Enveloping algebras of Malcev algebras

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Abstract. We first discuss the construction by Pérez-Izquierdo and Shestakov of universal nonassociative enveloping algebras of Malcev algebras. We then describe recent results on explicit structure constants for the universal enveloping algebras (both nonassociative and alternative) of the 4-dimensional solvable Malcev algebra and the 5-dimensional nilpotent Malcev algebra. We include a proof (due to Shestakov) that the universal alternative enveloping algebra of the real 7-dimensional simple Malcev algebra is isomorphic to the 8-dimensional division algebra of real octonions. We conclude with some brief remarks on tangent algebras of analytic Bol loops and monoassociative loops.

Keywords: Malcev algebras, universal enveloping algebras, universal alternative envelopes, differential operators, Bol algebras, analytic loops

Classification: Primary 17D10; Secondary 17B35, 17D05, 17D15, 20N05

1. Introduction

Moufang-Lie algebras were introduced by Malcev [12] as the tangent algebras of analytic Moufang loops. These structures were given the name Malcev algebras by Sagle [19]. Thus Malcev algebras are related to alternative algebras in the same way that Lie algebras are related to associative algebras.

In 2004, Pérez-Izquierdo and Shestakov [18] extended the famous PBW theorem (Poincaré-Birkhoff-Witt) from Lie algebras to Malcev algebras. For any Malcev algebra M, over a field of characteristic $\neq 2, 3$, they constructed a universal nonassociative enveloping algebra $U(M)$, which shares many properties of the universal associative enveloping algebras of Lie algebras: $U(M)$ is linearly isomorphic to the polynomial algebra $P(M)$ and has a natural (nonassociative) Hopf algebra structure. We will begin by describing the construction of $U(M)$. We will then show how this construction can be used to compute explicit structure constants for enveloping algebras of low-dimensional Malcev algebras. This involves differential operators on $P(M)$, which is the associated graded algebra of $U(M)$, and derivations defined by two elements of $N_{\mathrm{alt}}(M)$, the generalized alternative nucleus of $U(M)$.

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Since in general $U(M)$ is not alternative, it is interesting to calculate its alternator ideal $I(M)$ and its maximal alternative quotient $A(M) = U(M)/I(M)$, which is the universal alternative enveloping algebra of M. This produces new examples of infinite dimensional alternative algebras.

This program has been carried out for the 4-dimensional solvable Malcev algebra and for the 5-dimensional nilpotent Malcev algebra. It is work in progress for the one-parameter family of 5-dimensional solvable (non-nilpotent) Malcev algebras, and the 5-dimensional non-solvable Malcev algebra. The last algebra is especially interesting since it is the split extension of the simple 3-dimensional Lie algebra by its unique irreducible non-Lie module.

The ultimate goal of this research program is to calculate the structure constants for $U(\mathbb{M})$ where $\mathbb{M}$ is the 7-dimensional simple Malcev algebra. It is known that in this case the universal alternative enveloping algebra is just the octonion algebra; see Section 6 for a proof.

2. Lie algebras and Malcev algebras

Recall that a Lie algebra is a vector space L over a field F with a bilinear product $[a, b]$ satisfying anticommutativity and the Jacobi identity for all $a, b, c \in L$:

$$[a, a] = 0, \qquad J(a, b, c) = 0, \qquad J(a, b, c) = [[a, b], c] + [[b, c], a] + [[c, a], b].$$

If A is an associative algebra over F then we obtain a Lie algebra A^- by keeping the same underlying vector space but replacing the associative product ab by the commutator $[a, b] = ab - ba$. Conversely, the PBW theorem implies that every Lie algebra L has a universal associative enveloping algebra $U(L)$ for which the map $L \to U(L)$ is injective, so that L is isomorphic to a subalgebra of $U(L)^-$. If $\operatorname{char} F = 0$ then the image of L in $U(L)$ consists exactly of the elements of $U(L)$ which are primitive (in the Hopf algebra sense) with respect to the coproduct $\Delta\colon U(L) \to U(L) \otimes U(L)$.

A Malcev algebra is a vector space M over a field F with a bilinear product $[a, b]$ satisfying anticommutativity and the Malcev identity for all $a, b, c \in M$:

$$[a, a] = 0, \qquad [J(a, b, c), a] = J(a, b, [a, c]).$$

It is clear that every Lie algebra is a Malcev algebra.

Associative algebras are defined by the identity $(a, b, c) = 0$ where $(a, b, c) = (ab)c - a(bc)$ is the associator. We can weaken this condition by requiring only that the associator should be an alternating function of its arguments. This gives the left and right alternative identities, defining the variety of alternative algebras: $(a, a, b) = 0$ and $(b, a, a) = 0$. If A is an alternative algebra over F then we obtain a Malcev algebra A^- by keeping the same underlying vector space but replacing the alternative product ab by the commutator $[a, b] = ab - ba$. It is an open problem whether every Malcev algebra M has a universal alternative enveloping algebra $A(M)$ for which the map $M \to A(M)$ is injective. In other words, it is not known whether every Malcev algebra is special: that is, isomorphic to a subalgebra of A^- for some alternative algebra A.

A solution to a closely related problem was given in 2004 by Pérez-Izquierdo and Shestakov [18]: they constructed universal nonassociative enveloping algebras for Malcev algebras. This seems to be the natural generalization of the PBW theorem to Malcev algebras, since the universal nonassociative enveloping algebra $U(M)$ of a Malcev algebra M has a natural (nonassociative) Hopf algebra structure and an associated graded algebra which is a (commutative associative) polynomial algebra.

3. Theorem of Pérez-Izquierdo and Shestakov

For a nonassociative algebra A we define the generalized alternative nucleus by

$$N_{\mathrm{alt}}(A) = \big\{\, a \in A \mid (a,b,c) = -(b,a,c) = (b,c,a),\ \forall\, b,c \in A \,\big\}.$$

In general $N_{\mathrm{alt}}(A)$ is not a subalgebra of A but it is closed under the commutator (it is a subalgebra of A^-) and is a Malcev algebra.

Theorem 1 (Pérez-Izquierdo and Shestakov [18]). *For every Malcev algebra M over a field F of characteristic $\neq 2,3$ there exists a nonassociative algebra $U(M)$ and an injective algebra morphism $\iota\colon M \to U(M)^-$ such that $\iota(M) \subseteq N_{\mathrm{alt}}(U(M))$; furthermore, $U(M)$ is a universal object with respect to such morphisms.*

In general $U(M)$ is not alternative but it has a monomial basis of PBW type; over a field of characteristic 0, the elements $\iota(M)$ are exactly the primitive elements of $U(M)$ with respect to the coproduct $\Delta\colon U(M) \to U(M) \otimes U(M)$.

