\langle B, h \rangle$ be a renovation of A. By a k-*interpretation* in B (where k is a positive integer) we mean a mapping of the set $\{x_1, \dots, x_k\}$ into B. If α is a k-interpretation in B then for some terms $t \in T_k$ we define (by induction on the length of t) an element $\alpha^B(t)$ of B as follows: if t is a variable then $\alpha^B(t) = \alpha(t)$; if $t = Ft_1 \dots t_m$, if $\alpha^B(t_i)$ are defined for all i and if $F_B(\alpha^B(t_1), \dots, \alpha^B(t_m))$ is defined, put $\alpha^B(t) = F_B(\alpha^B(t_1), \dots, \alpha^B(t_m))$; in all other cases let $\alpha^B(t)$ be undefined. Clearly, $\mathrm{Dom}(\alpha^B)$ is a subset of T_k containing $\{x_1, \dots, x_k\}$ and all constants and closed under subterms. Clearly, α^B is a homomorphism of $T_k \restriction \mathrm{Dom}(\alpha^B)$ into B.

Let $\langle B, h \rangle$ be a renovation of A. For a term $t \in T_n$ we put $t^{B,h} = \alpha^B(t)$ where $\alpha(x_i) = h(a_i)$ for $i = 1, \dots, n$. The set of the terms $t \in T_n$ for which $t^{B,h}$ is defined will be denoted by $\mathrm{D}(B, h)$. We consider $\mathrm{D}(B, h)$ as a partial groupoid, $\mathrm{D}(B, h) = T_n \restriction \mathrm{D}(B, h)$.

3.1 Lemma. *Let $\langle B, h\rangle$ be a renovation of A. Then $t \mapsto t^{B,h}$ is a homomorphism of $\mathrm{D}(B,h)$ onto B.*

PROOF: Clearly, the mapping is a homomorphism. Since B is generated by $h(A)$, its range is the whole of B. □

Let $\langle B, h\rangle$ be a renovation of A and let b be an element of B. The least number i such that $b = t^{B,h}$ for a term $t \in T_n$ of length i will be called the *order* of b (in B, with respect to h), and denoted by $\mathrm{ord}_{B,h}(b)$ or just $\mathrm{ord}(b)$. Thus elements of order 1 are precisely the elements of $h(A)$; the elements of $B - h(A)$ have order ≥ 2. The following lemma provides a way of constructing (by induction on i) the set of all elements of B of order i without reference to terms.

3.2 Lemma. *Let $\langle B, h\rangle$ be a renovation of A. For every $i \geq 1$ define a subset O_i of B as follows: $O_1 = h(A)$; O_{i+1} is the set of those elements $b \in B - (O_1 \cup \dots \cup O_i)$ for which there exist an operation symbol F of arity $m \geq 1$ and elements $d_1 \in O_{j_1}, \dots, d_m \in O_{j_m}$ for some $j_1, \dots, j_m$ with $1 + j_1 + \dots + j_m = i$, such that $b = F_B(d_1, \dots, d_m)$. Then for any $i \geq 1$, O_i is just the set of elements of B of order i.*

PROOF: It is easy. □

3.3 Lemma. *Let $\langle B, h\rangle$ be a renovation of A and C be a partial algebra such that $B \subseteq C$, id_B is a homomorphism of B into C and C is generated by $h(A)$. Then $\langle C, h\rangle$ is also a renovation of A; we have $\mathrm{D}(B,h) \subseteq \mathrm{D}(C,h)$ and $\mathrm{ord}_{C,h}(b) \leq \mathrm{ord}_{B,h}(b)$ for all $b \in B$.*

PROOF: It is obvious. □

3.4 Lemma. *Let $\langle B, h\rangle$ be a renovation of A and r be a congruence of B; put $g(b) = b/r$ for $b \in B$. Then $\langle B/r, gh\rangle$ is a renovation of A, $\mathrm{D}(B,h) \subseteq \mathrm{D}(B/r, gh)$ and $\mathrm{ord}_{B/r,gh}(b/r) \leq \mathrm{ord}_{B,h}(b)$ for all $b \in B$.*

PROOF: It is obvious. □

4. Reductive steps

As before, let A be a fixed finite 0-complete partial algebra. Moreover, let E be a finite set of equations and V be the variety determined by E. Denote by N the least positive integer such that for any $\langle u, v\rangle \in E$, the set $\{x_1, \dots, x_N\}$ contains all variables occurring in either u or v.

