# On weakly monotonically monolithic spaces

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*Abstract.* In this note, we introduce the concept of weakly monotonically monolithic spaces, and show that every weakly monotonically monolithic space is a $D$-space. Thus most known conclusions on $D$-spaces can be obtained by this conclusion. As a corollary, we have that if a regular space $X$ is sequential and has a point-countable $wcs^*$-network then $X$ is a $D$-space.

*Keywords:* $D$-space, sequential space, $wcs^*$-network, weakly monotonically monolithic space

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## 1. Introduction

The notion of a $D$-space was first investigated by van Douwen and Pfeffer in [6]. A *neighborhood assignment* for a space $X$ is a function $\phi$ from $X$ to the topology of the space $X$ such that $x \in \phi(x)$ for any $x \in X$. A space $X$ is called a *$D$-space* if for any neighborhood assignment $\phi$ for $X$ there exists a closed discrete subspace $D$ of $X$ such that $X = \bigcup\{\phi(d) : d \in D\}$ (cf. [6]). By results of [3], we know that all semi-stratifiable spaces and all metrizable spaces are $D$-spaces. We also know that the union of a finite family of metrizable subspaces is a $D$-space and every space with a point-countable base is a $D$-space by results of [1] and [2], respectively.

In [5], it was proved that $C_p(X)$ is hereditarily a $D$-space if $X$ is compact. Some sufficiencies of $D$-spaces were discussed in [4], [8], [9], [16] and [19]. Let us recall that a space is called *monolithic* if $nw(\overline{A}) \leq \max\{|A|, \omega\}$ for any $A \subset X$. By results of [5] and [9], we know that many monolithic spaces have $D$-property. Thus V.V. Tkachuk introduced the concept of monotonically monolithic spaces in [19]. It was proved that every monotonically monolithic space is hereditarily a $D$-space (cf. [19]). Thus every Lindelöf $\Sigma$-space is hereditarily a $D$-space (cf. [9] and [19]).

In [4] and [15], it was proved that a space with a point-countable weak base is a $D$-space. In [16], Peng proved that a space with a point-countable $cs^*$-network is a $D$-space. In this paper, the idea of [19] and [16] is generalized and we introduce the concept of weakly monotonically monolithic spaces. It is proved that every weakly monotonically monolithic space is a $D$-space. Thus we have

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that if a regular space $X$ is sequential and has a point-countable $wcs^*$-network then $X$ is a $D$-space. In fact, most known results on $D$-spaces can be obtained by the conclusions of the paper.

All the spaces in this note are at least $T_1$-spaces. Let $\mathbb{N}$ be the set of all natural numbers and $\omega = \mathbb{N} \cup \{0\}$. In notation and terminology we will follow [7] and [10].

## 2. Main results

**Definition 1** (cf. [19])**.** Given a set $A$ in a space $X$ say that a family $\mathcal{N}$ of subsets of $X$ is an *external network* of $A$ in $X$ if for any $x \in A$ and $U$ the neighborhood of $x$ there exists $B \in \mathcal{N}$ such that $x \in B \subset U$. If $A = \{x\}$ for some $x \in X$ and $\mathcal{N}$ is an external network of $A$ in $X$, then we say that $\mathcal{N}$ is an *external network of* $x$.

**Definition 2** (cf. [19])**.** We say that a space $X$ is *monotonically monolithic* if for any $A \subset X$ we can assign an external network $\mathcal{O}(A)$ to the set $\overline{A}$ in such a way that the following conditions are satisfied:

(1) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if $\alpha$ is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of $X$ such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$ then $\mathcal{O}(\bigcup_{\beta<\alpha} A_\beta) = \bigcup_{\beta<\alpha} \mathcal{O}(A_\beta)$.

**Definition 3.** We say that a space $X$ is *weakly monotonically monolithic* if for any $A \subset X$ we can assign an external network $\mathcal{W}(A)$ of $A$ in such a way that the following conditions are satisfied:

(1) $|\mathcal{W}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$ then $\mathcal{W}(A) \subset \mathcal{W}(B)$;
(3) if $\alpha$ is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of $X$ such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$ then $\mathcal{W}(\bigcup_{\beta<\alpha} A_\beta) = \bigcup_{\beta<\alpha} \mathcal{W}(A_\beta)$;
(4) If $A \subset X$ is not closed in $X$ then there is some $x \in \overline{A} \setminus A$ such that $\mathcal{W}(A)$ is an external network of $x$.

Every monotonically monolithic space is a weakly monotonically monolithic space. We also know that if a monotonically monolithic space $X$ is separable then $X$ is hereditarily separable and hereditarily Lindelöf.

**Example 4.** Let $X = \mathbb{R}$, $\mathbb{Q} = \{x_n : n \in \mathbb{N}\}$ be the set of all rational numbers of the real set $\mathbb{R}$ and $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$.