Let $F(M)$ be the unital free nonassociative algebra on a basis of M. Let $R(M)$ be the ideal of $F(M)$ generated by the elements

$$ab - ba - [a,b], \qquad (a,x,y) + (x,a,y), \qquad (x,a,y) + (x,y,a),$$

for all $a,b \in M$ and all $x,y \in F(M)$. Define $U(M) = F(M)/R(M)$ with the natural mapping

$$\iota\colon M \to N_{\mathrm{alt}}(U(M)) \subseteq U(M), \qquad a \mapsto \iota(a) = \overline{a} = a + R(M).$$

Since ι is injective, we can identify M with $\iota(M) \subseteq U(M)$. We fix a basis $B = \{a_i \,|\, i \in \mathcal{I}\}$ of M and a total order $<$ on $\mathcal{I}$. Define

$$\Omega = \{\, (i_1,\ldots,i_n) \mid n \geq 0,\ i_1 \leq \cdots \leq i_n \,\}.$$

For $n = 0$ we have $\overline{a}_\emptyset = 1 \in U(M)$, and for $n \geq 1$ the n-tuple $(i_1,\ldots,i_n) \in \Omega$ defines a left-tapped monomial

$$\overline{a}_I = \overline{a}_{i_1}(\overline{a}_{i_2}(\cdots(\overline{a}_{i_{n-1}}\overline{a}_{i_n})\cdots)), \qquad |\overline{a}_I| = n.$$

In Pérez-Izquierdo and Shestakov [18] it is shown that the set of all $\overline{a}_I$ for $I \in \Omega$ is a basis of $U(M)$.

We write $P(M)$ for the polynomial algebra on the vector space M. It follows that there is a linear isomorphism

$$\phi\colon U(M) \to P(M), \qquad \overline{a}_{i_1}(\overline{a}_{i_2}(\cdots(\overline{a}_{i_{n-1}}\overline{a}_{i_n})\cdots)) \mapsto a_{i_1}a_{i_2}\cdots a_{i_{n-1}}a_{i_n}.$$

We now define linear mappings

$$\rho\colon U(M) \to \mathrm{End}_F P(M), \qquad \lambda\colon U(M) \to \mathrm{End}_F P(M).$$

For $x \in U(M)$ we write $\rho(x)$, respectively $\lambda(x)$, for the linear operator on $P(M)$ induced by the right bracket $y \mapsto [y,x]$ in $U(M)$, respectively the left multiplication $y \mapsto xy$ in $U(M)$:

$$\rho(x)(f) = \phi\big([\phi^{-1}(f), x]\big), \qquad \lambda(x)(f) = \phi\big(x\phi^{-1}(f)\big).$$

We use the linear operators $\rho(x)$ and $\lambda(x)$ to express commutation and multiplication in $U(M)$ in terms of differential operators on the polynomial algebra $P(M)$.

Since $M \subseteq N_{\mathrm{alt}}(U(M))$, for any $a, b \in M$ and $x \in U(M)$ we have

$$(a,b,x) = \tfrac{1}{6}[[x,a],b] - \tfrac{1}{6}[[x,b],a] - \tfrac{1}{6}[x,[a,b]].$$

From this follow the next three lemmas, which are implicit in Pérez-Izquierdo and Shestakov [18].

Lemma 1. *Let x be a basis monomial of $U(M)$ with $|x| \geq 2$ and write $x = by$ with $b \in M$. Then for any $a \in M$ we have*

$$[x,a] = [by,a] = [b,a]y + b[y,a] + \tfrac{1}{2}[[y,a],b] - \tfrac{1}{2}[[y,b],a] - \tfrac{1}{2}[y,[a,b]].$$

Lemma 2. *Let x be a basis monomial of $U(M)$ with $|x| \geq 2$ and write $x = by$ with $b \in M$. Then for any $a \in M$ we have*

$$ax = a(by) = b(ay) + [a,b]y - \tfrac{1}{3}[[y,a],b] + \tfrac{1}{3}[[y,b],a] + \tfrac{1}{3}[y,[a,b]].$$

Lemma 3. *Let y and z be basis monomials of $U(M)$ with $|y| \geq 2$. Write $y = ax$ with $a \in M$. Then*

$$yz = (ax)z = 2a(xz) - x(az) - x[z,a] + [xz,a].$$

We can use a result of Morandi, Pérez-Izquierdo and Pumplün [15] to express these lemmas in terms of derivations. Let A be a nonassociative algebra, and let $a, b \in N_{\mathrm{alt}}(A)$ be any two elements of the generalized alternative nucleus. As usual we define the following operators on A:

$$L_a(x) = ax, \qquad R_a(x) = xa, \qquad \mathrm{ad}_a(x) = [a,x].$$

Then the following operator is a derivation of A:

$$D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b] = \mathrm{ad}_{[a,b]} - 3[L_a, R_b] = \tfrac{1}{2}\mathrm{ad}_{[a,b]} + \tfrac{1}{2}[\mathrm{ad}_a, \mathrm{ad}_b].$$

To apply this to the universal enveloping algebra of a Malcev algebra M, we take $A = U(M)$ and $a, b \in M \subseteq N_{\mathrm{alt}}(A)$. It is clear that $D_{a,a} = 0$ for all a, and that $D_{b,a} = -D_{a,b}$ for all a, b. For any $a, b \in M$ and any $x \in U(M)$ we have

$$6(b, x, a) = -2D_{a,b}(x) - 2[x, [a, b]].$$

Using this, we can rewrite the first two lemmas as follows:

$$\begin{aligned} [x, a] &= [b, a]y + b[y, a] - D_{a,b}(y) - [y, [a, b]], \\ ax &= b(ay) + [a, b]y + \tfrac{2}{3}D_{a,b}(y) + \tfrac{2}{3}[y, [a, b]]. \end{aligned}$$

These equations allow us to inductively build up the structure constants for the universal nonassociative enveloping algebra $U(M)$.

4. The 4-dimensional Malcev algebra

The first application of the work of Pérez-Izquierdo and Shestakov to the computation of explicit structure constants for $U(M)$ was done by Bremner, Hentzel, Peresi and Usefi [3] in the case of the 4-dimensional Malcev algebra. We write $\mathbb{S}$ for this algebra; it is the smallest non-Lie Malcev algebra, and it is solvable. It has basis $\{a, b, c, d\}$ with structure constants in Table 1.

$[-,-]$	a	b	c	d
a	0	$-b$	$-c$	d
b	b	0	$2d$	0
c	c	$-2d$	0	0
d	$-d$	0	0	0

Table 1. The 4-dimensional Malcev algebra $\mathbb{S}$

In $\mathbb{S}$ the sets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$ span 2-dimensional solvable Lie subalgebras, and the set $\{b, c, d\}$ spans a 3-dimensional nilpotent Lie subalgebra. Therefore these sets generate associative subalgebras of $U(\mathbb{S})$ with the following structure constants.

