Octonionic Cayley spinors and E_6

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Abstract. Attempts to extend our previous work using the octonions to describe fundamental particles lead naturally to the consideration of a particular real, noncompact form of the exceptional Lie group E_6, and of its subgroups. We are therefore led to a description of E_6 in terms of 3×3 octonionic matrices, generalizing previous results in the 2×2 case. Our treatment naturally includes a description of several important subgroups of E_6, notably G_2, F_4, and (the double cover of) $SO(9, 1)$. An interpretation of the actions of these groups on the squares of 3-component *Cayley spinors* is suggested.

Keywords: octonions, E_6, exceptional Lie groups, Dirac equation

Classification: 17C90, 17A35, 22E70

1. Introduction

In previous work [10], [5], we used a formalism involving 2×2 octonionic matrices to describe the Lorentz group in 10 spacetime dimensions, and then applied this formalism to the Dirac equation. We developed a mechanism for reducing 10 dimensions to 4 without compactification, thus reducing the 10-dimensional massless Dirac equation to a unified treatment of massive and massless fermions in 4 dimensions. This description involves both vectors (momentum) and spinors (solutions of the Dirac equation), which we here combine into a single, 3-component object. This leads to a representation of the Dirac equation in terms of 3×3 octonionic matrices, revealing a deep connection with the exceptional Lie group E_6.

2. The Lorentz group

In earlier work [13], we gave an explicit octonionic representation of the finite Lorentz transformations in 10 spacetime dimensions, which we now summarize in somewhat different language.

Matrix groups are usually defined over the complex numbers $\mathbb{C}$, such as the Lie group $SL(n;\mathbb{C})$, consisting of the $n \times n$ complex matrices of determinant 1, or its subgroup $SU(n;\mathbb{C})$, the unitary (complex) matrices with determinant 1. It is well-known that $SL(2,\mathbb{C})$ is the the double cover of the Lorentz group $SO(3,1)$ in 4 spacetime dimensions, $\mathbb{R}^{3+1}$. One way to see this is to represent elements of $\mathbb{R}^{3+1}$ as 2×2 complex Hermitian matrices $X \in \mathbf{H}_2(\mathbb{C})$, noting that $\det X$ is just the Lorentzian norm. Elements $M \in SL(2;\mathbb{C})$ act on $X \in \mathbf{H}_2(\mathbb{C})$ via linear transformations of the form

$$T_M(X) = MXM^\dagger \tag{1}$$

and such transformations preserve the determinant. The set of transformations of the form (1) with $M \in SL(2;\mathbb{C})$ is a group under composition, and is therefore isomorphic to, and can be identified with, $SO(3,1)$. However, the map

$$\begin{aligned} SL(2;\mathbb{C}) &\longrightarrow SO(3,1) \\ M &\longmapsto T_M \end{aligned} \tag{2}$$

which takes M to the linear transformation defined by (1), is not one-to-one; in fact, this map is easily seen to be a two-to-one homomorphism with kernel $\{\pm I\}$. We call such a homomorphism a *double cover*. Restricting M to the subgroup $SU(2;\mathbb{C}) \subset SL(2;\mathbb{C})$ similarly leads to the well-known double cover

$$SU(2;\mathbb{C}) \longrightarrow SO(3) \tag{3}$$

of the rotation group in three dimensions. It is straightforward to restrict the maps above to the reals, obtaining the double covers

$$SL(2;\mathbb{R}) \longrightarrow SO(2,1) \tag{4}$$

$$SU(2;\mathbb{R}) \longrightarrow SO(2). \tag{5}$$

Since determinants of non-Hermitian matrices over the division algebras $\mathbb{H}$ and $\mathbb{O}$ are not well-defined, we seek alternative characterizations of these complex matrix groups which do not involve such determinants. The key idea is that the determinant of (2×2 and 3×3) Hermitian matrices over any division algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is well-defined, and therefore so is the notion of determinant-preserving transformations. We therefore define

$$TSL(2;\mathbb{H}) := \big\{T_M : \det\big(T_M(X)\big) = \det X \quad \forall\, X \in \mathbf{H}_2(\mathbb{H})\big\} \tag{6}$$

to be the set of determinant-preserving transformations in the quaternionic case, where M is now a quaternionic 2×2 matrix. It is straightforward to verify that $TSL(2;\mathbb{H})$ is a group under composition, and that

$$TSL(2;\mathbb{H}) \cong SO(5,1) \tag{7}$$

under which we identify quaternionic linear transformations of the form (1) with the corresponding Lorentz transformations in $\mathbb{R}^{5+1}$.

