Toward ternary C

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Abstract. In which the binary product algebra of complex numbers, **C**, is generalized to a ternary product algebra, $\mathbf{C}_3$.

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Why?

For decades my explorations into mathematics and physics have been based on the opinion that mathematics is primary, physics secondary. Mathematics is intellectually profligate, and its practitioners are occasionally proud of the purity and inapplicability of their work. This is a good thing. In being less fettered than physics, it has the potential to generate ideas and objects of no presently perceivable value, but which in future may be seen by those with an interest in applications as just what they needed to add flesh to their intuition. Riemann geometry and Lie group theory are outstanding examples.

For me, initially, it was the real normed division algebras: real numbers, **R**; complex numbers, **C**; quaternions, **H**; octonions, **O** (although by the time I came on the scene only **H** and **O** were having any difficulty gaining traction in physics). However, it is not the algebras that are special, it's their dimensions (over **R**): 1, 2, 4 and 8. These dimensions are associated with those division algebras, with the parallelizable spheres, the 4 sequences of classical Lie groups, and so much more. They are mathematically resonant, and I have never had any doubt that our physical reality requires this kind of seminal resonance in its mathematical underpinnings.

There is another finite sequence of resonant dimensions associated with lattice theory: 0, 2, 8 and 24. The last of these, 24, is the dimension of the Leech lattice, Λ_{24}, accounted by those in the know as one of the most special objects in mathematics [1]. Since $24 = 3 \times 8$, some have pursued representations of Λ_{24} over $\mathbf{O}^3$ [2].

My work in this area was heavily influenced by work I'd done connecting the octonion X-product [3] and XY-product [4] to the 8- and 16-dimensional laminated lattices, $E_8 = \Lambda_8$ and Λ_{16} [5]. I wondered if Λ_{24} were also associated with some product, with luck involving **O**. But the factor 3 in its dimensionality was problematic, unless, perhaps, the hypothetical product was ternary. I knew next to nothing of ternary products, but made an initial foray on my own [6].

Eventually, finding the possible paths into this thicket too numerous, I decided to attempt to construct a ternary analog of $\mathbf{C}$ (which, of course, has a binary product). The results of this effort are presented here. Because this is a construction from scratch of what is (at least to me) a new mathematical object, the ideas are presented primarily in the form of a series of Assumptions and Motivations.

What?

Assumption 1. $\mathbf{C}_3$ is 3-dimensional, with basis

$$\{i_0, i_1, i_2\}.$$

Motivation: This seems a natural generalization of the structure of $\mathbf{C}$, with basis $\{1, i\}$. (I think of i_0 as being the $\mathbf{C}_3$ version of the unit 1, but will refrain from writing it like that.) The point of making it 3-dimensional is the desire to be able to express $3^3 + 4^3 + 5^3 = 6^3$ as an equation in $\mathbf{C}_3$ in a way similar to $(3 + i4)(3 + i4)^* = 3^2 + 4^2 = 5^2$. More generally, the smallest number of positive integers the sum of whose cubes is an integer cubed, is 3.

Assumption 2. $\mathbf{C}_3$ is a complex linear space, every element $X \in \mathbf{C}_3$ of the form

$$X = ui_0 + vi_1 + wi_2,\ u, v, w \in \mathbf{C}.$$

Motivation: Actually, I tried to make it a real linear space, but failed. More on this below.

Assumption 3. Complex conjugation effects only elements of $\mathbf{C}$. That is,

$$X^* = (ui_0 + vi_1 + wi_2)^* = u^* i_0 + v^* i_1 + w^* i_2.$$

Assumption 4. There is a conjugation on $\mathbf{C}_3$, denoted $X^\#$, which has no effect on $\mathbf{C}$. That is,

$$X^\# = (ui_0 + vi_1 + wi_2)^\# = ui_0^\# + vi_1^\# + wi_2^\#.$$

Assumption 5. For all $X \in \mathbf{C}_3$,

$$X^{\#\#\#} = X.$$

Motivation: This generalizes $u^{**} = u$, for all $u \in \mathbf{C}$.

Assumption 6. For all $X \in \mathbf{C}_3$, there exists $u \in \mathbf{C}$ such that

$$X + X^\# + X^{\#\#} = ui_0.$$

Motivation: This generalizes $u + u^* \in \mathbf{R}$ for all $u \in \mathbf{C}$.