Lemma 4. *For $e \in \{b, c\}$ these equations hold in $U(\mathbb{S})$:*

$$(a^i e^j)(a^k e^\ell) = a^i (a+j)^k e^{j+\ell}, \quad (a^i d^j)(a^k d^\ell) = a^i (a-j)^k d^{j+\ell}.$$

Lemma 5. *This equation holds in $U(\mathbb{S})$:*

$$(b^i c^j d^k)(b^\ell c^m d^n) = \sum_{h=0}^{\ell} (-1)^h 2^h \binom{\ell}{h} \frac{j!}{(j-h)!} b^{i+\ell-h} c^{j+m-h} d^{k+n+h}.$$

We define the following operators on the polynomial algebra $P(\mathbb{S})$:

- I is the identity;
- M_x is multiplication by $x \in \{a,b,c,d\}$;
- D_x is differentiation with respect to $x \in \{a,b,c,d\}$;
- S is the shift $a \mapsto a{+}1$: $S(a^i b^j c^k d^\ell) = (a{+}1)^i b^j c^k d^\ell$.

Lemma 6. *For $x, y \in \{a,b,c,d\}$ we have*

$$\begin{aligned} &[D_x, M_x] = I, && [D_x, M_y] = 0 \ (x \neq y), && [D_x, D_y] = [M_x, M_y] = 0, \\ &[M_a, S] = -S, && [M_x, S] = 0 \ (x \neq a), && [D_x, S] = [D_x, S^{-1}] = 0, \\ &[M_a, S^{-1}] = S^{-1}, && [M_x, S^{-1}] = 0 \ (x \neq a). && \end{aligned}$$

We can now determine $\rho(x)$ and $\lambda(x)$ for $x \in \{a,b,c,d\}$ as differential operators on the polynomial algebra $P(\mathbb{S})$; see Table 2. The following lemma gives an example of our inductive proof techniques.

x	$\rho(x)$	$\lambda(x)$
a	$M_b D_b + M_c D_c - M_d D_d - 3 M_d D_b D_c$	M_a
b	$(I-S)M_b + \left(S-I-2S^{-1}\right) M_d D_c$	$S M_b + \left(S^{-1}-S\right) M_d D_c$
c	$(I-S)M_c + \left(S-I+2S^{-1}\right) M_d D_b$	$S M_c - \left(S^{-1}+S\right) M_d D_b$
d	$\left(I-S^{-1}\right) M_d$	$S^{-1} M_d$

Table 2. Differential operators $\lambda(x)$ and $\rho(x)$ for $\mathbb{S}$

Lemma 7. *We have $[b^n c^p d^q, a] = (n{+}p{-}q) b^n c^p d^q - 3np b^{n-1} c^{p-1} d^{q+1}$.*

Proof: By induction on n. The basis $n = 0$ is $[c^p d^q, a] = (p{-}q) c^p d^q$, since a, c, d span a Lie subalgebra of $\mathbb{M}$. Now let $n \geq 0$ and use the right-bracket lemma:

$$\begin{aligned} &[b^{n+1} c^p d^q, a] = \\ &[b,a] b^n c^p d^q + b[b^n c^p d^q, a] + \tfrac{1}{2}\big([[b^n c^p d^q, a], b] - [[b^n c^p d^q, b], a] - [b^n c^p d^q, [a,b]]\big). \end{aligned}$$

We use $[a,b] = -b$ and then apply the structure constants for the nilpotent Lie subalgebra spanned by b, c, d:

$$\begin{aligned} &[b^{n+1} c^p d^q, a] = \\ &b^{n+1} c^p d^q + b[b^n c^p d^q, a] + \tfrac{1}{2}[[b^n c^p d^q, a], b] + p[b^n c^{p-1} d^{q+1}, a] - p b^n c^{p-1} d^{q+1}. \end{aligned}$$

We now apply the inductive hypothesis to obtain

$$\begin{aligned} &[b^{n+1} c^p d^q, a] = b^{n+1} c^p d^q + (n{+}p{-}q) b^{n+1} c^p d^q - 3np b^n c^{p-1} d^{q+1} \\ &+ \tfrac{1}{2}(n{+}p{-}q)[b^n c^p d^q, b] - \tfrac{3}{2} np [b^{n-1} c^{p-1} d^{q+1}, b] + (n{+}p{-}q{-}2) p b^n c^{p-1} d^{q+1} \\ &- 3np(p{-}1) b^{n-1} c^{p-2} d^{q+2} - p b^n c^{p-1} d^{q+1}. \end{aligned}$$

We use the nilpotent Lie subalgebra again to get

$$
\begin{aligned}
&[b^{n+1}c^pd^q, a] = b^{n+1}c^pd^q + (n{+}p{-}q)b^{n+1}c^pd^q - 3np\, b^nc^{p-1}d^{q+1} \\
&- (n{+}p{-}q)pb^nc^{p-1}d^{q+1} + 3np(p{-}1)b^{n-1}c^{p-2}d^{q+2} + (n{+}p{-}q{-}2)pb^nc^{p-1}d^{q+1} \\
&- 3np(p{-}1)b^{n-1}c^{p-2}d^{q+2} - pb^nc^{p-1}d^{q+1}.
\end{aligned}
$$

Combining terms now gives

$$
[b^{n+1}c^pd^q, a] = (n{+}1{+}p{-}q)b^{n+1}c^pd^q - 3(n{+}1)pb^nc^{p-1}d^{q+1},
$$

and this completes the proof. □

It is not hard to show that $[a^mb^nc^pd^q, a] = a^m[b^nc^pd^q, a]$; combining this with the last result gives the formula for $\rho(a)$ in Table 2.

Our next task is to determine $\lambda(x)$ for $x = a^ib^jc^kd^\ell$ as a differential operator on $P(\mathbb{S})$. The proofs use the fact that if linear operators E and F satisfy $[[E,F],F] = 0$ then $[E, F^k] = k[E,F]F^{k-1}$ for $k \geq 1$. Since c, d span an abelian Lie subalgebra $\mathbb{A} \subset \mathbb{S}$, associativity gives $\lambda(c^kd^\ell) = \lambda(c)^k\lambda(d)^\ell$ on $U(\mathbb{A})$; this also holds on $U(\mathbb{S})$.

Lemma 8. *In $U(\mathbb{S})$ we have $\lambda(c^kd^\ell) = \lambda(c)^k\lambda(d)^\ell$.*

Since b, c, d span a nilpotent Lie subalgebra $\mathbb{N} \subset \mathbb{S}$, associativity gives $\lambda(b^jc^kd^\ell) = \lambda(b)^j\lambda(c)^k\lambda(d)^\ell$ on $U(\mathbb{N})$; this fails on $U(\mathbb{S})$.