We also have the *spinor action* of 2×2 quaternionic matrices M on 2-component column vectors, namely

$$S_M(v) = Mv \tag{8}$$

with $v \in \mathbb{H}^2$. We now *define* $SL(2;\mathbb{H})$ to be the spinor transformations S_M such that the corresponding (vector) transformation T_M is determinant-preserving, that is,

$$SL(2;\mathbb{H}) := \{S_M : T_M \in TSL(2;\mathbb{H})\} \tag{9}$$

and it is straightforward to verify that this set of linear transformations is a group under composition. Furthermore, the map

$$\begin{aligned} SL(2;\mathbb{H}) &\longrightarrow TSL(2;\mathbb{H}) \\ S_M &\longmapsto T_M \end{aligned} \tag{10}$$

is again easily seen to be a two-to-one homomorphism, this time with kernel $\{S_{\pm I}\}$, leading to the double cover

$$SL(2;\mathbb{H}) \longrightarrow SO(5,1). \tag{11}$$

Requiring in addition that $\operatorname{tr}(MXM^\dagger) = \operatorname{tr} X$ for all $X \in \mathbf{H}_2(\mathbb{H})$, and repeating the above construction, leads to the subgroup $SU(2;\mathbb{H}) \subset SL(2;\mathbb{H})$ and the double cover

$$SU(2;\mathbb{H}) \longrightarrow SO(5). \tag{12}$$

Generalizing these groups to $\mathbb{O}$ must be done with some care due to the lack of associativity; for this reason, most authors discuss the corresponding Lie algebras instead. However, since composition of transformations of the forms (1) or (8) is associative, the above construction can indeed be generalized [13], provided care is taken that (1) itself is well-defined, that is, provided we require M to satisfy

$$M(XM^\dagger) = (MX)M^\dagger \tag{13}$$

for all $X \in \mathbf{H}_2(\mathbb{O})$. In order to be able to later combine spinor transformations S_M with vector transformations T_M, we also require our transformations to be *compatible* [13], [11] with the mapping from spinors to vectors given by $v \mapsto vv^\dagger$. Explicitly, we require

$$S_M(v)\big(S_M(v)\big)^\dagger = T_M(vv^\dagger) \tag{14}$$

or in other words

$$(Mv)(v^\dagger M^\dagger) = M(vv^\dagger)M^\dagger \tag{15}$$

for all $v \in \mathbb{O}^2$. Conditions (13) and (15) turn out to be equivalent to the assumption that M is complex [1] and that

$$\det M \in \mathbb{R}. \tag{16}$$

[1] A *complex matrix* is one whose elements lie in a complex subalgebra of the division algebra in question, in this case $\mathbb{O}$. Each such matrix has a well-defined determinant. It is important to note that there is no requirement that the elements of two such matrices lie in the *same* complex subalgebra.

We therefore let $\mathbf{M}_2(\mathbb{O})$ denote the set of *complex* 2×2 octonionic matrices which have real determinant, and note that the corresponding vector transformations (1) are determinant-preserving precisely when $\det(M) = \pm 1$.

We are finally ready to define the octonionic transformation groups by generalizing (6), noting that the composition of linear transformations is associative even when the underlying matrices are not (since the order of operation is fixed). However, in order to generate the entire group, (compatible) transformations must be *nested*; the action of a composition of transformations cannot in general be represented by a single transformation. We therefore generalize (6) by defining

$$TSL(2;\mathbb{O}) := \Big\langle \{T_M : M \in \mathbf{M}_2(\mathbb{O}), \det(M) = \pm 1\} \Big\rangle \tag{17}$$

where the angled brackets denote the span of the listed elements under composition, and it is of course then straightforward to verify that $TSL(2;\mathbb{O})$ is a group under composition. A similar definition can be given for the spinor transformations, namely

$$SL(2;\mathbb{O}) := \Big\langle \{S_M : M \in \mathbf{M}_2(\mathbb{O}), \det(M) = \pm 1\} \Big\rangle. \tag{18}$$

Since each transformation in $TSL(2;\mathbb{O})$ preserves the determinant of elements of $\mathbf{H}_2(\mathbb{O})$, it is clearly (isomorphic to) a subgroup of $SO(9,1)$. Manogue and Schray [13] showed, in slightly different language, that in fact

$$TSL(2;\mathbb{O}) \cong SO(9,1) \tag{19}$$

by giving an explicit set of basis elements which correspond to the standard rotations and boosts in $SO(9,1)$. Furthermore, it is easy to see that the map

$$\begin{aligned} SL(2;\mathbb{O}) &\longrightarrow TSL(2;\mathbb{O}) \\ S_M &\longmapsto T_M \end{aligned} \tag{20}$$

is a two-to-one homomorphism with kernel $\{S_{\pm I}\}$, which establishes the double covers

$$SL(2;\mathbb{O}) \longrightarrow SO(9,1) \tag{21}$$

$$SU(2;\mathbb{O}) \longrightarrow SO(9) \tag{22}$$

(where $SU(2;\mathbb{O})$ is defined as for $SU(2;\mathbb{H})$ by restricting to trace-preserving transformations), which are known results usually stated at the Lie algebra level.