For each $n \in \mathbb{N}$, let $\mathcal{B}(x_n) = \{\{x_n\}\}$ be a neighborhood base of the point $x_n$.

For each $x \in \mathbb{I}$, let $\mathcal{B}(x) = \{\{x\} \cup A : A \subset Q \text{ and } Q \setminus A \text{ is finite }\}$ be a neighborhood base of the point $x$. We denote the topology of $X$ by $\mathcal{T}$. Thus $(X, \mathcal{T})$ is a $T_1$-space and $\mathbb{I}$ is a closed discrete subspace of $X$. So $(X, \mathcal{T})$ is separable but it is not Lindelöf. Thus it is not monolithic.

Let $y_1 \in \mathbb{I}$. For each $B \subset X$, we let $\mathcal{W}(B) = \{\{x\} : x \in B\}$ if $B \subset \mathbb{I}$, otherwise $\mathcal{W}(B) = \{\{x\} \cup \{y_1\} : x \in B\} \bigcup \{\{x\} : x \in B\}$.

We can see that $\mathcal{W}$ witnesses weak monotonic monolithity of $X$. Thus $X$ is a weakly monotonically monolithic $T_1$-space but it is not a monotonically monolithic space.

Let us recall that a family $\mathcal{F}$ of subsets of $X$ is a *$cs^*$-network* of $X$, if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges to a point $x$ and for any open set $U$ which contains $x$, there is some $F \in \mathcal{F}$ such that $x \in F \subset U$ and $|\{n : x_n \in F\}| = \omega$ (cf. [13] and [14]). A space is called *sequential* if for any non-closed subset $A$ of $X$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in A$ for each $n \in \mathbb{N}$, such that $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in \overline{A} \setminus A$. We also know that every space with a point-countable weak base is sequential and has a point-countable $cs^*$-network.

**Lemma 5.** *If $X$ is a sequential space with a point-countable $cs^*$-network, then $X$ is a weakly monotonically monolithic space.*

Proof: Let $\mathcal{F}$ be a point-countable $cs^*$-network of $X$. For any $A \subset X$, we let $\mathcal{W}(A) = \{F : F \cap A \neq \emptyset \text{ and } F \in \mathcal{F}\}$. By the sequential property we see that $\mathcal{W}(A)$ satisfies the conditions which appear in the Definition 3. □

**Corollary 6.** *Let $X$ be a space. If $X$ has a point-countable weak base, then $X$ is a weakly monotonically monolithic space.*

**Example 7** ([11, Example 9.3). Let $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $Y = [0,1] \times S$. Let $Y' = [0,1] \times \{\frac{1}{n} : n \in \mathbb{N}\}$ have the usual Euclidean topology as a subspace of $[0,1] \times S$. Define a typical neighborhood $(t,0)$ in $Y$ to be the form $\{(t,0)\} \cup (\bigcup\{U(t,\frac{1}{k}) : k \geq n\})$, where $U(t,\frac{1}{k})$ is a neighborhood of $(t,\frac{1}{k})$ in $[0,1] \times \{\frac{1}{k}\}$.

In [11], it is pointed out that $Y$ is a completely regular separable space but it is not Lindelöf, and it is also pointed out that $Y$ is a two-to-one quotient image of the topological sum $M$ of compact metric spaces $\{[0,1] \times \{\frac{1}{n}\} : n \in \mathbb{N}\} \cup \{\{t\} \times S : t \in [0,1]\}$.

The space $Y$ has a point-countable weak base (this is pointed out in [12, p. 26]). So we know that the space $Y$ is not a monotonically monolithic space but $Y$ is a weakly monotonically monolithic regular space by Corollary 6.

In [12], Lin did not give a proof that the space $Y$ has a point-countable weak base. To assist the reader, we give a short proof.

Let $f : M \to Y$ be the two-to-one quotient map, where $M$ is the topological sum of compact metric spaces $\{[0,1] \times \{\frac{1}{n}\} : n \in \mathbb{N}\} \cup \{\{t\} \times S : t \in [0,1]\}$. For each $y \in Y$, we let $y = (a_1, a_2)$. Suppose $\{y_n\}_{n\in\mathbb{N}}$ is a sequence of $Y$ such that $\{y_n\}_{n\in\mathbb{N}}$ converges to the point $y$ and we assume that $y_n \neq y$ for each $n \in \mathbb{N}$.

(1) If $a_2 \neq 0$ then there is some $m \in \mathbb{N}$ such that $y_n \in [0,1] \times \{a_2\}$ for each $n > m$, since $Y$ is determined by the collection $\{[0,1] \times \{\frac{1}{n}\} : n \in \mathbb{N}\} \cup \{\{t\} \times S : t \in [0,1]\}$.
(2) If $a_2 = 0$ then there is some $m \in \mathbb{N}$ such that $y_n \in \{a_1\} \times S$ for each $n > m$, since $[0,1] \times \{0\}$ is a closed discrete subspace of $Y$ and $Y$ is determined by the collection $\{[0,1] \times \{\frac{1}{n}\} : n \in \mathbb{N}\} \cup \{\{t\} \times S : t \in [0,1]\}$.