Lemma 9. *In $U(\mathbb{S})$ the operator $\lambda(b^jc^kd^\ell)$ equals*

$$
\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta}\alpha! \binom{\alpha}{\beta}\binom{j}{\alpha}\binom{k}{\alpha} S^{-\beta}\lambda(b)^{j-\alpha}\lambda(c)^{k-\alpha}M_d^\alpha\lambda(d)^\ell.
$$

The inductive proof of this result is similar to, but much more complicated than, the example given above.

Multiplication by the general basis monomial $a^ib^jc^kd^\ell$ is given by the following formula.

Lemma 10. *In $U(\mathbb{S})$ the operator $\lambda(a^ib^jc^kd^\ell)$ equals*

$$
\begin{gathered}
\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{i}\sum_{\delta=0}^{i-\gamma}\sum_{\epsilon=0}^{i-\gamma-\delta} (-1)^{i+\alpha-\beta-\gamma-\delta}\alpha!\delta!\epsilon! \binom{\alpha}{\beta}\binom{j}{\alpha,\epsilon}\binom{k}{\alpha,\delta} \times \\
X_i(\gamma,\delta,\epsilon)\lambda(a)^\gamma S^{-\beta-\delta-\epsilon}\lambda(b)^{j-\alpha-\epsilon}D_b^\delta D_c^\epsilon\lambda(c)^{k-\alpha-\delta}M_d^{\alpha+\delta+\epsilon}\lambda(d)^\ell,
\end{gathered}
$$

where $X_i(\gamma,\delta,\epsilon)$ is a polynomial in $\alpha{-}\beta$ satisfying the recurrence

$$
\begin{aligned}
X_{i+1}(\gamma,\delta,\epsilon) &= (\alpha{-}\beta{+}\delta{+}\epsilon)X_i(\gamma,\delta,\epsilon) \\
&\quad + X_i(\gamma{-}1,\delta,\epsilon) + X_i(\gamma,\delta{-}1,\epsilon) + X_i(\gamma,\delta,\epsilon{-}1),
\end{aligned}
$$

with the initial conditions $X_0(0,0,0) = 1$ and $X_i(\gamma,\delta,\epsilon) = 0$ unless $0 \leq \gamma \leq i$, $0 \leq \delta \leq i{-}\gamma$, $0 \leq \epsilon \leq i{-}\gamma{-}\delta$.

The unique solution to the recurrence in this lemma is

$$X_i(\gamma,\delta,\epsilon) = \binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i-\gamma-\delta-\epsilon} \binom{i}{\gamma,\zeta} \left\{ {i-\gamma-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^\zeta,$$

where the Stirling numbers of the second kind are defined by

$$\left\{ {r \atop s} \right\} = \frac{1}{s!} \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} t^r.$$

We are almost ready to write down the structure constants for the universal nonassociative enveloping algebra $U(\mathbb{S})$. We need the differential coefficients defined by

$$\begin{bmatrix} r \\ 0 \end{bmatrix} = 1, \ \begin{bmatrix} r \\ s \end{bmatrix} = r(r-1)\cdots(r-s+1), \text{ so that } D_x^s(x^r) = \begin{bmatrix} r \\ s \end{bmatrix} x^{r-s}.$$

Theorem 2 (Bremner, Hentzel, Peresi and Usefi [3]). *In $U(\mathbb{S})$, the universal nonassociative enveloping algebra of the 4-dimensional Malcev algebra $\mathbb{S}$, the product of basis monomials $(a^i b^j c^k d^\ell)(a^m b^n c^p d^q)$ equals*

$$\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{i} \sum_{\delta=0}^{i-\gamma} \sum_{\epsilon=0}^{i-\gamma-\delta} \sum_{\zeta=0}^{i-\gamma-\delta-\epsilon} \sum_{\eta=0}^{j-\alpha-\epsilon} \sum_{\theta=0}^{j-\alpha-\epsilon-\eta} \sum_{\lambda=0}^{k-\alpha-\delta} \sum_{\mu=0}^{k-\alpha-\delta-\lambda} \sum_{\nu=0}^{m}$$
$$(-1)^{i+j+k+\alpha-\beta-\gamma-\epsilon-\eta-\theta-\lambda} (\alpha-\beta)^\zeta \alpha! \binom{\alpha}{\beta} (\delta+\epsilon)! \omega^\nu \binom{i}{\gamma,\zeta} \times$$
$$\left\{ {i-\gamma-\zeta \atop \delta+\epsilon} \right\} \binom{j}{\alpha,\epsilon,\eta,\theta} \binom{k}{\alpha,\delta,\lambda,\mu} \binom{m}{\nu} \begin{bmatrix} n \\ k-\alpha-\lambda \end{bmatrix} \begin{bmatrix} p+\lambda \\ j-\alpha-\eta \end{bmatrix} \times$$
$$a^{m+\gamma-\nu} b^{-k+n+\alpha+\eta+\lambda} c^{-j+p+\alpha+\eta+\lambda} d^{j+k+\ell+q-\alpha-\eta-\lambda},$$

where $\omega = j+k-\ell-2\alpha-\beta-2\delta-2\epsilon-2\theta-2\mu$. (We make the convention that when $\alpha=\beta$ and $\zeta=0$ we set $(\alpha-\beta)^\zeta = 1$.)

It is not difficult to show that $U(\mathbb{S})$ is not alternative; in fact, it is not even power-associative. So it is interesting to determine the largest alternative quotient of $U(\mathbb{S})$. The alternator ideal in a nonassociative algebra is the ideal generated by all elements of the form (x,x,y) and (y,x,x). Let M be a Malcev algebra, $U(M)$ its universal enveloping algebra, and $I(M) \subseteq U(M)$ the alternator ideal. The universal alternative enveloping algebra of M is $A(M) = U(M)/I(M)$.

Lemma 11. *We have the following nonzero alternators in $U(\mathbb{S})$:*

$$(a,bc,bc) = 2d^2, \qquad (b,ac,ac) = cd, \qquad (c,ab,ab) = -bd.$$

PROOF: From the structure constants for $U(\mathbb{S})$ we obtain

$$\big(a(bc)\big)(bc) = ab^2c^2 - 2abcd + 2d^2, \quad a\big((bc)(bc)\big) = ab^2c^2 - 2abcd,$$

which imply the first equation. The others are similar. □

Let $J \subseteq U(\mathbb{S})$ be the ideal generated by $\{d^2, cd, bd\}$. In $U(\mathbb{S})/J$ it suffices to consider two types of monomials: $a^i d$ and $a^i b^j c^k$. If m is one of these, we write m when we mean $m + J$ in the next theorem.