Despite the separate definitions presented above for $SL(2;\mathbb{C})$, $SL(2;\mathbb{H})$, and $SL(2;\mathbb{O})$, a uniform definition can be given for any division algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, modeled on the definition over $\mathbb{O}$. The basis used by Manogue and Schray [13] consists of only two types of transformations: single transformations corresponding to matrices of determinant $+1$, and compositions of two transformations, each corresponding to matrices of determinant -1; in this sense, each basis transformation can be thought of as being "of determinant $+1$". If we now define

$$\begin{aligned} SL_1(2;\mathbb{K}) &:= \{S_M : M \in \mathbf{M}_2(\mathbb{K}),\ \det M = +1\} \\ SL_2(2;\mathbb{K}) &:= \{S_P \circ S_Q : P, Q \in \mathbf{M}_2(\mathbb{K}),\ \det P = -1 = \det Q\} \\ SL(2;\mathbb{K}) &:= \langle SL_1(2;\mathbb{K}) \cup SL_2(2;\mathbb{K})\rangle \end{aligned} \tag{23}$$

where $\mathbf{M}_2(\mathbb{K})$ denotes the set of *complex* 2×2 matrices over $\mathbb{K}$, we recover the above definitions when $\mathbb{K} = \mathbb{H}, \mathbb{O}$, while retaining agreement with the standard definitions when $\mathbb{K} = \mathbb{R}, \mathbb{C}$ (under the usual identification of matrices with linear transformations). A similar definition can be made for $SU(2;\mathbb{K})$ by restricting to trace-preserving transformations.

We can extend this treatment to the higher rank groups: There is a natural action of $SL(n;\mathbb{C})$ as determinant-preserving linear transformations of $n \times n$ Hermitian (complex) matrices, with the unitary matrices $SU(n;\mathbb{C})$ additionally preserving the trace of $n \times n$ Hermitian (complex) matrices, since

$$\operatorname{tr}(MXM^\dagger) = \operatorname{tr}(M^\dagger MX) \tag{24}$$

and $M^\dagger M = I$ for $M \in SU(n;\mathbb{C})$, and these groups could be defined as (the covering groups of) those groups of transformations. Analogous results hold for $SL(n;\mathbb{H})$ and $SU(n;\mathbb{H})$ (and of course also for $SL(n;\mathbb{R})$ and $SU(n;\mathbb{R})$).

When extending these results to octonionic Hermitian matrices, we consider only the 2×2 case discussed above and the 3×3 case, constituting the *exceptional Jordan algebra* $\mathbf{H}_3(\mathbb{O})$, also known as the *Albert algebra*. In both cases, the determinant is well defined (see below). The group preserving the determinant in the 3×3 case is known to be (a particular noncompact real form of) E_6; we can interpret this as

$$E_6 := TSL(3;\mathbb{O}) \cong SL(3;\mathbb{O}). \tag{25}$$

Furthermore, the identity (24) from the complex case still holds for $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$ (and suitable $\mathcal{M} \in E_6$, as discussed below) in the form

$$\operatorname{tr}(\mathcal{M}\mathcal{X}\mathcal{M}^\dagger) = \operatorname{Re}\Big(\operatorname{tr}(\mathcal{M}^\dagger\mathcal{M}\mathcal{X})\Big) \tag{26}$$

where the right-hand side reduces to $\operatorname{tr}(\mathcal{X})$ if $\mathcal{M}^\dagger\mathcal{M} = \mathcal{I}$. The group which preserves the trace of matrices in $\mathbf{H}_3(\mathbb{O})$ is just (the compact real form of) F_4 [8], which we can interpret as

$$F_4 \cong SU(3;\mathbb{O}). \tag{27}$$

(There is no double-cover involved in (25) and (27), since these real forms are simply-connected.) At the Lie algebra level, this has been explained by Sudbery [19] and at the group level this has been discussed by Ramond [15] and Freudenthal [7].

The remainder of this paper uses the above results from the 2×2 case to provide an explicit construction of both F_4 and E_6 at the group level, and discusses their properties.

3. Generators of E_6

We consider octonionic 3×3 matrices $\mathcal{M}$ acting on octonionic Hermitian 3×3 matrices $\mathcal{X}$, henceforth called *Jordan matrices*, in analogy with (1), that is

$$T_{\mathcal{M}}(\mathcal{X}) = \mathcal{M}\mathcal{X}\mathcal{M}^\dagger. \tag{28}$$

For this to be well-defined, $\mathcal{M}\mathcal{X}\mathcal{M}^\dagger$ must be Hermitian and hence independent of the order of multiplication. Just as was noted by Manogue and Schray [13] in the 2×2 case, the necessary and sufficient conditions for this are either that $\mathcal{M}$ be complex or that the columns of the imaginary part of $\mathcal{M}$ be (real) multiples of each other. As with $SL(2;\mathbb{O})$, we will restrict ourselves to the case where $\mathcal{M}$ is complex; this suffices to generate all of E_6.