A renovation $\langle B, h\rangle$ of A is said to be *admissible* if for any homomorphism f of A into any algebra $G \in V$ there exists a homomorphism g of B into G with $f = gh$.

Let $\langle B, h\rangle$ be a renovation of A. We denote by $\Gamma(B)$ the set of the ordered triples $\langle u, v, \alpha\rangle$ such that $\langle u, v\rangle \in E \cup E^{-1}$, α is an N-interpretation in B, $\alpha^B(u)$, $\alpha^B(v)$ are both defined and $\alpha^B(u) \neq \alpha^B(v)$. Denote by $\mathrm{C}(B)$ the congruence of B generated by the pairs $\langle \alpha(u), \alpha(v)\rangle$ with $\langle u, v, \alpha\rangle \in \Gamma(B)$. Put $B^r = B/\,\mathrm{C}(B)$ and define $h^r : A \to B^r$ by $h^r(a) = h(a)/\,\mathrm{C}(B)$ for all $a \in A$.

4.1 Lemma. *Let $\langle B, h\rangle$ be a renovation of A. Then $\langle B^r, h^r\rangle$ is also a renovation of A. If $\langle B, h\rangle$ is admissible then $\langle B^r, h^r\rangle$ is also admissible.*

Proof: It follows from 3.4 and 2.1 that $\langle B^r, h^r\rangle$ is a renovation and it is sufficient to prove that whenever f is a homomorphism of B into an algebra $G \in V$ then the generating relation for the congruence $\mathrm{C}(B)$ is contained in $\ker(f)$ (so that then the congruence itself is contained in $\ker(f)$). Let $\langle u, v, \alpha\rangle \in \Gamma(B)$. Denote by g the homomorphism of T_N into G with $g(x_i) = f\alpha(x_i)$ for $i = 1, \dots, N$. Since $\langle u, v\rangle$ is satisfied in G, we have $g(u) = g(v)$. It is easy to check by induction on the length of t that $g(t) = f\alpha^B(t)$ for all $t \in T_N$. Thus $f\alpha^B(u) = g(u) = g(v) = f\alpha^B(v)$, i.e., $\langle \alpha^B(u), \alpha^B(v)\rangle \in \ker(f)$. □

Starting with a renovation $\langle B, h\rangle$ such that $\Gamma(B)$ is nonempty, it may happen that $\Gamma(B^r)$ is still nonempty. Then we need to repeat the process and to create a sequence of renovations $\langle B_i, h_i\rangle$ where $\langle B_0, h_0\rangle = \langle B, h\rangle$ and $\langle B_{i+1}, h_{i+1}\rangle = \langle B_i^r, h_i^r\rangle$. This sequence necessarily terminates with a renovation $\langle B_k, h_k\rangle$ such that $\Gamma(B_k)$ is empty. This last member of the sequence will be denoted by $\langle B^\rho, h^\rho\rangle$. The transition from $\langle B, h\rangle$ to $\langle B^\rho, h^\rho\rangle$ will be called a *reductive step*.

5. Extensive steps

Let A and E be as above. By a *spark* in a renovation $\langle B, h\rangle$ of A (or just in B) we mean an $(m+1)$-tuple $\langle F, b_1, \dots, b_m\rangle$ where $m \geq 1$, F is an m-ary operation symbol and $b_1, \dots, b_m$ are elements of B such that $F_B(b_1, \dots, b_m)$ is not defined. For any spark $e = \langle F, b_1, \dots, b_m\rangle$ define $\langle B^e, h^e\rangle$ as follows: the underlying set of B^e is the union $B \cup \{c\}$ for an element $c \notin B$; the operations of B^e coincide with the operations of B, with only the addition of $F_{B^e}(b_1, \dots, b_m) = c$; $h^e = h$.