Theorem 3 (Bremner, Hentzel, Peresi and Usefi [3]). *In $U(\mathbb{S})/J$ we have*

$$\begin{aligned}
(a^i d)(a^m d) &= 0,\\
(a^i b^j c^k)(a^m d) &= \delta_{j0}\delta_{k0}\, a^{i+m} d,\\
(a^i d)(a^m b^n c^p) &= \delta_{n0}\delta_{p0}\, a^i (a{-}1)^m d,\\
(a^i b^j c^k)(a^m b^n c^p) &= a^i (a{+}j{+}k)^m b^{j+n} c^{k+p} + \delta_{j+n,1}\delta_{k+p,1} T^{im}_{jk},
\end{aligned}$$

where

$$T^{im}_{jk} = \begin{cases} 0 & \text{if } (j,k) = (0,0),\\ (a{-}1)^{i+m} d - a^i (a{+}1)^m d & \text{if } (j,k) = (1,0),\\ -(a{-}1)^{i+m} d - a^i (a{+}1)^m d & \text{if } (j,k) = (0,1),\\ a^i (a{-}1)^m d - a^i (a{+}2)^m d & \text{if } (j,k) = (1,1). \end{cases}$$

It can be shown by direct calculation that the associator in $U(\mathbb{S})/J$ is an alternating function of its arguments. It follows that J equals the alternator ideal $I(\mathbb{S})$, and that $U(\mathbb{S})/J$ is isomorphic to the universal alternative enveloping algebra $A(\mathbb{S})$. The first proof of this result was given by Shestakov [21], [22] using different methods. Since $I(\mathbb{S})$ contains no elements of degree 1, the natural mapping from $\mathbb{S}$ to $A(\mathbb{S})$ is injective, and hence $\mathbb{S}$ is special. This also follows directly from the isomorphism $\mathbb{S} \cong \mathbb{A}^-$ where $\mathbb{A}$ is the 4-dimensional alternative algebra in Table 3.

$\cdot$	a	b	c	d
a	a	0	0	d
b	b	0	d	0
c	c	$-d$	0	0
d	0	0	0	0

TABLE 3. The 4-dimensional alternative algebra $\mathbb{A}$ for which $\mathbb{A}^- = \mathbb{S}$

5. Malcev algebras of dimensions 5, 6 and 7

The 5-dimensional Malcev algebras were classified by Kuzmin [10]: there is one nilpotent algebra, a one-parameter family of solvable (non-nilpotent) algebras, and one non-solvable algebra. A recent preprint by Gavrilov [8] gives a new approach to the classification of the solvable algebras. The structure constants of the nilpotent Malcev algebra $\mathbb{T}$ are given in Table 4.

$[-,-]$	a	b	c	d	e
a	0	c	0	0	0
b	$-c$	0	0	0	0
c	0	0	0	e	0
d	0	0	$-e$	0	0
e	0	0	0	0	0

Table 4. The 5-dimensional nilpotent Malcev algebra $\mathbb{T}$

Theorem 4 (Bremner and Usefi [4]). *In $U(\mathbb{T})$, the universal nonassociative enveloping algebra of the 5-dimensional nilpotent Malcev algebra $\mathbb{T}$, the product of basis monomials $(a^i b^j c^k d^\ell e^m)(a^p b^q c^r d^s e^t)$ equals*

$$\sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{i}\sum_{\gamma=0}^{\beta}\sum_{\delta=0}^{\gamma}\sum_{\epsilon=0}^{j-\alpha-\delta}\sum_{\zeta=0}^{j-\alpha-\delta-\epsilon}\sum_{\eta=0}^{\ell-\alpha-(\gamma-\delta)}\sum_{\theta=0}^{\ell-\alpha-(\gamma-\delta)-\eta}\sum_{\lambda=0}^{\eta}$$
$$(-1)^{\beta+\zeta+\ell-\alpha-\gamma-\eta}\frac{\alpha!\beta!\lambda!}{2^{\alpha+\gamma}3^{j-\epsilon-\zeta+\ell-\alpha-\eta-\theta}}\times$$
$$\binom{\alpha}{\beta-\gamma}\binom{i}{\beta}\binom{j}{\alpha,\delta,\epsilon,\zeta}\binom{j-\alpha-\epsilon-\zeta}{\lambda}\binom{\ell}{\alpha,\gamma-\delta,\eta,\theta}\binom{\eta}{\lambda}\times$$
$$\begin{bmatrix}p\\ j-\beta-\epsilon+\ell-\alpha-\eta-\theta\end{bmatrix}\begin{bmatrix}q\\ \ell-\alpha-\eta-\theta\end{bmatrix}\begin{bmatrix}r\\ \theta\end{bmatrix}\begin{bmatrix}s\\ j-\alpha-\epsilon-\zeta-\lambda\end{bmatrix}\times$$
$$a^{i+p-j+\epsilon-\ell+\alpha+\eta+\theta}b^{\epsilon+q-\ell+\alpha+\eta+\theta}c^{\zeta+k+r-\theta}d^{\eta+s+\alpha+\epsilon+\zeta-j}e^{j-\alpha-\epsilon-\zeta+\ell-\eta+m+t}.$$

Lemma 12. *We have the following nonzero alternators in $U(\mathbb{T})$:*

$$(ab,ab,d)=-\tfrac{1}{6}ce,\qquad (bd,bd,a^2)=\tfrac{1}{18}e^2.$$

A basis of the ideal J generated by $\{ce,e^2\}$ consists of the monomials

$$\{a^i b^j c^k d^\ell e^m \mid m\geq 2\}\cup\{a^i b^j c^k d^\ell e \mid k\geq 1\}.$$

Theorem 5 (Bremner and Usefi [4]). *The algebra $U(\mathbb{T})/J$ is alternative; we have*

$$(a^i b^j d^\ell e)(a^p b^q d^s e)=0,$$

$$
\begin{aligned}
&(a^i b^j c^k d^\ell)(a^p b^q d^s e) = \delta_{k0} a^{i+p} b^{j+q} d^{\ell+s} e,\\
&(a^i b^j d^\ell e)(a^p b^q c^r d^s) = \delta_{r0} a^{i+p} b^{j+q} d^{\ell+s} e,\\
&(a^i b^j c^k d^\ell)(a^p b^q c^r d^s) = \sum_{\mu=0}^{j} (-1)^\mu \mu! \binom{j}{\mu}\binom{p}{\mu} a^{i+p-\mu} b^{j+q-\mu} c^{k+r+\mu} d^{\ell+s}\\
&\quad + \delta_{k0}\delta_{r0}\left(\tfrac{1}{6}ijs - \tfrac{1}{6}i\ell q + \tfrac{1}{2}j\ell p + \tfrac{1}{3}jps - \tfrac{1}{3}\ell pq\right) a^{i+p-1} b^{j+q-1} d^{\ell+s-1} e\\
&\quad - \delta_{k0}\delta_{r1}\ell a^{i+p} b^{j+q} d^{\ell+s-1} e.
\end{aligned}
$$

The remaining 5-dimensional algebras are an infinite family of solvable non-nilpotent algebras and a non-solvable algebra. Bremner and Tvalavadze are studying the universal enveloping algebras for these algebras. Before discussing the non-solvable algebra, we will describe the 7-dimensional simple Malcev algebra, which is closely related to the 8-dimensional alternative algebra of octonions.