3.1 Jordan matrices. The Jordan matrices form the exceptional Jordan algebra $\mathbf{H}_3(\mathbb{O})$ under the commutative (but not associative) *Jordan product* (see e.g. [8], [16])

$$\mathcal{X}\circ\mathcal{Y} = \frac{1}{2}(\mathcal{X}\mathcal{Y}+\mathcal{Y}\mathcal{X}). \tag{29}$$

The *Freudenthal product* of two Jordan matrices is given by

$$\mathcal{X}*\mathcal{Y} = \mathcal{X}\circ\mathcal{Y} - \frac{1}{2}(\mathcal{X}\ \mathrm{tr}(\mathcal{Y}) + \mathcal{Y}\ \mathrm{tr}(\mathcal{X})) - \frac{1}{2}\Big(\mathrm{tr}(\mathcal{X}\circ\mathcal{Y}) - \mathrm{tr}(\mathcal{X})\ \mathrm{tr}(\mathcal{Y})\Big) \tag{30}$$

where the identity matrix is implicit in the last term. The *triple product* of 3 Jordan matrices is defined by

$$[\mathcal{X},\mathcal{Y},\mathcal{Z}] = (\mathcal{X}*\mathcal{Y})\circ\mathcal{Z}. \tag{31}$$

Finally, the *determinant* of a Jordan matrix is defined by

$$\det\mathcal{X} = \frac{1}{3}\ \mathrm{tr}[\mathcal{X},\mathcal{X},\mathcal{X}]. \tag{32}$$

Remarkably, Jordan matrices satisfy the usual characteristic equation

$$\mathcal{X}^3 - (\mathrm{tr}\,\mathcal{X})\,\mathcal{X}^2 + \sigma(\mathcal{X})\,\mathcal{X} - (\det\mathcal{X})\,\mathcal{I} = 0 \tag{33}$$

where we must be careful to define

$$\mathcal{X}^3 := \mathcal{X}^2\circ\mathcal{X} \equiv \mathcal{X}\circ\mathcal{X}^2 \tag{34}$$

and where the coefficient $\sigma(\mathcal{X})$ is given by

$$\sigma(\mathcal{X}) := \mathrm{tr}(\mathcal{X}*\mathcal{X}) = \frac{1}{2}\Big((\mathrm{tr}\,\mathcal{X})^2 - \mathrm{tr}(\mathcal{X}^2)\Big). \tag{35}$$

3.2 $SO(9,1)$**.** Consider first matrices of the form

$$\mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \tag{36}$$

where $M \in SL(2;\mathbb{O})$ is one of the generators given by Manogue and Schray [13]. These generators include straightforward generalizations of the standard representation of $SL(2;\mathbb{C})$ in terms of 3 rotations and 3 boosts, yielding 15 rotations and 9 boosts, together with particular nested phase transformations (imaginary multiples of the identity matrix), yielding the remaining 21 rotations corresponding to rotations of the imaginary units ($SO(7)$). Each such generator is complex and has real determinant; for further details, and an explicit list of generators, see [13]. Since

$$\mathcal{M} \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \mathcal{M}^\dagger = \begin{pmatrix} MXM^\dagger & M\theta \\ (M\theta)^\dagger & n \end{pmatrix} = \begin{pmatrix} T_M(X) & S_M(\theta) \\ \big(S_M(\theta)\big)^\dagger & n \end{pmatrix} \tag{37}$$

and using the fact that

$$\det \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} = (\det X)n + 2X \cdot \theta\theta^\dagger \tag{38}$$

where

$$X \cdot Y = \frac{1}{2}\Big(\mathrm{tr}(X \circ Y) - \mathrm{tr}(X)\ \mathrm{tr}(Y)\Big) \tag{39}$$

is the Lorentzian inner product in $9+1$ dimensions, it is straightforward to verify that $T_{\mathcal{M}}$ preserves the determinant of a Jordan matrix $\mathcal{X}$, and is hence in E_6, if $\mathcal{M}$ is of the form (36). This shows that

$$SL(2;\mathbb{O}) \subset E_6 \tag{40}$$

as expected, where, as already noted, $SL(2;\mathbb{O})$ is the double cover of $SO(9,1)$. Since $SL(2;\mathbb{O})$ acts as $SO(9,1)$ on X in (37), we will somewhat loosely describe $SO(9,1)$ itself (and its subgroups) as being "in" E_6.