Thus we have the following conclusion:

If $y \in Y$ then there is some point $x_y \in f^{-1}(y)$ such that whenever a sequence $\{y_n\}_{n\in\mathbb{N}}$ of $Y$ converges to the point $y$ and $y_n \neq y$ for each $n \in \mathbb{N}$ then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of $X$ such that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to the point $x_y$ and $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$. (This is pointed out in [12, p. 26]).

Let $\mathcal{B}$ be a point-countable base of $M$. For each $y \in Y$, let $\mathcal{B}'_y = \{B : x_y \in B$ and $B \in \mathcal{B}\}$ and let $\mathcal{B}^*_y = \{f(B) : B \in \mathcal{B}'_y\}$. We will show that $\mathcal{B}^* = \bigcup\{\mathcal{B}^*_y : y \in Y\}$ is a point-countable weak base of $Y$.

We only need to prove that $U$ is open if for any $y \in U$ there is some $B^* \in \mathcal{B}^*_y$ such that $y \in B^* \subset U$.

Suppose $U$ is not open. Then there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $\{y_n\}_{n\in\mathbb{N}}$ converges to a point $y \in U$ and $y_n \notin U$ for each $n \in \mathbb{N}$, since $Y$ is sequential. Thus there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of $M$ such that $x_n \in f^{-1}(y_n)$ and $\{x_n\}_{n\in\mathbb{N}}$ converges to $x_y$. There is some $B^* \in \mathcal{B}^*_y$ such that $y \in B^* \subset U$. We let $B^* = f(B)$ for some $B \in \mathcal{B}'_y$. Thus $x_y \in B$, and hence there is some $m \in \mathbb{N}$ such that $x_n \in B$ for each $n > m$. So $y_n = f(x_n) \in f(B) = B^* \subset U$. This contradicts $y_n \notin U$. Thus $\mathcal{B}^*$ is a weak base of $Y$. Since $\mathcal{B}$ is point-countable and $|f^{-1}(y)| = 2$ for each $y \in Y$, we know that $\mathcal{B}^*$ is point-countable in $Y$. Thus $Y$ has a point-countable weak base.

A family $\mathcal{P}$ of subsets of $X$ is called a *$wcs^*$-network* of $X$ if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges to a point $x$, and any open set $U$ which contains $x$, there is some $P \in \mathcal{P}$ such that $P \subset U$ and $|\{n : x_n \in P\}| = \omega$ (cf. [14]). We know that every $k$-network of $X$ is a $wcs^*$-network of $X$ and if $X$ is regular and $\mathcal{P}$ is a $wcs^*$-network of $X$ then $\{\overline{P} : P \in \mathcal{P}\}$ is a $cs^*$-network of $X$.

**Example 8** ([18, Example 78])**.** Let $\mathcal{T}$ be the usual Euclidean topology of $\mathbb{R}^2$. Let $S_1 = \{(x, y) : x, y \in \mathbb{R}, y > 0\}$, $L = \{(x, 0) : x \in \mathbb{R}\}$ and $X = S_1 \cup L$. Let $\mathcal{T}^* = \{\mathcal{T}|X\} \cup \{\{x\} \cup (S_1 \cap U) : x \in L, x \in U$ and $U \in \mathcal{T}\}$, where $\mathcal{T}|X = \{U \cap X : U \in \mathcal{T}\}$. The space $(X, \mathcal{T}^*)$ is a non-regular $T_2$-space.

In [12, p. 28], it is pointed out that the space $X$ which appears in Example 8 has a locally countable $k$-network but it has no point-countable $cs^*$-network. Thus it has a point-countable $wcs^*$-network but it has no point-countable $cs^*$-network.

**Lemma 9.** *If a regular space $X$ is sequential and has a point-countable $wcs^*$-network, then $X$ is a weakly monotonically monolithic space.*

Proof: Let $\mathcal{F}$ be a point-countable $wcs^*$-network of $X$. For any $A \subset X$, let $\mathcal{W}(A) = \{\overline{F} : F \cap A \neq \emptyset$ and $F \in \mathcal{F}\}$. We see that $\mathcal{W}(A)$ satisfies the conditions which appear in the Definition 3, since $X$ is regular and sequential. □

Similarly to Definition 2.11 from [19], we have:

**Definition 10.** Assume that $Y$ is a weakly monotonically monolithic space and fix an operator $\mathcal{W}$ which witnesses that. Let $\phi$ be any neighborhood assignment on $X$. For every $P \subset X$, we denote $P^* = \{x : x \in P$ *and* $P \subset \phi(x)\}$. For any open set $U \subset X$ say that a set $A \subset X$ is *$U$-saturated* if $P^* \subset \phi(A) \cup U$ for any $P \in \mathcal{W}(A)$, where $\phi(A) = \bigcup\{\phi(x) : x \in A\}$. If $U = \emptyset$ then $U$-saturated sets will be called *saturated*.