The 8-dimensional alternative division algebra $\mathbb{O}$ of real octonions has basis 1, i, j, k, l, m, n, p with structure constants obtained from the Cayley-Dickson doubling process. We replace the product in $\mathbb{O}$ by the commutator $[x,y] = xy - yx$ and denote the resulting Malcev algebra by $\mathbb{O}^-$. We have the direct sum of ideals $\mathbb{O}^- = \mathbb{R} \oplus \mathbb{M}$, where $\mathbb{R}$ denotes the span of 1, and $\mathbb{M}$ denotes the span of the other basis elements. The subalgebra $\mathbb{M}$ is a simple (non-Lie) Malcev algebra with structure constants in Table 5. We complexify $\mathbb{M}$ and introduce a new basis where $\epsilon^2 = -1$:

$$
\begin{aligned}
h = \epsilon l, \quad x &= \tfrac{1}{2}(i - \epsilon m), \qquad y = \tfrac{1}{2}(k - \epsilon p), \qquad z = \tfrac{1}{2}(j - \epsilon n),\\
x' &= -\tfrac{1}{2}(i + \epsilon m), \quad y' = -\tfrac{1}{2}(k + \epsilon p), \quad z' = -\tfrac{1}{2}(j + \epsilon n).
\end{aligned}
$$

For this new basis we have structure constants in Table 6.

We observe that the subalgebra $\mathbb{L}$ of $\mathbb{M}$ with basis $\{h, x, x'\}$ is isomorphic to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$: $[h,x] = 2x$, $[h,x'] = -2x'$, $[x,x'] = h$. The subspace $\mathbb{V}$ (resp. $\mathbb{W}$) with basis $\{y, z'\}$ (resp. $\{z, y'\}$) has trivial product $[y,z'] = 0$ (resp. $[z,y'] = 0$) and is an $\mathbb{L}$-module:

$$
\begin{array}{llll}
[h,y] = 2y, & [h,z'] = -2z', & [h,z] = 2z, & [h,y'] = -2y',\\
[x,y] = 2z', & [x,z'] = 0, & [x,z] = -2y', & [x,y'] = 0,\\
[x',y] = 0, & [x',z'] = 2y, & [x',z] = 0, & [x',y'] = -2z.
\end{array}
$$

We have $\mathbb{M} = \mathbb{L} \oplus \mathbb{V} \oplus \mathbb{W}$: the direct sum of three $\mathbb{L}$-modules. The $\mathbb{L}$-modules $\mathbb{V}$ and $\mathbb{W}$ are isomorphic by $y \leftrightarrow -z$, $z' \leftrightarrow y'$; these are 2-dimensional irreducible representations of $\mathbb{L}$ which are not isomorphic to the natural representation of $\mathfrak{sl}_2(\mathbb{C})$. This is possible because $\mathbb{V}$ and $\mathbb{W}$ are Malcev modules which are not Lie modules. (Carlsson [5] has shown that this is the only case in which a simple Lie algebra has a Malcev module which is not a Lie module. Elduque and Shestakov [7] determined all irreducible non-Lie modules for Malcev algebras without any

$[-,-]$	i	j	k	l	m	n	p
i	0	$-2k$	$2j$	$2m$	$-2l$	$-2p$	$2n$
j	$-2k$	0	$2i$	$2n$	$2p$	$-2l$	$-2m$
k	$2j$	$-2i$	0	$2p$	$-2n$	$2m$	$-2l$
l	$-2m$	$-2n$	$-2p$	0	$2i$	$2j$	$2k$
m	$2l$	$-2p$	$2n$	$-2i$	0	$-2k$	$2j$
n	$2p$	$2l$	$-2m$	$-2j$	$2k$	0	$-2i$
p	$-2n$	$2m$	$2l$	$-2k$	$-2j$	$2i$	0

TABLE 5. The 7-dimensional simple Malcev algebra $\mathbb{M}$ (non-split form)

$[-,-]$	h	x	y	z	x'	y'	z'
h	0	$2x$	$2y$	$2z$	$-2x'$	$-2y'$	$-2z'$
x	$-2x$	0	$2z'$	$-2y'$	h	0	0
y	$-2y$	$-2z'$	0	$2x'$	0	h	0
z	$-2z$	$2y'$	$-2x'$	0	0	0	h
x'	$2x'$	$-h$	0	0	0	$-2z$	$2y$
y'	$2y'$	0	$-h$	0	$2z$	0	$-2x$
z'	$2z'$	0	0	$-h$	$-2y$	$2x$	0

TABLE 6. The 7-dimensional simple Malcev algebra $\mathbb{M}$ (split form)

restriction on the dimension.) The subalgebras $\mathbb{L} \oplus \mathbb{V}$ and $\mathbb{L} \oplus \mathbb{W}$ are isomorphic to the 5-dimensional non-solvable (non-Lie) Malcev algebra.

The paper of Kuzmin [10] also classifies the 6-dimensional nilpotent Malcev algebras. It would be interesting to extend this to a classification of all 6-dimensional Malcev algebras, and to study their enveloping algebras.

6. Universal property of the octonions

The goal of this research program is to study the universal nonassociative enveloping algebra of the 7-dimensional simple Malcev algebra. Elduque [6] showed that this Malcev algebra does not have a 6-dimensional subalgebra, so we have to jump directly from the 5-dimensional non-solvable algebra to the 7-dimensional simple algebra. The universal nonassociative enveloping algebra is linearly isomorphic to the polynomial algebra in 7 variables, hence infinite dimensional. But the universal alternative enveloping algebra is the octonion algebra; the following proof was explained to us by Shestakov.

Theorem 6. *The universal alternative enveloping algebra $U(\mathbb{M})$ of the 7-dimensional simple Malcev algebra $\mathbb{M}$ over $\mathbb{R}$ is isomorphic to the division algebra $\mathbb{O}$ of real octonions: $U(\mathbb{M}) \cong \mathbb{O}$.*

Proof: The octonion algebra $\mathbb{O}$ has a basis $\{1, e_1, \ldots, e_7\}$ for which the multiplication is completely determined by the following conditions for $1 \leq i \neq j \leq 7$ with the convention that $e_{i+7} = e_i$; see Kuzmin and Shestakov [11, p. 217]:

$$(1) \quad e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad e_i e_{i+1} = e_{i+3}, \quad e_{i+1} e_{i+3} = e_i, \quad e_{i+3} e_i = e_{i+1}.$$

In the Malcev algebra $\mathbb{O}^-$, the subalgebra of traceless (or pure imaginary) octonions has basis $\{e_1, \ldots, e_7\}$; its multiplication is completely determined by the following conditions:

$$(2) \qquad [e_i, e_{i+1}] = 2e_{i+3}, \qquad [e_{i+1}, e_{i+3}] = 2e_i, \qquad [e_{i+3}, e_i] = 2e_{i+1}.$$

The 7-dimensional simple Malcev algebra $\mathbb{M}$ is isomorphic to the Malcev algebra of traceless octonions; see Kuzmin [9, Theorem 10]. Therefore, $\mathbb{M}$ has basis $\{e_1, \ldots, e_7\}$ with the structure constants (2).