This construction in fact yields three obvious copies of $SO(9,1)$ contained in E_6, corresponding to the three natural ways of embedding a 2×2 matrix inside a 3×3 matrix. These three copies are related by the *cyclic permutation matrix*

$$\mathcal{T} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{41}$$

which satisfies

$$\mathcal{T}^{-1} = \mathcal{T}^2 = \mathcal{T}^\dagger \tag{42}$$

and which is clearly in E_6.

Conversely, all elements of E_6 can be built up out of these (three sets of) $SO(9,1)$ transformations.

3.3 $SO(8)$, triality, and F_4. Since each copy of $SO(9,1)$ is 45-dimensional, but the dimension of E_6 is only 78, it is clear that our description so far must contain some redundancy. We note first of all that (37) contains not only the vector representation (1) of $SO(9,1)$, but also the dual spinor representations

$$\begin{aligned} \theta &\longmapsto M\theta \\ \theta^\dagger &\longmapsto \theta^\dagger M^\dagger \end{aligned} \tag{43}$$

and therefore combines 2×2 vector and spinor representations into a single 3×3 representation. *Triality* says that, for $SO(8)$, these three representations are isomorphic.[2]

To see explicitly why triality holds, we begin with the description of $SO(8)\subset SO(9,1)$ from [13]. Since $SO(8)$ transformations of the form (1) leave the diagonal of X invariant, these transformations correspond to the "transverse" degrees of freedom in $SO(9,1)$. One might therefore expect $SO(8)$ transformations to take the form $\left(\begin{smallmatrix} q & 0 \\ 0 & r \end{smallmatrix}\right)$ with $|q| = |r| = 1$. However, the essential insight of [13] was to require that all $SO(9,1)$ transformations be *compatible*, that is, that they (be generated by matrices which) satisfy (15); we will see the importance of this requirement below. This condition restricts the allowed form of $SO(8)$ transformations to those which can be constructed from (2×2) diagonal matrices which are either imaginary multiples of the identity matrix, or of the form

$$M = \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix} \tag{44}$$

where $|q| = 1$, so that $q = e^{s\theta}$ for some imaginary unit $s\in\mathbb{O}$ with $s^2 = -1$. As discussed in [13], the matrix (44) induces a rotation in the $(1,s)$-plane through an angle 2θ. Furthermore, $SO(7)$ transformations, namely those leaving invariant the identify element 1, can be constructed by suitably nesting an even number of purely imaginary matrices of the form (44), that is, matrices of this form for which $\theta = \frac{\pi}{2}$. This allows us to generate all of $SO(8)$ using matrices which have determinant 1. Alternatively, $SO(7)$ transformations can also be obtained by nesting imaginary multiples of the identity matrix (which have determinant -1), since this involves an even number of sign changes when compared with the above description.

[2]The term *triality* appears to have first been used by Cartan [2], who used it to describe the symmetries of the Dynkin diagram of $SO(8)$. An infinitesimal *principle of triality* in the language of derivations is proved in [16], which credits Jacobson [9] with the analogous theorem for Lie groups. Baez [1] describes the four *normed trialities* as trilinear maps on representations of (particular) Lie algebras, and discusses their relationship to the four normed division algebras and their automorphisms. A similar treatment can be found in Conway and Smith [3].

Inserting (44) into (36), the resulting E_6 transformation $\mathcal{M}$ leaves the diagonal of a Jordan matrix $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$ invariant. Explicitly, writing

$$\mathcal{X} = \begin{pmatrix} p & \overline{a} & c \\ a & m & \overline{b} \\ \overline{c} & b & n \end{pmatrix} \tag{45}$$

we see that the action (28) leaves p, m, n invariant, and acts on the octonions a, b, c via

$$\begin{aligned} a &\longmapsto \overline{q}a\overline{q} \\ b &\longmapsto bq \\ c &\longmapsto qc. \end{aligned} \tag{46}$$

These three transformations are precisely the standard description of the (vector and two spinor) representations of $SO(8)$ in terms of symmetric, left, and right multiplication by unit octonions. The actions (46) provide an implicit mapping between these three representations (obtained by using the same q in each case),[3] which is clearly both a (local) diffeomorphism and a 1-to-1 map between the two spinor representations, and a 2-to-1 map between either spinor representation and the vector representation. For us, *triality* is this explicit relationship between the three representations.

In our language, this means that if M in (36) is an $SO(8)$ transformation, then not only does $\mathcal{M}$ generate an $SO(8)$ transformation on $\mathcal{X}$ via (37), but so do $\mathcal{T}\mathcal{M}\mathcal{T}^2$ and $\mathcal{T}^2\mathcal{M}\mathcal{T}$, since each of these latter two transformations differs from $\mathcal{M}$ merely in which representation of $SO(8)$ acts on each of a, b, c. Even though these individual transformations are different, the collection of all of them is the same in each case. Note that this identification of the three copies of $SO(8)$ is only possible because the original $SO(9,1)$ transformation was assumed to be compatible. Thus, (the double cover of) $SO(8)$ is precisely the subgroup of E_6 which leaves the diagonal of every Jordan matrix $\mathcal{X}$ invariant.