**Lemma 11.** *Let $\phi$ be any neighborhood assignment for $X$. If $X$ is a weakly monotonically monolithic space, $A \subset X$ is a countable closed discrete subspace of $X$ and $U$ is any open subset of $X$, then there is a closed discrete subspace $B \subset X \setminus U \cup \phi(A)$, such that $A \cup B$ is $U$-saturated and $|B| \leq \omega$.*

PROOF: $X$ is a weakly monotonically monolithic space. We fix an operator $\mathcal{W}$ which witnesses that. Let $\mathcal{F}_0^* = \{P^* : P \in \mathcal{W}(A)\}$. Thus $\mathcal{F}_0^*$ is a countable family. Enumerate it by prime numbers $p$. We take the first member $F^*$ of $\mathcal{F}_0^*$ such that $F^* \setminus U \cup \phi(A) \neq \emptyset$. We choose a point $x_1 \in F^* \setminus U \cup \phi(A)$. Then $F \subset \phi(x_1)$. The family $\mathcal{W}(A \cup \{x_1\})$ is countable. We denote $\mathcal{F}_1^* = \{F^* : F \in \mathcal{W}(A \cup \{x_1\}) \setminus \mathcal{W}(A)\}$. We enumerate $\mathcal{F}_1^*$ by the squares $p^2$ of prime numbers.

Suppose we have finished $n$ steps. We have $\phi(A)$, $U \cup \phi(A \cup \{x_1\})$, $\dots, U \cup \phi(A \cup \{x_1, \dots, x_n\})$, and families $\mathcal{F}_i^*$ of subsets of $X$, the family $\mathcal{F}_i^*$ is enumerated by the $i$-th powers of prime numbers for each $0 \leq i \leq n$.

If $U \cup \phi(A \cup \{x_1, \dots, x_n\}) = X$, then we stop the induction and let $B = \{x_i : 1 \leq i \leq n\}$. So we assume that $U \cup \phi(A \cup \{x_1, \dots, x_n\}) \neq X$. If $\bigcup\{\bigcup \mathcal{F}_i^* : 0 \leq i \leq n\}$ is contained in $U \cup \phi(A \cup \{x_1, \dots, x_n\})$ then we choose a point $x_{n+1} \in X \setminus U \cup \phi(A \cup \{x_1, \dots, x_n\})$. If $\bigcup\{\bigcup \mathcal{F}_i^* : 0 \leq i \leq n\} \setminus (U \cup \phi(A \cup \{x_1, \dots, x_n\})) \neq \emptyset$ then we take the first member $F^*$ of $\bigcup\{\mathcal{F}_i^* : 0 \leq i \leq n\}$ such that $F^* \setminus U \cup \phi(A \cup \{x_1, \dots, x_n\}) \neq \emptyset$ and choose a point $x_{n+1} \in F^* \setminus U \cup \phi(A \cup \{x_1, \dots, x_n\})$. Thus $F^* \subset F \subset \phi(x_{n+1})$. We let $\mathcal{F}_{n+1}^* = \{F^* : F \in \mathcal{W}(A \cup \{x_1, \dots, x_{n+1}\}) \setminus \mathcal{W}(A \cup \{x_1, \dots, x_n\})\}$ and enumerate it by the $(n+1)$-st powers of prime numbers.

In this way, we have a set $B = \{x_n : n \in \mathbb{N}\}$ and $B \subset X \setminus U \cup \phi(A)$. To prove that $B$ is a closed discrete subspace of $X$, we only need to prove that $B$ is closed since $x_m \notin \phi(x_n)$ whenever $m > n$.

Suppose $\overline{B} \setminus B \neq \emptyset$. Then there is some $x \in \overline{B} \setminus B$ such that $\mathcal{W}(B)$ is an external network of $x$. Thus there is some $P \in \mathcal{W}(B)$ such that $x \in P \subset \phi(x)$ and hence $x \in P^*$. Since $B = \bigcup\{\{x_1, \dots, x_n\} : n \in \mathbb{N}\} \subset \bigcup\{A \cup \{x_1, \dots, x_n\} : n \in \mathbb{N}\}$, we see that $P \in \mathcal{W}(\bigcup\{A \cup \{x_1, \dots, x_n\} : n \in \mathbb{N}\})$. So $P \in \mathcal{W}(A \cup \{x_1, \dots, x_n\})$ for some $n \in \mathbb{N}$. Thus $P^* \in \mathcal{F}_i^*$ for some $0 \leq i \leq n$. So there is some $m > n$ such that $P^* \subset U \cup \phi(A \cup \{x_1, \dots, x_m\})$. Thus $x \in \phi(A \cup \{x_1, \dots, x_m\})$. So $x \notin \overline{B}$. This contradicts the fact $x \in \overline{B}$. So $B$ is closed and hence it is a closed discrete subspace of $X$ and $B \subset X \setminus U \cup \phi(A)$, $|B| \leq \omega$.