Assume that $\mathbb{M}$ is embedded in an alternative algebra $\mathbb{A}$ in which multiplication is denoted by juxtaposition, and consider the injective Malcev algebra homomorphism sending $e_i \in \mathbb{M} \to e_i \in \mathbb{A}^-$. The following identity holds in $\mathbb{A}$ since it holds in every alternative algebra as can be easily shown by direct expansion:

$$(3) \qquad (x, y, z) = \tfrac{1}{6} J(x, y, z), \qquad J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y].$$

Using (3) and the structure constants (2) we obtain

$$(4) \qquad \begin{aligned} (e_{i+1}, e_{i+2}, e_{i+5}) &= \tfrac{1}{6} J(e_{i+1}, e_{i+2}, e_{i+5}) = 2e_i, \\ (e_{i+4}, e_{i+1}, e_{i+6}) &= \tfrac{1}{6} J(e_{i+4}, e_{i+1}, e_{i+6}) = 2e_i, \\ (e_{i+1}, e_{i+3}, e_{i+2}) &= \tfrac{1}{6} J(e_{i+1}, e_{i+3}, e_{i+2}) = 2e_{i+6}. \end{aligned}$$

Every alternative algebra satisfies the polynomial identity

$$(5) \qquad [x, y] \circ (x, y, z) = 0,$$

where $\circ$ denotes the product $x \circ y = xy + yx$; see Zhevlakov et al. [25], page 35 and Lemma 9, page 145. From this identity, using (2) and (4), we obtain

$$(6) \qquad e_i \circ e_{i+4} = 0, \qquad e_i \circ e_{i+2} = 0, \qquad e_i \circ e_{i+6} = 0.$$

Therefore (recalling that $e_{i+7} = e_i$) we have $e_i \circ e_j = 0$ for all $i \neq j$, or equivalently

$$(7) \qquad e_i e_j = -e_j e_i.$$

From (2) and (7) it follows that

$$(8) \qquad e_i e_{i+1} = e_{i+3}, \qquad e_{i+1} e_{i+3} = e_i, \qquad e_{i+3} e_i = e_{i+1}.$$

Therefore for all $i \neq j$ we have $e_i e_j = \pm e_k$ for some k. Linearizing (5) gives

$$[x,y] \circ (r,s,z) + [r,y] \circ (x,s,z) + [x,s] \circ (r,y,z) + [r,s] \circ (x,y,z) = 0. \tag{9}$$

We set $x = e_{i+2}$, $y = e_{i+6}$, $r = e_{i+1}$, $s = e_{i+2}$, $z = e_{i+5}$. On the left side of (9) the two middle terms vanish, and we obtain

$$[e_{i+2}, e_{i+6}] \circ (e_{i+1}, e_{i+2}, e_{i+5}) + [e_{i+1}, e_{i+2}] \circ (e_{i+2}, e_{i+6}, e_{i+5}) = 0. \tag{10}$$

Now (2), (4) and (10) give

$$2e_i \circ 2e_i - 2e_{i+4} \circ 2e_{i+4} = 0, \quad \text{hence} \quad 8e_i^2 - 8e_{i+4}^2 = 0. \tag{11}$$

Therefore $e_i^2 = e_{i+4}^2$ for all i, and hence $e_i^2 = e_j^2$ for all i, j; we denote the common value by a.

Let $\mathbb{A}_0 \subseteq \mathbb{A}$ be the subalgebra generated by $e_1, \dots, e_7$; then $\mathbb{A}_0$ is also generated by e_1, e_2, e_3 since (8) gives $e_1e_2 = e_4$, $e_2e_3 = e_5$, $e_3e_4 = e_6$, $e_4e_5 = e_7$. Now consider the following subspace of $\mathbb{A}_0$:

$$\mathbb{A}_0' = \mathbb{R}a + \sum_{i=1}^{7} \mathbb{R}e_i.$$

We claim that $\mathbb{A}_0'$ is a subalgebra of $\mathbb{A}_0$ and that $-a$ is the identity element of this subalgebra. The proof is in three parts. *Part 1:* We have already seen that $e_i^2 = a$ for all i. By (8) we have $e_ie_{i+1} = e_{i+3}$. Multiplying this equation on the left by e_i we obtain $e_i^2 e_{i+1} = e_i e_{i+3}$. But equations (7) and (8) imply that $e_i e_{i+3} = -e_{i+1}$, and so $ae_{i+1} = -e_{i+1}$. In a similar manner, we have $e_{i+1}a = -e_{i+1}$. *Part 2:* For a^2, we use $e_i^2 = a$ for all i and Part 1 to obtain $a^2 = ae_i^2 = (ae_i)e_i = (-e_i)e_i = -e_i^2$. Hence $a^2 = -a$. *Part 3:* We have already observed that for all $i \neq j$ we have $e_ie_j = \pm e_k$ for some k. This completes the proof of the claim.

Since $\mathbb{A}_0'$ is also generated by e_1, e_2, e_3 we see that $\mathbb{A}_0 = \mathbb{A}_0'$. It is now clear that $\mathbb{A}_0$ and $\mathbb{O}$ are isomorphic over $\mathbb{R}$ since they have the same structure constants. □

Remark. Theorem 6 also follows easily from Example 2 in Shestakov [21]. In fact, in the notation of [21] we have

$$8 \leq \dim U(\mathbb{M}) = \dim \operatorname{gr} U(\mathbb{M}) \leq \dim \widetilde{S}(\mathbb{M}) = 8,$$

hence $\dim U(\mathbb{M}) = 8$ and $U(\mathbb{M}) \cong \mathbb{O}$. (We thank the referee for pointing this out.)

Shestakov and Zhelyabin [24] determined the nucleus N (elements which associate with all elements) and center Z (elements which commute and associate with all elements) for the universal nonassociative enveloping algebras over a field F of characteristic 0 of the 4-dimensional solvable algebra $\mathbb{S}$, the 5-dimensional nilpotent algebra $\mathbb{T}$, the 5-dimensional non-solvable algebra $\mathbb{U}$, and the 7-dimensional simple algebra $\mathbb{M}$:

$$
\begin{array}{llll}
N(U(\mathbb{S})) = F[d], & N(U(\mathbb{T})) = F[c,e], & N(U(\mathbb{U})) = F, & N(U(\mathbb{M})) = F[c], \\
Z(U(\mathbb{S})) = F, & Z(U(\mathbb{T})) = F[e], & Z(U(\mathbb{U})) = F, & Z(U(\mathbb{M})) = F[c].
\end{array}
$$

For $U(\mathbb{M})$, we set $c = e_1^2 + \cdots + e_7^2$, so that $N(U(\mathbb{M})) = Z(U(\mathbb{M})) = F[c]$, which is isomorphic to the polynomial ring in one variable.