The dimension of the single resulting copy of $SO(8)$ is 28. Adding in the $3 \times 8 = 24$ additional rotations in (the three copies of) $SO(9)$ yields 52, the correct dimension for F_4. Including the $3 \times 9 - 1 = 26$ independent boosts gives the full 78 generators of E_6. Thus, triality fully explains the redundancy in our original 135 generators.

At the Lie algebra level, the dimension of E_6 can be determined by first noting that the diagonal elements are not independent. It turns out that an independent set can be taken to be the 64 independent tracefree matrices, together with the 14 generators of G_2 (see below), for a total of 78 generators. Of these, 26 (24

[3] To verify this assertion, one must first argue that the implicit map between representations in (46) is well-defined. At issue is the uniqueness of representations such as $a \longmapsto q_1(\dots(q_m a)\dots)$ for a particular $SO(8)$ transformation, and specifically whether this notion of uniqueness is the same for the three representations. One way to show this is to explicitly construct the maps between the representations.

non-diagonal + 2 diagonal) are Hermitian and hence boosts (the third diagonal Hermitian generator is not independent); the remaining 52 generators yield F_4. In fact, the (complex) generators of $SO(9,1)$ as given in [13] all satisfy either $MM^\dagger = I$ (rotations) or $M = M^\dagger$ (boosts). But F_4 is generated precisely by the unitary elements of E_6, and hence is generated by the (3 sets of) rotations in $SO(9)$.

3.4 G_2**.** As discussed in [13], the automorphism group of the octonions, G_2, can be constructed by suitably combining rotations of the octonionic units, thus providing explicit verification that G_2 is a subgroup of $SO(7)$. In particular, a copy of G_2 sits naturally inside each $SO(9,1)$, generated by 14 (nested) imaginary multiples of the identity matrix (the "additional transverse rotations" of [13], also denoted "flips").[4] We thus appear to have three copies of G_2 sitting inside E_6, one for each copy of $SO(9,1)$.

As further shown in [13], the automorphisms of $\mathbb{O}$ can be generated by octonions of the form

$$e^{\hat{q}\theta} = \cos(\theta) + \hat{q}\sin(\theta) \tag{47}$$

with $\hat{q}$ a pure imaginary, unit octonion, but where θ must be restricted to be a multiple of $\pi/3$, corresponding to the sixth roots of unity. But, as can be verified by direct computation, multiplying the identity matrix by such an automorphism leads to an element of E_6, thus giving us yet another apparent copy of G_2 in E_6.

Remarkably, due to triality, all four of these subgroups are the same.

To see this, consider the rotations by $\frac{\pi}{2}$ ("flips") used in [13] to generate the transverse rotations. Using the identification (36), such a transformation takes the form

$$\mathcal{Q}_{\hat{q}} = \begin{pmatrix} \hat{q}I & 0 \\ 0 & 1 \end{pmatrix} \tag{48}$$

and we have

$$\mathcal{Q}_{\hat{q}} \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \mathcal{Q}_{\hat{q}}^\dagger = \begin{pmatrix} -\hat{q}X\hat{q} & \hat{q}\theta \\ -\theta^\dagger\hat{q} & n \end{pmatrix}. \tag{49}$$

Under this transformation, X, θ, and $\theta^\dagger$ undergo separate $SO(7)$ transformations, related by triality. We emphasize that, in general, the off-diagonal elements of $\mathcal{X}$ undergo *different* $SO(7)$ transformations. (The diagonal elements are of course fixed by any such transformation.)

Acting on a single octonion, nested sequences of these $SO(7)$ transformations can be used to generate G_2. For instance, conjugating successively with

$$\hat{q} = i, i\cos\theta + i\ell\sin\theta, j, j\cos\theta - j\ell\sin\theta \tag{50}$$

[4] Even though these three copies of $SO(7)$ all live in the single copy of $SO(8)$ described above, they are not the same.

yields a G_2 transformation which leaves the quaternionic subalgebra generated by k and ℓ fixed. What happens when this sequence of $\hat{q}$'s is applied as E_6 transformations, that is, in the form $\mathcal{Q}_{\hat{q}}$? Remarkably, direct computation shows in this case that the elements of $\mathcal{X}$ all undergo the same G_2 transformation. Since all G_2 transformations can be generated by such transformations, triality is, in this sense, the identity map on G_2! The G_2 transformation obtained by suitably nesting $\mathcal{Q}_{\hat{q}}$'s is therefore the same as the G_2 transformation obtained by replacing $\mathcal{Q}_{\hat{q}}$ by $\hat{q}\mathcal{I}$ at each step. This shows that the three G_2 subgroups contained in the three copies of $SO(9,1)$ are all identical to the "diagonal" G_2 subgroup, as claimed above.