For any $P \in \mathcal{W}(A \cup B)$, since $A \cup B = \bigcup\{A \cup \{x_1, \dots, x_n\} : n \in \mathbb{N}\}$, we have $P \in \mathcal{W}(A \cup \{x_1, \dots, x_n\})$ for some $n \in \mathbb{N}$. Hence $P^* \in \mathcal{F}_i^*$ for some $0 \leq i \leq n$. So $P^*$ will be covered by $U \cup \phi(A \cup \{x_1, \dots, x_m\})$ for some $m > n$. Thus $A \cup B$ is $U$-saturated. □

**Lemma 12.** *Let $\phi$ be any neighborhood assignment for $X$. If $X$ is a weakly monotonically monolithic space and $A \subset X$ is a closed discrete subspace of $X$ and $U$ is any open subset of $X$, then there is a closed discrete subspace $D \subset X \setminus U \cup \phi(A)$, such that $A \cup D$ is $U$-saturated and $|D| \leq \max\{|A|, \omega\}$.*

PROOF: We will use induction on the cardinal $\kappa = |A|$. If $|A| \leq \omega$, then it is true by Lemma 11. Now assume that $\kappa$ is an uncountable cardinal and our theorem is proved whenever $U$ an open subset of $X$ and $A \subset X$, $|A| < \kappa$.

Take an arbitrary set $A \subset X$ such that $|A| = \kappa$ and let $A = \{x_\alpha : \alpha < \kappa\}$. Let $A_\alpha = \{x_\beta : \beta < \alpha\}$ for each $\alpha \in [\omega, \kappa)$ and let $U' = U \cup \phi(A)$. Proceeding inductively assume that $\omega < \alpha < \kappa$ and we have a family $\{D_\beta : \omega \le \beta < \alpha\}$ of closed discrete subspaces of $X$ with the following properties:

(1) $A_\omega \cup D_\omega$ is $U'$-saturated;
(2) If $\omega < \beta < \alpha$ then $U_\beta = U' \cup \phi(\bigcup\{D_\gamma : \omega \le \gamma < \beta)$, $D_\beta \subset X \setminus U_\beta$ and $|D_\beta| \le |\beta|$;
(3) If $B_\beta = A_\beta \cup (\bigcup\{D_\gamma : \omega \le \gamma < \beta\})$ then $B_\beta \cup D_\beta$ is $U_\beta$-saturated whenever $\omega < \beta < \alpha$;
(4) The set $\bigcup\{D_\gamma : \omega \le \gamma < \beta\}$ is a closed discrete subspace of $X$ for each $\beta \in (\omega, \alpha)$.

We first prove that $\bigcup\{D_\beta : \omega \le \beta < \alpha\}$ is a closed discrete subspace of $X$. We only need to prove that $\bigcup\{D_\beta : \beta < \alpha\}$ is a closed subspace of $X$. If $\alpha = \gamma + 1$, then $\bigcup\{D_\beta : \omega \le \beta < \alpha\} = \bigcup\{D_\beta : \omega \le \beta < \gamma\} \cup D_\gamma$. The sets $D\gamma$ and $\bigcup\{D_\beta : \omega \le \beta < \gamma\}$ are closed discrete and hence $\bigcup\{D_\beta : \omega \le \beta < \alpha\}$ is closed discrete. Let $\alpha$ be a limit ordinal. Suppose $\bigcup\{D_\beta : \omega \le \beta < \alpha\}$ is not closed in $X$. Then there is some $x \in \overline{\bigcup\{D_\beta : \omega \le \beta < \alpha\}} \setminus \bigcup\{D_\beta : \omega \le \beta < \alpha\}$ such that $\mathcal{W}(\bigcup\{D_\beta : \omega \le \beta < \alpha\})$ is an external network of the point $x$. So there is some $P \in \mathcal{W}(\bigcup\{D_\beta : \omega \le \beta < \alpha\})$ such that $x \in P \subset \phi(x)$ and hence $x \in P^*$. The set $\bigcup\{D_\beta : \omega \le \beta < \alpha\} \subset \bigcup\{A_\beta \cup (\bigcup\{D_\gamma : \omega \le \gamma \le \beta\}) : \omega \le \beta < \alpha\}$, so $P \in \mathcal{W}(A_{\beta_1} \cup (\bigcup\{D_\gamma : \omega \le \gamma \le \beta_1\}))$ for some $\beta_1 \in [\omega, \alpha)$.