Notation. For all $X, Y, Z \in \mathbf{C}_3$, we denote their ordered ternary product by

$$\langle X, Y, Z \rangle$$

(and for the sake of clarity I assume this satisfies

$$\langle X + A, Y, Z \rangle = \langle X, Y, Z \rangle + \langle A, Y, Z \rangle$$

for all $A \in \mathbf{C}_3$ (along with this equation's cyclic permutations), and for all complex scalars, u, v, w,

$$\langle uX, vY, wZ \rangle = uvw\langle X, Y, Z \rangle).$$

Assumption 7. $\mathbf{C}_3$ is cyclic commutative:

$$\langle X, Y, Z \rangle = \langle Z, X, Y \rangle.$$

Motivation: Well, for one, without some assumptions of this sort we will never have a chance of getting to a multiplication table. In addition, $\mathbf{C}$ is commutative (so also cyclic commutative). Another ternary example of this is found in the octonion associator, $p(qr) - (pq)r,\ p, q, r \in \mathbf{O}$. This is invariant with respect to cyclic (symmetric) permutations of p, q, r. (Only in dimensions 4 or higher is the set of symmetric permutations bigger than the set of cyclic permutations.)

Assumption 8. For all $X, Y, Z \in \mathbf{C}_3$,

$$\langle X, Y, Z \rangle^{\#} = \langle Z^{\#}, X^{\#}, Y^{\#} \rangle.$$

Note: Due to assumption 7 this can be replaced with

$$\langle X, Y, Z \rangle^{\#} = \langle X^{\#}, Y^{\#}, Z^{\#} \rangle.$$

I include this non-effective assumption because I am used to dealing with division algebras where that kind of thing is important. In $\mathbf{C}$ one generally sets $(uv)^* = u^*v^*$. This works because $\mathbf{C}$ is commutative, but $\mathbf{H}$ and $\mathbf{O}$ are not, and in those cases we need to permute the right hand side to read $(uv)^* = v^*u^*$. This works for $\mathbf{C}$ too. Someday, however, I may get around to seeing if there might be ternary algebras $\mathbf{H}_3$ or $\mathbf{O}_3$, and at that point Assumption 8 may play a crucial role.

Assumption 9. For all $X = ui_0 + vi_1 + wi_2 \in \mathbf{C}_3$,

$$X^{\#} = X \Longrightarrow X = ui_0.$$

Note: This is consistent with Assumption 6, since

$$(X + X^{\#} + X^{\#\#})^{\#} = X^{\#} + X^{\#\#} + X^{\#\#\#} = X + X^{\#} + X^{\#\#},$$

($X^{\#\#\#} = X$). This was one of my original Assumptions, but it is actually a consequence of earlier Assumptions. If $X^{\#} = X$, then $X + X^{\#} + X^{\#\#} = 3X$, so by Assumption 6, $3X$, hence X, is linear in i_0.

Assumption 10. For all $X = ui_0 + vi_1 + wi_2 \in \mathbf{C}_3$,

$$\langle X, X^{\#}, X^{\#\#} \rangle = (u^3 + v^3 + w^3) i_0.$$

Motivation: This is a straightforward generalization of

$$(a + ib)(a + ib)^* = a^2 + b^2, \ a, b \in \mathbf{R}.$$

Without the factor 3 in the dimensionality of the Leech lattice, and that $3^2 + 4^2 = 5^2$ and $3^3 + 4^3 + 5^3 = 6^3$, I would never have attempted to construct a ternary generalization of $\mathbf{C}$. As to those two sums of powers, it may be just coincidental, but mathematics is not known for such coincidences lacking any deeper meaning.