An explicit example of triality-related automorphisms is given by [5]

$$k(j(iq)) = k(j(iq\bar{\imath})\bar{\jmath})\overline{k} = (((q\bar{\imath})\bar{\jmath})\overline{k}) \tag{51}$$

with $q \in \mathbb{O}$, which realizes "ℓ-conjugation" as a linear map.

4. Cayley spinors

We have argued elsewhere [10], [5] that the ordinary momentum-space (massless and massive) Dirac equation in $3+1$ dimensions can be obtained via dimensional reduction from the Weyl (massless Dirac) equation in $9+1$ dimensions. The dimensional reduction is accomplished by the simple expedient of choosing a preferred complex subalgebra of the octonions, thus reducing $SL(2;\mathbb{O})$ to $SL(2;\mathbb{C})$, and hence the Lorentz group in 10 spacetime dimensions to that in 4 dimensions.

The massless Dirac equation in 10 spacetime dimensions can be written in momentum space as the eigenvalue problem

$$\widetilde{P}\psi = 0 \tag{52}$$

where P is a 2×2 octonionic Hermitian matrix corresponding to the 10-dimensional momentum vector, $\psi \in \mathbb{O}^2$ is a 2-component octonionic column, corresponding to a Majorana-Weyl spinor, and where tilde denotes trace reversal, that is

$$\widetilde{P} = P - \mathrm{tr}(P)\, I. \tag{53}$$

The general solution of (52) is

$$\begin{aligned} P &= \pm\theta\theta^\dagger && (54)\\ \psi &= \theta\xi && (55) \end{aligned}$$

where $\theta \in \mathbb{O}^2$ is a 2-component octonionic vector whose components lie in the same complex subalgebra of $\mathbb{O}$ as do those of P, and where $\xi \in \mathbb{O}$ is arbitrary. (Such a θ must exist since $\det(P) = 0$.)

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[5]Further examples can be found in [11].

In [5], we further showed how to translate the standard treatment of the Dirac equation in terms of gamma matrices into octonionic language, pointing out that a 2-component *quaternionic* formalism is of course isomorphic to the traditional 4-component complex formalism. Remarkably, the above solutions to the *octonionic* Dirac equation must be quaternionic, as they only involve 2 independent octonionic directions. This allows solutions of the octonionic Dirac equation to be interpreted as standard fermions — and one can fit precisely 3 "families" of such quaternionic solutions into the octonions, which we interpret as generations. For further details, see [5], or the more recent treatment in [12].

As outlined in [4], it is natural to introduce a 3-component formalism; this approach was first suggested to us by Fairlie and Corrigan [6], and later used by Schray [18], [17] for the superparticle. Defining

$$\Psi = \begin{pmatrix} \theta \\ \xi \end{pmatrix} \tag{56}$$

we have first of all that

$$\mathcal{P} := \Psi\Psi^\dagger = \begin{pmatrix} P & \psi \\ \psi^\dagger & |\xi|^2 \end{pmatrix} \tag{57}$$

so that Ψ combines the bosonic and fermionic degrees of freedom. Lorentz transformations on both the vector P and the spinor ψ now take the elegant form (37), which we used to view $SO(9,1)$ as a subgroup of E_6; the rotation subgroup $SO(9)$ lies in F_4. We refer to Ψ as a *Cayley spinor*.

Direct computation shows that the Dirac equation (52) is equivalent to the equation

$$\mathcal{P} * \mathcal{P} = 0 \tag{58}$$

whose solutions are precisely *quaternionic* matrices of the form (57), that is, (the components of) θ and ξ must lie in a quaternionic subalgebra of $\mathbb{O}$; Ψ is a quaternionic Cayley spinor. But the *Cayley plane* $\mathbb{OP}^2$ consists of those elements $\mathcal{P} \in \mathbf{H}_3(\mathbb{O})$ which satisfy

$$\mathcal{P} \circ \mathcal{P} = \mathcal{P}; \qquad \operatorname{tr} \mathcal{P} = 1 \tag{59}$$

and this turns out to be equivalent to requiring $\mathcal{P}$ to be a (normalized) solution of (58). Thus, in this interpretation, the Cayley plane consists precisely of normalized, quaternionic Cayley spinors, and these are precisely the (normalized) solutions of the Dirac equation.