The set $A_{\beta_1} \cup (\bigcup\{D_\gamma : \omega \le \gamma \le \beta_1\})$ is $U_{\beta_1}$-saturated, so $P^* \subset U_{\beta_1} \cup \phi(D_{\beta_1})$ and hence $x \in U' \cup \phi(A_{\beta_1} \cup (\bigcup\{D_\gamma : \omega \le \gamma \le \beta_1\}))$. We have $x \in \phi(\bigcup\{D_\gamma : \omega \le \gamma \le \beta_1\})$, since $x \in \overline{\bigcup\{D_\beta : \omega \le \beta < \alpha\}} \setminus \bigcup\{D_\beta : \omega \le \beta < \alpha\}$ and $U' \cap (\bigcup\{D_\beta : \omega \le \beta < \alpha\}) = \emptyset$. Let $\gamma_x$ be the smallest ordinal such that $x \in \phi(D_{r_x})$. Since $x \in \overline{\bigcup\{D_\beta : \omega \le \beta < \alpha\}}$, we see that $\gamma_x \ne \omega$. The set $\bigcup\{D_\gamma : \omega \le \gamma < \gamma_x\}$ is a closed discrete subspace of $X$ by induction and $x \notin \phi(\bigcup\{D_\gamma : \omega \le \gamma < \gamma_x\})$. If $V_x = (X \setminus \bigcup\{D_\gamma : \omega \le \gamma < \gamma_x\}) \cap \phi(D_{\gamma_x}) \cap O_x$, where $x \in O_x$ and $|O_x \cap D_{\gamma_x}| \le 1$ for an open set $O_x$, then $|V_x \cap (\bigcup\{D_\beta : \omega \le \beta < \alpha\})| \le 1$. This contradicts $x \in \overline{\bigcup\{D_\beta : \omega \le \beta < \alpha\}} \setminus \bigcup\{D_\beta : \omega \le \beta < \alpha\}$. Thus $\bigcup\{D_\beta : \omega \le \beta < \alpha\}$ is a closed discrete subspace of $X$.

Let $U_\alpha = U' \cup \phi(A_\alpha \cup (\bigcup\{D_\beta : \omega \le \beta < \alpha\}))$, so $|A_\alpha \cup (\bigcup\{D_\beta : \omega \le \beta < \alpha\})| \le |\alpha| < \kappa$ and $A_\alpha \cup (\bigcup\{D_\beta : \omega \le \beta < \alpha\}) \subset U_\alpha$. Thus by induction, there exists a closed discrete subspace $D_\alpha \subset X \setminus U_\alpha$ such that $(A_\alpha \cup (\bigcup\{D_\beta : \omega \le \beta < \alpha\})) \cup D_\alpha$ is $U_\alpha$-saturated.

If $D = \bigcup\{D_\alpha : \omega \le \alpha < \kappa\}$ then $D \subset X \setminus U \cup \phi(A)$ and $|D| \le \kappa$. In what follows, we show that $D$ is a closed discrete subspace of $X$.

For any $\beta \in (\omega, \kappa)$, the set $\bigcup\{D_\alpha : \omega \le \alpha < \beta\}$ is closed and $\phi(D_\beta) \cap D_\gamma = \emptyset$ for any $\gamma \in (\beta, \kappa)$. Thus to prove that $D$ is a closed discrete subspace of $X$ we only need to prove that $D$ is closed in $X$. Suppose $D$ is not closed; then there is some $x \in \overline{D} \setminus D$ such that $\mathcal{W}(D)$ is an external network of $x$. Since $\bigcup\{D\gamma : \omega \le \gamma < \beta\}$ is closed discrete for each $\beta < \kappa$, we can see that $x \notin \phi(D)$. There is some $P \in \mathcal{W}(D)$ such that $x \in P \subset \phi(x)$ and hence $x \in P^*$. The set

$D \subset \bigcup\{A_\alpha \cup (\bigcup\{D_\beta : \omega \leq \beta \leq \alpha\}) : \omega \leq \alpha < \kappa\}$, so there exists some $\gamma < \kappa$ such that $P \in \mathcal{W}(A_\gamma \cup (\bigcup\{D_\beta : \omega \leq \beta \leq \gamma\}))$. Let $U_\gamma = U' \cup \phi(A_\gamma \cup (\bigcup\{D_\beta : \omega \leq \beta < \gamma\}))$, we know that $A_\gamma \cup (\bigcup\{D_\beta : \omega \leq \beta < \gamma\}) \cup D_\gamma$ is $U_\gamma$-saturated. So $P^* \subset U' \cup \phi(A_\gamma \cup (\bigcup\{D_\beta : \omega \leq \beta \leq \gamma\}))$. Thus $x \in \phi(D)$. This contradicts the fact $x \notin \phi(D)$. So $D$ is a closed discrete subspace of $X$ and $D \subset X \setminus U'$.