5.1 Lemma. *Let $\langle B, h\rangle$ be a renovation of A and let $e = \langle F, b_1, \dots, b_m\rangle$ be a spark in B. Then $\langle B^e, h^e\rangle$ is also a renovation of A. If $\langle B, h\rangle$ is admissible then $\langle B^e, h^e\rangle$ is also admissible.*

Proof: It follows from 3.3 that $\langle B^e, h^e\rangle$ is a renovation of A. Let $\langle B, h\rangle$ be admissible and let f be a homomorphism of B into an algebra $G \in V$. It is sufficient to prove that f can be extended to a homomorphism of B^e to G. Such an extension f' can be defined by $f'(b) = g(b)$ for all $b \in B$ and $f'(c) = F_G(f(b_1), \dots, f(b_m))$. □

The transition from $\langle B, h\rangle$ to $\langle B^e, h^e\rangle$ will be called an *extensive step*.

6. The algorithm

Let A be a fixed finite partial algebra and E be a finite set of equations. We want to construct the reflection of A in the variety V determined by E, whenever the reflection is finite. We can assume that A is 0-complete, because if it is not, we can replace it by its 0-completion.

By a *building sequence* we mean a sequence $\langle B_i, h_i\rangle$ $(i = 0, 1, \dots)$ of renovations of A with the following properties:

(1) $B_0 = A$ and $h_0 = \mathrm{id}_A$;
(2) for each $i \geq 1$, either $\Gamma(B_{i-1}) \neq \emptyset$ and $\langle B_i, h_i\rangle = \langle B^{\rho}_{i-1}, h^{\rho}_{i-1}\rangle$, or else $\Gamma(B_{i-1}) = \emptyset$ and $\langle B_i, h_i\rangle = \langle B^{e}_{i-1}, h^{e}_{i-1}\rangle$ for a spark e in B_{i-1};
(3) the sequence is either infinite or terminates with a (complete) algebra B_k such that $\Gamma(B_k)$ is empty.

Of course, there may exist various building sequences for the given partial algebra A. If we find one of them that terminates then our task is finished according to the following theorem.

6.1 Theorem. *Let $\langle B_i, h_i\rangle$ $(i = 0, 1, \dots, k)$ be a terminating building sequence starting with $\langle A, \mathrm{id}_A\rangle$. Then $\langle B_k, h_k\rangle$ is a reflection of A in V.*

Proof: The algebra B_k belongs to V, since the set $\Gamma(B_k)$ is empty. The fact that $\langle B_k, h_k\rangle$ is a reflection of A in K follows from 4.1 and 5.1. □

6.2 Example. Let the signature be the signature of groupoids and let E consist of these equations:

$$\langle x_1x_2, x_1(x_2x_2)\rangle \quad \text{and} \quad \langle x_1, x_1(x_2x_2)\rangle\,.$$

Let A be the partial groupoid with the underlying set $\{0, 1\}$ and no products defined. Since the two equations imply $\langle x_1x_2, x_1\rangle$, we can immediately see that the groupoid with the same underlying set as A and with multiplication $ab = a$ is, together with the identity, the reflection of A in V. However, let us pretend that we do not know it and let us try to find the reflection by constructing the building sequence starting with A as above. With an inept choice of the sparks for the extensive steps we could obtain the following infinite sequence: $B_i = \{0, 1, \dots, i + 1\}$ with multiplication on B_i defined by $ab = c$ if and only if $a = 0$, $1 \leq b < i$ and $c = b + 1$.

We see that if we want to be successful, at least the sparks for the extensive steps must be chosen in a not arbitrary way.

A building sequence $\langle B_i, h_i\rangle, \dots$ $(i = 0, 1, \dots)$ will be called *smart* if for each its extensive step from $\langle B_{i-1}, h_{i-1}\rangle$ to $\langle B_i, h_i\rangle$ the spark $\langle F, b_1, \dots, b_m\rangle$ is selected in such a way that the sum $\mathrm{ord}_{B_{i-1},h_{i-1}}(b_1) + \dots + \mathrm{ord}_{B_{i-1},h_{i-1}}(b_m)$ is the least possible.