Furthermore, any Jordan matrix can be decomposed in the form [14]

$$\mathcal{A} = \sum_{i=1}^{3} \lambda_i \mathcal{P}_i \tag{60}$$

where

$$\mathcal{A} \circ \mathcal{P}_i = \lambda_i \mathcal{P}_i \tag{61}$$

and

$$\mathcal{P}_i \circ \mathcal{P}_j = 0 \qquad (i \neq j) \tag{62}$$

so that the $\mathcal{P}_i$ are orthogonal eigenmatrices of $\mathcal{A}$ with eigenvalue λ_i. As discussed in [4], we refer to the decomposition (60) as a *p-square decomposition* of $\mathcal{A}$, with p denoting the number of with p denoting the number of nonzero eigenvalues λ_i, and hence the number of nonzero primitive idempotents in the decomposition. As shown in [4], if $\det(\mathcal{A}) \neq 0$, then $\mathcal{A}$ is a 3-square, while if $\det(\mathcal{A}) = 0 \neq \sigma(\mathcal{A})$, then $\mathcal{A}$ is a 2-square. Finally, if $\det(\mathcal{A}) = 0 = \sigma(\mathcal{A})$, then $\mathcal{A}$ is a 1-square (unless also $\operatorname{tr}(\mathcal{A}) = 0$, in which case $\mathcal{A} \equiv 0$). It is intriguing that, since E_6 preserves both the determinant and the condition $\sigma(\mathcal{A}) = 0$, E_6 therefore preserves the class of p-squares for each p. But solutions of the Dirac equation (58) are 1-squares! Thus, the Dirac equation in 10 dimensions admits E_6 as a symmetry group.

The particle interpretation described in [10], [5] suggests regarding 1-squares as representing three generations of leptons. If 1-squares correspond to leptons, could it be that 2-squares are mesons and 3-squares are baryons?

5. The structure of E_6

We have shown that the massless 10-dimensional Dirac equation, originally posed as an eigenvalue problem for 2×2 octonionic Hermitian matrices, is in fact equivalent to the defining condition for the Cayley plane. This suggests that the natural arena for the Dirac equation is a 3-component formalism involving Cayley spinors, which explicitly incorporates both bosonic and fermionic degrees of freedom, suggesting a natural supersymmetry. Furthermore, the symmetry group of the Dirac equation has been shown to be E_6, suggesting that E_6 (or possibly one of its larger cousins, E_7 or E_8) is the natural symmetry group of fundamental particles.

Understanding the structure of (this particular real representation of) E_6 may therefore be of great importance to an ultimate understanding of fundamental particles. In this regard, we call the reader's attention to the recent work of Aaron Wangberg [20], [21], which describes the real representations of E_6 and its physically important subgroups. A "map" of E_6, excerpted from [20], appears in Figure 1.

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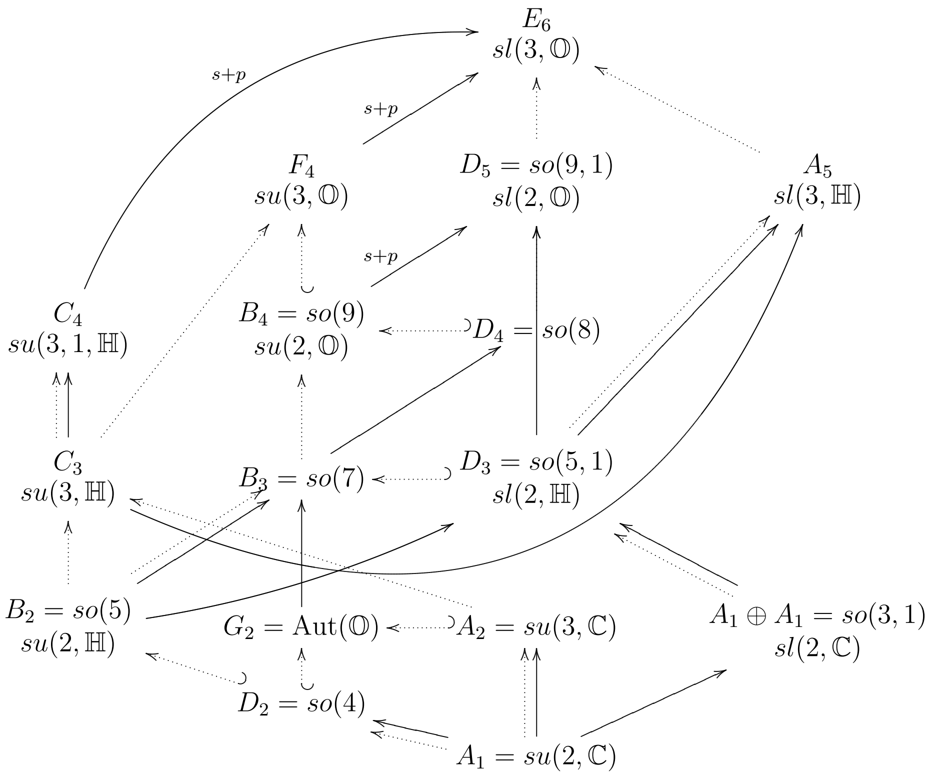

Figure 1. A map of E_6 (taken from [20]).

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