Definition. Let

$$\begin{aligned} \epsilon &= -\tfrac{1}{2} + \tfrac{i\sqrt{3}}{2} = \sqrt[3]{1}, \\ \epsilon^2 &= -\tfrac{1}{2} - \tfrac{i\sqrt{3}}{2} = \epsilon^* = \epsilon^{-1} = \sqrt[3]{1}. \end{aligned}$$

These are the two nontrivial cube roots of unity in $\mathbf{C}$. That is,

$$\epsilon^3 = (\epsilon^2)^3 = \epsilon^6 = 1.$$

Note that

$$1 + \epsilon + \epsilon^2 = 0.$$

Notation. For any integer k, let $k\%3$ be shorthand for k modulo 3 (this is a notation derived from computer coding, which is replete with notational conventions that could enrich mathematics). Note that for integers k, m, n,

$$\epsilon^k \epsilon^m \epsilon^n = \epsilon^{k+m+n} = \epsilon^{(k+m+n)\%3}.$$

Assumption 11. This is a big one:

$$i_k^{\#} = \epsilon^k i_k, \ k = 0, 1, 2.$$

So,

$$i_k^{\#\#} = \epsilon^{2k} i_k = \epsilon^{(2k)\%3} i_k,$$

and

$$i_k^{\#\#\#} = \epsilon^{3k} i_k = \epsilon^{(3k)\%3} i_k = i_k.$$

So, in general, and less specifically, $i_k^{\#} = \sqrt[3]{1} \cdot i_k$. Likewise, in $\mathbf{C}$ we have $1^* = \sqrt[2]{1} \cdot 1$ and $i^* = \sqrt[2]{1} \cdot i$, the former square root being $+1$, and the latter -1.
Discussion: This takes us back to the assumption that $\mathbf{C}_3$ is a complex linear

space. I tried to make it real, but couldn't find a conjugation that gave me Assumption 10. In the end I found the introduction of the complex cube root of unity to be natural. Finally note that $i_k + i_k^{\#} + i_k^{\#\#} = (1+\epsilon+\epsilon^2)i_k = 0$, $k = 1, 2$, so $X + X^{\#} + X^{\#\#}$ is linear in i_0 for all $X \in \mathbf{C}_3$.

Assumption 12 (toward a multiplication table). For all $k, m, n \in \{0, 1, 2\}$, there exists some $j \in \{0, 1, 2\}$, and some $u_{kmn} \in \mathbf{C}$ with norm 1, such that

$$\langle i_k, i_m, i_n\rangle = u_{kmn} i_j.$$

Motivation: This is a natural generalization of what we observe in the multiplication tables of **C**, **H** and **O**.

Consequence. For all $k, m, n \in \{0, 1, 2\}$,

$$\langle i_k, i_m, i_n\rangle = u_{kmn} i_{(k+m+n)\%3}.$$

Proof:

$$\begin{aligned}
\langle i_k, i_m, i_n\rangle^{\#} &= \langle i_k^{\#}, i_m^{\#}, i_n^{\#}\rangle \\
&= \epsilon^{(k+m+n)\%3}\langle i_k, i_m, i_n\rangle \\
&= u_{kmn}\epsilon^{(k+m+n)\%3} i_j \\
&= (u_{kmn} i_j)^{\#} \\
&= u_{kmn} i_j^{\#}.
\end{aligned}$$

Therefore,

$$i_j^{\#} = \epsilon^{(k+m+n)\%3} i_j.$$

By assumption 11, $j = (k + m + n)\%3$. □

Final results and assumptions. As usual, let $X = ui_0 + vi_1 + wi_2$, and to simplify the notation, let

$$\langle kmn\rangle \equiv \langle i_k, i_m, i_n\rangle.$$

Then, using cyclic commutivity ($\langle kmn\rangle = \langle nkm\rangle$),

$$\begin{aligned}
\langle X, X^{\#}, X^{\#\#}\rangle &= \langle ui_0{+}vi_1{+}wi_2, ui_0{+}v\epsilon i_1{+}w\epsilon^2 i_2, ui_0{+}v\epsilon^2 i_1{+}w\epsilon i_2\rangle \\
&= u^3\langle 000\rangle + v^3\langle 111\rangle + w^3\langle 222\rangle \\
&\quad + (1{+}\epsilon{+}\epsilon^2)(u^2v\langle 001\rangle + uv^2\langle 011\rangle + u^2w\langle 002\rangle \\
&\quad + uw^2\langle 022\rangle + v^2w\langle 112\rangle + vw^2\langle 122\rangle) \\
&\quad + uvw(\epsilon^2 + \epsilon^2 + \epsilon^2)\langle 012\rangle \\
&\quad + uvw(\epsilon + \epsilon + \epsilon)\langle 210\rangle \\
&= u^3\langle 000\rangle + v^3\langle 111\rangle + w^3\langle 222\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ uvw(3\epsilon^2)\langle 012\rangle \\
&+ uvw(3\epsilon)\langle 210\rangle,
\end{aligned}
$$

since $1+\epsilon+\epsilon^2 = 0$. It follows from Assumption 10 that

$$\langle kkk\rangle = i_0,\ k = 0, 1, 2,$$

and

$$\langle 210\rangle = -\epsilon\langle 012\rangle.$$

We can achieve this latter condition rather nicely with two new assumptions.