6.3 Theorem. *Let $\langle G, h\rangle$ be the reflection of the partial algebra A in V and let G be finite. Then any smart building sequence starting with $\langle A, \mathrm{id}_A\rangle$ is finite and its last member is isomorphic to G.*

Proof: Suppose, on the contrary, that a smart building sequence $\langle B_i, h_i\rangle$ $(i \geq 0)$ starting with $\langle A, \mathrm{id}_A\rangle$ is infinite. For every $i \geq 0$ put $D_i = \mathrm{D}(B_i, h_i)$, so that $D_0 \subseteq D_1 \subseteq \dots$ by 3.3 and 3.4. Denote by m_i the least positive integer such that there exists a term of length m_i not belonging to D_i. We have $m_0 \leq m_1 \leq \dots$.

Since every step must be later followed by an extensive step, it follows from the smartness of the sequence that for any $t \in T_n$ there exists an m with $t \in D_i$ for all $i \geq m$. (Recall that $n = |A|$ and $A = \{a_1, \dots, a_n\}$.)

Denote by g the extension of $x_i \mapsto h(a_i)$ to a homomorphism of T_n onto G. It is easy to see that for $u, v \in T_n$, $g(u) = g(v)$ if and only if there exist a nonnegative integer k and terms $u_0, \dots, u_k$ such that $u_0 = u$, $u_k = v$ and for every $I = 1, \dots, k$ one of the following two cases takes place:

(i) $\langle u_{I-1}, u_I \rangle = \langle L\beta(p), L\beta(q) \rangle$ for some lift L in T_n, some $\langle p, q \rangle \in E \cup E^{-1}$ and some homomorphism β of T into T_n;

(ii) $\{u_{I-1}, u_I\} = \{L(Fx_{j_1} \dots x_{j_k}), L(x_j)\}$ for some lift L in T_n and some defined situation $F_A(a_{j_1}, \dots, a_{j_k}) = a_j$ in A.

For every pair u, v of elements of T_n such that $g(u) = g(v)$ let us fix one such sequence from u to v and denote it by $S_{u,v}$.

Let i be so large that for any pair u, v of elements of T_n such that $g(u) = g(v)$ and the depths of both u and v are at most $|G|$, all members of $S_{u,v}$ belong to D_i; moreover, we may assume that $\mathrm{C}(B_i) = \mathrm{id}_{B_i}$.

Since B_i is not a complete algebra, there exists a term $t \in T_n$ not belonging to D_i. Among all such terms t fix one of minimal possible depth, and among those of the minimal depth one of minimal length. Denote by m the depth of t, so that $m > |G|$. There exists a sequence $w_0, \dots, w_m$ of terms from T_n such that $w_0 \in \{x_1, \dots, x_n\}$, $w_m = t$ and for every $i = 1, \dots, m$, w_i is obtained from w_{i-1} by an elementary lift in T_n. Thus w_i is of depth i. Since $g(w_0), \dots, g(w_{|G|})$ are $|G|+1$ elements of G, there exist $0 \le j_1 < j_2 \le |G|$ with $g(w_{j_1}) = g(w_{j_2})$. Denote by $u_0, \dots, u_k$ the sequence $S_{w_{j_1}, w_{j_2}}$. These terms all belong to D_i.

Let $0 < I \le k$. One of the two cases, either (i) or (ii), takes place. Let (i) take place. It follows from 4.1 and 5.1 that there is a (unique) homomorphism $h' : B_i \to G$ with $h = h'h_i$. Where f_i is the homomorphism of D_i onto B_i defined by $f_i(s) = s^{B_i, h_i}$ (as in 3.1), we have $h' f_i(s) = g(s)$ for all $s \in D_i$. Since u_{I-1} and u_I belong to D_i, we have $\beta(p) \in D_i$ and $\beta(q) \in D_i$. Easily, there exists a lift L' in B_i such that $f_i L\beta(p) = L' f_i \beta(p)$ and $f_i L\beta(q) = L' f_i \beta(q)$. It follows from the definition of $\mathrm{C}(B_i)$ that $\langle f_i\beta(p), f_i\beta(q) \rangle \in \mathrm{C}(B_i)$. Since $\mathrm{C}(B_i)$ is a congruence, we get $\langle L' f_i\beta(p), L' f_i\beta(q) \rangle \in \mathrm{C}(B_i)$, i.e., $\langle u_{I-1}, u_I \rangle \in \mathrm{C}(B_i)$. But $\mathrm{C}(B_i) = \mathrm{id}_{B_i}$, so $u_{I-1} = u_I$. By transitivity we get $w_{j_1} = w_{j_2}$. In the case (ii) we get $w_{j_1} = w_{j_2}$ similarly. Denote by t' the term obtained from t by replacing w_{j_2} with w_{j_1}. Then t' is shorter than t and of depth at most that of t, so that $f_i(t')$ is defined. But then clearly $f_i(t)$ is defined (and equals $f_i(t')$), so that $t \in D_i$, a contradiction. □