Consider the set $A \cup D = \bigcup\{A_\alpha \cup (\bigcup\{D_\beta : \omega \leq \beta \leq \alpha\}) : \omega \leq \alpha < \kappa\}$. For any $P \in \mathcal{W}(A \cup D)$, there exists some $\alpha < \kappa$ such that $P \in \mathcal{W}(A_\alpha \cup (\bigcup\{D_\beta : \omega \leq \beta \leq \alpha\}))$. The set $A_\alpha \cup (\bigcup\{D_\beta : \omega \leq \beta \leq \alpha\})$ is $U_\alpha$-saturated. Thus $P^* \subset U' \cup \phi(A_\alpha \cup (\bigcup\{D_\beta : \omega \leq \beta \leq \alpha\})) \subset U \cup \phi(A \cup D)$. Thus $A \cup D$ is $U$-saturated and $|D| \leq \max\{|A|, \omega\}$. □

**Corollary 13.** *Let $X$ be a space and $\phi$ be a neighborhood assignment for $X$. If $X$ is a weakly monotonically monolithic space and $A$ is a closed discrete subspace of $X$, then there is a closed discrete subspace $D \subset X \setminus \phi(A)$ such that $A \cup D$ is saturated.*

**Theorem 14.** *If $X$ is a weakly monotonically monolithic space, then $X$ is a $D$-space.*

Proof: Let $|X| = \kappa$ and $X = \{x_\alpha : \alpha \in \kappa\}$. Let $\phi$ be any neighborhood assignment for $X$ and $D_0 = \{x_0\}$. Then by Corollary 13 there is a closed discrete subspace $D'_1 \subset X \setminus \phi(D_0)$ such that $D_0 \cup D'_1$ is saturated. Let $D_1 = D_0 \cup D'_1$ if $x_1 \in \phi(D_0 \cup D'_1)$, otherwise $D_1 = \{x_1\} \cup D_0 \cup D'_1$.

Let $0 < \alpha < \kappa$ and assume we have closed discrete subspaces $D_\beta$ and $D'_\beta$ for each $\beta < \alpha$ such that:

(1) $D'_0 = \emptyset$;
(2) $D_{\beta_1} \subset D_{\beta_2}$ if $\beta_1 < \beta_2 < \beta$;
(3) $x_\beta \in \phi(D_\beta)$;
(4) $\bigcup\{D_\gamma : \gamma < \beta\}$ is closed discrete and $D'_\beta \subset X \setminus \phi(\bigcup\{D_\gamma : \gamma < \beta\})$ is such that $\bigcup\{D_\gamma : \gamma < \beta\} \cup D'_\beta$ is saturated;
(5) $D_\beta = \bigcup\{D_\gamma : \gamma < \beta\} \cup D'_\beta$ if $x_\beta \in \phi(\bigcup\{D_\gamma : \gamma < \beta\} \cup D'_\beta)$, otherwise $D_\beta = \bigcup\{D_\gamma : \gamma < \beta\} \cup D'_\beta \cup \{x_\beta\}$.

In what follows, we will show that $\bigcup\{D_\beta : \beta < \alpha\}$ is a closed discrete subspace of $X$.

If $\alpha = \gamma + 1$ for some $\gamma$ then $\bigcup\{D_\beta : \beta < \alpha\} = \bigcup\{D_\beta : \beta < \gamma\} \cup D_\gamma$ is a closed discrete subspace of $X$. If $\alpha$ is a limit ordinal and $\bigcup\{D_\beta : \beta < \alpha\}$ is not closed, then there is some $x \in \overline{\bigcup\{D_\beta : \beta < \alpha\}} \setminus \bigcup\{D_\beta : \beta < \alpha\}$ such that $\mathcal{W}(\bigcup\{D_\beta : \beta < \alpha\})$ is an external network of the point $x$. So there is some $P \in \mathcal{W}(\bigcup\{D_\beta : \beta < \alpha\})$ such that $x \in P \subset \phi(x)$ and hence $x \in P^*$. We know that $\mathcal{W}(\bigcup\{D_\beta : \beta < \alpha\}) = \bigcup\{\mathcal{W}(D_\beta) : \beta < \alpha\}$. So $P \in \mathcal{W}(D_{\beta_1})$ for some $\beta_1 < \alpha$. The set $D_{\beta_1} \cup D'_{\beta_1+1}$ is saturated. Thus $P^* \subset \phi(D_{\beta_1} \cup D'_{\beta_1+1})$ and hence $x \in \phi(D_{\beta_1+1})$. Thus $\phi(D_{\beta_1+1}) \cap D'_\gamma = \emptyset$ for any $\gamma \in (\beta_1 + 1, \kappa)$. The set $D_{\beta_1+1}$ is a closed discrete subspace of $X$, so $x \notin \overline{\bigcup\{D_\beta : \beta < \alpha\}}$. This contradicts $x \in \overline{\bigcup\{D_\beta : \beta < \alpha\}}$. So $\bigcup\{D_\beta : \beta < \alpha\}$ is a closed discrete subspace of $X$.