Assumption 13. Anticyclic permutations on the units results in a complex conjugate. That is,

$$\langle nmk\rangle = \langle kmn\rangle^*.$$

Consequence. For all $k, m \in \{0, 1, 2\}$,

$$\langle kkm\rangle = \pm i_{(2k+m)\%3}.$$

PROOF: The cyclic and anticyclic permutations of $: kkm :$ are the same. In combination with Assumptions 12 and 13, this is sufficient. □

Assumption 14. Since $(0 + 1 + 2)\%3 = 0$, $\langle 012\rangle$ and $\langle 210\rangle$ are linear in i_0. Because it works, and it makes things pretty, we set

$$\langle 012\rangle = i\epsilon i_0.$$

In combination with Assumption 13 this implies

$$\langle 210\rangle = \langle 012\rangle^* = -i\epsilon^2 i_0 = -\epsilon\langle 012\rangle,$$

as required. As a further consequence,

$$\langle i_k, i_m^{\#}, i_n^{\#\#}\rangle = \pm i i_0, \quad : kmn :\ \text{an even/odd permutation of :012:.}$$

Discussion: The only thing needed to complete the multiplication table would be to determine the signs in $\langle kkm\rangle = \pm i_{(2k+m)\%3}$. My preference, paralleling the structure of $\mathbf{C}$, is for $\langle 00m\rangle = +i_m$, and $\langle kkm\rangle = -i_{(2k+m)\%3},\ k = 1, 2$. However, this has not yet been checked for consistency. And beyond this is the question of ternary associativity. For example, does

$$
\begin{aligned}
&\langle\langle\langle A, B, C\rangle, \langle D, E, F\rangle, \langle G, H, I\rangle\rangle, \langle J, K, M\rangle, \langle N, P, Q\rangle\rangle \\
&= \langle\langle A, B, C\rangle, \langle\langle D, E, F\rangle, \langle G, H, I\rangle, \langle J, K, M\rangle\rangle, \langle N, P, Q\rangle\rangle ?
\end{aligned}
$$

It is clear the results presented here just scratch the surface.

Conclusion

I do not presently know if something useful could be achieved by replacing **C** with **H** or **O** in the above. Even at this point one could, for example, replace ϵ with

$$-\frac{1}{2} + \frac{q_1 + q_2 + q_3}{2} \in \mathbf{H},$$

where q_1, q_2, q_3 are the anticommuting imaginary units of **H**. This element is also a cube root of unity, but nothing new comes out of such a replacement, as the subalgebra of **H** generated by that element and its conjugate is isomorphic to **C**. Allowing arbitrary coefficients in $\mathbf{C}_3$ from **H** or **O** (instead of **C**) would require great care due to noncommutativity, and, in the case of **O**, nonassociativity (which of course arises in products of 3 or more elements). It is clear much thought would have to be given to possible modifications of the conditions underlying $\mathbf{C}_3$, perhaps involving the XY-product in the case of **O** (in my experience the XY-product is useful at unraveling all sorts of octonion problems [5]). On the other hand, mayhaps **H** would find greater use as coefficients in a ternary quaternion generalization.

However, my motivation for looking into this is rooted in physics, and in particular into my assumption that the full spinor space for all three families of leptons and quarks is [7]

$$\mathbf{T^6} = \mathbf{C} \otimes \mathbf{H^2} \otimes \mathbf{O^3}.$$

Since the dimensionality of this space is wrong for a conventional spinor space, some new direction must be sought to make full sense of it, and perhaps at the same time to give the Leech lattice a ternary product structure. And does one require a ternary product to construct a Lagrangian for the 3 families of quark/ lepton/ antiquark/ antilepton fields of $\mathbf{T^6}$? Theoretical physics owes its successes to occasional transfusions of new mathematics. It remains to be seen if these ideas are the right blood type.

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