7. Examples

Since it is easy (using 3.2) to design a way of selecting sparks for the extensive steps in an efficient way, theorems 6.1 and 6.3 give us an algorithm that outputs a reflection of A in the variety V determined by E whenever the input is such that the reflection of A in V is finite, and works for ever in the opposite case. We cannot expect the algorithm to work fast. We can give no estimate for the number of steps in the successful cases, because any recursive upper bound would imply the existence of an algorithm deciding for any finite set of equations whether the corresponding variety is trivial, and it is known that such an algorithm does not exist. (It follows from [3], where it is proved that the set of equations having a nontrivial model, and also the set of equations having a nontrivial finite model, are both undecidable.) So, it may be interesting to give at least some examples to get an idea of how it works.

The examples were obtained by the computer program Alg, in which the above described algorithm is implemented. We must remark that in the computer program the steps do not correspond precisely to the steps as described above. This has been done for the sake of speeding up the execution a little bit. Namely, in each step we also try to partially complete the partial algebra in the following way. Whenever $\langle u,v\rangle \in E \cup E^{-1}$ and α is an N-interpretation in B such that $\alpha^B(u)$ is defined, $\alpha^B(v)$ is not defined and $v = Fv_1 \dots v_k$ where $\alpha^B(v_i)$ are all defined, we can set $F_{B'}(\alpha^B(v_1),\dots,\alpha^B(v_k)) = \alpha^B(u)$ in the new partial algebra B'. This can be done in various ways, so the theory would become even more technical if we wanted to explain it here in full detail. And each such partial completion can be replaced by a combination of a reductive with an extensive step, so that they are not interesting from the theoretical point of view.

In each of the following examples let ε be the number of extensive steps and κ be the cardinality of the largest partial algebra that needed to be constructed. Evidently, ε must be at least $|G| - |A|$ where A is the partial algebra and G is its reflection in the appropriate variety.

The free distributive lattice with 3 generators can be obtained as the reflection of the 3-element partial algebra with partial operations empty, in the variety determined by seven equations for distributive lattices. It has 18 elements; here $\varepsilon = 15$ and $\kappa = 18$ (so that the process goes straight up). For 4 generators, the free distributive lattice has 166 elements; here $\varepsilon = 162$ and $\kappa = 166$.

We can obtain the free modular lattice with 3 generators in a similar way. It has 28 elements; here $\varepsilon = 25$ and $\kappa = 28$.

The free modular lattice over the partial lattice that is the cardinal sum of the 3-element chain and the 4-element chain has 124 elements; here $\varepsilon = 117$ and $\kappa = 124$.

The free lattice over the partial lattice that is the cardinal sum of the 3-element chain and the 1-element lattice has 20 elements; here $\varepsilon = 16$ and $\kappa = 20$.

The free Boolean algebra with 2 generators has 16 elements; here $\varepsilon = 233$ and $\kappa = 106$. For 3 generators we should obtain 256 elements; however, we did not obtain this from the program; its execution would take a long time.

The 8-element quaternion group can be given by the so-called defining relations, which means that it is the free group over a certain 6-element partial algebra; here $\varepsilon = 14$ and $\kappa = 9$.

The free idempotent semigroup with 3 generators has 159 elements; here $\varepsilon = 156$ and $\kappa = 159$.

The variety generated by the commutative fork is based on six equations that can be found in [2]. The free algebra in this variety with 3 generators has 13 elements; here $\varepsilon = 44$ and $\kappa = 14$. (Of course, since the variety is finitely generated, the free algebra can be found faster, using a different algorithm without reference to any equations; the program Alg also contains this more simple function.)

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