Thus there is a closed discrete subspace $D'_\alpha \subset X \setminus \phi(\bigcup\{D_\beta : \beta < \alpha\})$ such that $\bigcup\{D_\beta : \beta < \alpha\} \cup D'_\alpha$ is saturated by Corollary 13. Let $D_\alpha = \bigcup\{D_\beta : \beta < \alpha\} \cup D'_\alpha$ if $x_\alpha \in \phi(\bigcup\{D_\beta : \beta < \alpha\} \cup D'_\alpha)$, otherwise $D_\alpha = \bigcup\{D_\beta : \beta < \alpha\} \cup D'_\alpha \cup \{x_\alpha\}$.

Let $D = \bigcup\{D_\alpha : \alpha < \kappa\}$. We see that $X = \bigcup\{\phi(d) : d \in D\}$. For each $x \in X$, let $\alpha_x = \min\{\alpha : x \in \phi(D_\alpha)\}$. Thus $\phi(D_{\alpha_x}) \cap D'_\beta = \emptyset$ for each $\beta \in (\alpha_x, \kappa)$. Let $O_x$ be an open set of $x$ such that $|O_x \cap D_{\alpha_x}| \leq 1$. Thus $|(O_x \cap \phi(D_{\alpha_x})) \cap D| \leq 1$. Thus $D$ is a closed discrete subspace of $X$. So $X$ is a $D$-space. □

By Lemma 9 and Theorem 14, we have:

**Corollary 15.** *If a regular space $X$ is sequential and has a point-countable wcs$^*$-network, then $X$ is a $D$-space.*

**Theorem 16.** *Let $X$ be a space and $\mathcal{F}(x)$ be a countable family of subsets of $X$ for each $x \in X$. If for any non-closed subset $A \subset X$ there exists some point $x \in \overline{A} \setminus A$ such that for every open neighborhood $U$ of $x$ there exists some $y \in A$ and some $F \in \mathcal{F}(y)$ such that $x \in F \subset U$, then $X$ is a $D$-space.*

Proof: For any $A \subset X$, we let $\mathcal{W}(A) = \bigcup\{\mathcal{F}(a) : a \in A\}$. We can see that $\mathcal{W}$ witnesses weak monotonic monolithity of $X$. Thus $X$ is a weakly monotonically monolithic space and hence $X$ is a $D$-space by Theorem 14. □

Recall that a space $X$ satisfies *open* $(G)$ if each point $x \in X$ has a countable neighborhood base $\mathcal{B}_x$ such that whenever $x \in \overline{A}$ and $N(x)$ is a neighborhood of $x$, then there is an $a \in A$ and $B \in \mathcal{B}_a$ for which $x \in B \subset N(x)$.

By Theorem 16, we have:

**Corollary 17** (cf. [9])**.** *Any space satisfying open $(G)$ is a $D$-space.*

**Corollary 18** (cf. [16])**.** *Let a space $X$ have a point-countable family $\mathcal{F}$ of subsets of $X$, such that for any non-closed subset $A \subset X$ there exists some point $x \in \overline{A} \setminus A$ such that for every open neighborhood $U$ of $x$ there exists some $F \in \mathcal{F}$ with $x \in F \subset U$ and $F \cap A \neq \emptyset$. Then $X$ is a $D$-space.*

If $X$ is a sequential space and $x \in W \subset X$ we say that $W$ is a *weak-neighborhood* of $x$ if whenever a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ then $\{x_n\}_{n\in\mathbb{N}}$ is eventually in $W$. A collection $\mathcal{W}$ of subsets of a sequential space $X$ is said to be a $\mathcal{W}$-system for the topology if whenever $x \in U \subset X$, with $U$ open, there exists a subcollection $\mathcal{V} \subset \mathcal{W}$ such that $x \in \bigcap \mathcal{V}$, $\bigcup \mathcal{V}$ is a weak-neighborhood of $x$ and $\bigcup \mathcal{V} \subset U$ (cf. [4]).

**Corollary 19** (cf. [4])**.** *A sequential space with a point-countable $\mathcal{W}$-system is a $D$-space.*

**Corollary 20** (cf. [4] and [15])**.** *If $X$ has a point-countable weak base, then $X$ is a $D$-space.*

By Lemma 5 and Theorem 14, we have:

**Corollary 21** (cf. [16])**.** *If $X$ is a sequential space with a point-countable cs$^*$-network, then $X$ is a $D$-space.*

Every $k$-network of $X$ is a $wcs^*$-network of $X$, so by Lemma 9 and Theorem 14, we have:

**Corollary 22** (cf. [17]). *If a regular space $X$ is sequential and has a point-countable $k$-network, then $X$ is a $D$-space.*

Every monotonically monolithic space is a weakly monotonically monolithic space and every subspace of monotonically monolithic space is monotonically monolithic. So we have:

**Corollary 23** (cf. [19]). *If $X$ is a monotonically monolithic space, then $X$ is hereditarily a $D$-space.*

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