7. Bol algebras

In an associative algebra, the commutator $[a,b]$ satisfies the Jacobi identity, and the associator (a,b,c) is identically zero. In an alternative algebra, $[a,b]$ satisfies the Malcev identity, and (a,b,c) is an alternating function of its arguments; moreover, the associator can be expressed in terms of the commutator:

$$(a,b,c) = \tfrac{1}{6}\big([[a,b],c] + [[b,c],a] + [[c,a],b]\big).$$

The next step is right alternative algebras, which satisfy only $(b,a,a) = 0$. In this case we need both operations, commutator and associator. In fact, it is more convenient to consider not the usual associator but rather the Jordan associator:

$$\langle a,b,c\rangle = (a \circ b) \circ c - a \circ (b \circ c), \qquad a \circ b = \tfrac{1}{2}(ab + ba).$$

In any right alternative algebra, $[a,b]$ and $[a,b,c] = \langle b,c,a\rangle$ (note the cyclic shift) satisfy the following identities, which define the variety of Bol algebras:

$$
\begin{aligned}
&[a,a] = 0, \qquad [a,a,c] = 0, \qquad [a,b,c] + [b,c,a] + [c,a,b] = 0, \\
&[a,b,[c,d,e]] - [[a,b,c],d,e] - [c,[a,b,d],e] - [c,d,[a,b,e]] = 0, \\
&[[a,b],[c,d]] - [a,b,[c,d]] + [c,d,[a,b]] + [[a,b,c],d] - [[a,b,d],c] = 0.
\end{aligned}
$$

That is, a Bol algebra is a Lie triple system with respect to the operation $[a,b,c]$, and also has an anticommutative operation $[a,b]$; the two are related by the last identity above. Any Lie triple system becomes a Bol algebra by defining $[a,b] = 0$ for all a,b. Universal nonassociative enveloping algebras for Bol algebras have been obtained by Pérez-Izquierdo [16]. The structure constants have not been worked out explicitly even for the 2-dimensional simple Lie triple system with basis e,f and relations $[e,f,e] = 2e$ and $[e,f,f] = -2f$.

8. Beyond Bol algebras

To go further we need to recall the origins of Lie, Malcev and Bol algebras in differential geometry. A Lie algebra is the natural algebraic structure on the tangent space at the identity element of a Lie group. Similarly, Malcev (resp. Bol) algebras arise as tangent algebras of analytic Moufang (resp. Bol) loops. A Lie algebra determines a local Lie group uniquely up to isomorphism, and similarly a Malcev (resp. Bol) algebra determines a local Moufang (resp. Bol) loop uniquely up to isomorphism. In general, one may consider any analytic loop and the operations induced by its multiplicative structure on the tangent space at the identity. The resulting algebraic structure was described from the geometric point of view by Mikheev and Sabinin [14], and was clarified from the algebraic point of view by Shestakov and Umirbaev [23] and Pérez-Izquierdo [17]. These algebraic structures are now called Sabinin algebras.

In the general case, a Sabinin algebra has an infinite sequence of operations; from the algebraic point of view these operations correspond to the elements of the free nonassociative algebra which are primitive in the sense of (nonassociative) Hopf algebras. Following this approach, we see that Lie, Malcev and Bol algebras are the Sabinin algebras corresponding to associative, alternative, and right alternative algebras. Beyond right alternative algebras, the next weaker form of associativity leads to third-power associative algebras defined by the identity $(a, a, a) = 0$; these algebras correspond to monoassociative loops, defined by the identity $a^2a = aa^2$. In order to determine a local monoassociative loop uniquely up to isomorphism, the corresponding tangent algebra requires not only a binary operation (the commutator) and a ternary operation (the associator), but also a quaternary operation, called a quaternator by Akivis and Goldberg [1].

These binary-ternary-quaternary structures have been studied from the geometric point of view by Akivis and Shelekhov [2], Shelekhov [20], and Mikheev [13]. We will say a few words about them from an algebraic point of view. Let A be the free nonassociative algebra on a set X of generators. Define a coproduct $\Delta\colon A \to A \otimes A$ by setting $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in X$ and extending to all of A by assuming that Δ is an algebra morphism: $\Delta(fg) = \Delta(f)\Delta(g)$. The multilinear operations on A correspond to the polynomials f which are primitive in the Hopf algebra sense: $\Delta(f) = f \otimes 1 + 1 \otimes f$. Shestakov and Umirbaev [23] gave a complete set of these primitive elements. In degrees 2 and 3 there are only the commutator and associator, but in degree 4 there are two quaternary operations,

$$
\begin{aligned}
f &= \langle a, b, c, d\rangle = (ab, c, d) - a(b, c, d) - (a, c, d)b,\\
g &= \{a, b, c, d\} = (a, bc, d) - b(a, c, d) - (a, b, d)c,
\end{aligned}
$$

which measure the deviation of the associators $(-, c, d)$ and $(a, -, d)$ from being derivations. In degree 4, we also have six elements (called Akivis elements) that are generated by the operations of lower degree (the commutator and associator):

$$
[[[a, b], c], d], \quad [(a, b, c), d], \quad [[a, b], [c, d]], \quad ([a, b], c, d), \quad (a, [b, c], d), \quad (a, b, [c, d]).
$$

Every primitive multilinear nonassociative polynomial of degree 4 is a linear combination of permutations of these six Akivis elements and the two non-Akivis elements f and g. At this point we have two quaternary operations, f and g, but we have not yet imposed third-power associativity. If we assume $(a, a, a) = 0$ then g is a consequence of f, in the sense that g is a linear combination of permutations of f, the Akivis elements, and the consequences in degree 4 of the linearization of the third-power associative identity $(a,a,a)=0$:

$$([a,d],b,c)+([a,d],c,b)+(b,[a,d],c)+(b,c,[a,d])+(c,[a,d],b)+(c,b,[a,d])=0,$$
$$[(a,b,c),d]+[(a,c,b),d]+[(b,a,c),d]+[(b,c,a),d]+[(c,a,b),d]+[(c,b,a),d]=0.$$

The next step is to find the polynomial identities relating these three operations: $[a,b]$, (a,b,c), $\langle a,b,c,d\rangle$. These identities will define a variety of nonassociative multioperator algebras generalizing Bol algebras.

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