Nonassociativity in VOA theory and finite group theory

Robert L. Griess, Jr.

Abstract. We discuss some examples of nonassociative algebras which occur in VOA (vertex operator algebra) theory and finite group theory. Methods of VOA theory and finite group theory provide a lot of nonassociative algebras to study. Ideas from nonassociative algebra theory could be useful to group theorists and VOA theorists.

Keywords: nonassociative algebra, nonassociative commutative algebra, groups of Lie type, sporadic groups, vertex operator algebras, lattice type vertex operator algebras, axioms, (B, N)-pair, monster, $2A$-involutions, Jordan algebra, pairwise orthogonal idempotents, E_8, E_6, polynomial identity

Classification: 20D06, 20D08, 17A01, 17B69

1. Introduction

This article is motivated by concerns from finite group theory.

The first is that we do not have good axiom systems for all of the finite simple groups. The second is that we do not really understand how the sporadic simple groups fit into mathematics.

An answer to the first concern could help us with the second. A good theory of some relevant nonassociative algebras might be a suitable answer.

We hope that this article will encourage nonassociative theorists to think about connections with finite simple groups, especially the sporadic groups. Technicalities are kept to a minimum and references are provided.

1.1 A condensed list of the finite simple groups.

The finite simple groups
The alternating groups, $\mathrm{Alt}_n, n \geq 5$
Finite groups of Lie type, $A_n(q), B_n(q), \dots, E_8(q), {}^2A_n(q), \dots, {}^2F_4(q)$ (q is a prime power) (for example, $A_n(q)$ is $\mathrm{PSL}(n+1, q)$, determinant 1 matrices mod scalars over $\mathbb{F}_q$, the finite field of q elements; ${}^2A_n(q)$ is $\mathrm{PSU}(n+1, q)$,

The author acknowledges support from grant NSF (DMS-0600854).

$B_n(q)$ is $\mathrm{PSO}(2n+1,q)$, etc.)

The 26 sporadic groups:

M_{11} (this is the first Mathieu group; it is the smallest sporadic group, order $7920 = 2^4 3^2 5{\cdot}11$)

...

$F_1 = \mathbb{M}$ (the largest, order $2^{46}3^{20}5^9 7^6 11^2 13^3 17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}41{\cdot}47{\cdot}59{\cdot}71 \sim 8\times 10^{53}$)

2. Theory for groups of Lie type

Most finite simple groups are groups of Lie type, so we concentrate on them first. These are, roughly, analogues over finite fields of the simple real and complex Lie groups. The starting point is the definition of a Lie algebra, given by a few simple axioms. One can quickly derive consequences to make a structure theory. In a one-term course, it is possible to classify all the simple finite dimensional Lie algebras over the complex numbers and describe significant results in representation theory.

For groups of Lie type, we have an axiom system derived from Lie theory:

Definition 2.1. A (B,N) *pair* is a pair of subgroups B and N of a group G such that the following axioms hold.

G is generated by B and N.
The intersection $H := B\cap N$ is a normal subgroup of N.
The group $W := N/H$ is generated by a nonempty set S of elements of order 2 such that:

- if sH is one of the generators of W and n is any element of N, then $sBn \subseteq BsnB \cup BnB$;
- if $sH \in S$ then sH contains no element which normalizes B.

Example 2.2. $G = \mathrm{GL}(m,K)$, $B =$ invertible upper triangular matrices, $N=$ all invertible monomial matrices (diagonal times permutation matrix), $H=$ diagonal matrices in G, $W \cong \mathrm{Sym}_m$.

These axioms lead to uniform proofs for structure theory, representation theory, conjugacy, etc. They predict and explain a lot. For example, there are uniform arguments for many aspects of GL_n, orthogonal, symplectic groups, G_2, F_4, E_6, etc. which complement or replace theories of these groups as isometry groups of forms (e.g., orthogonal and symplectic groups), as automorphism groups of algebras (e.g., groups of type G_2 and F_4) or otherwise.

3. Theory for sporadic groups

We would like a theory with similar uniform qualities for sporadic groups. None is known at this time, despite decades of study.

There are reasons for hope that the world of commutative associative algebras will offer help to finding a theory. The following is generally true. Let G be any finite group and Ω a G-set. Let K be a field. Form the permutation module $K\Omega$. We can define an algebra structure on the permutation module by making $\alpha \cdot \beta =: \delta_{a\beta}\,\alpha$, for $\alpha, \beta \in \Omega$. The permutation module is then a direct sum of fields, indexed by Ω, and G permutes the indecomposable summands and acts as a group of automorphisms.

Notice that there is a G-invariant form based on making Ω an orthonormal basis.

Now suppose that S is any algebra and that T is a subspace. Let U complement T, so that $S = T \oplus U$ as vector spaces. Let π be the projection to T. We may define an algebra structure on T as follows. For $x, y \in T$, define $x * y = \pi(xy)$, where xy means the product in S. The algebra $(T, *)$ is called the *contraction* of S to T (more correctly, *the contraction with respect to the direct sum decomposition* $S = T \oplus U$). Note that if S is commutative, so is the contraction. However, the contraction of an associative algebra may not be associative.

We now return to the earlier situation with G and $K\Omega$. Take any G-submodule A of $K\Omega$. Let us assume that A is nonsingular and let π be the orthogonal projection $K\Omega \to A$. As above, we define the contraction $(A, *)$ of $K\Omega$. While the product is usually not associative, it has an associative bilinear form $(x, y*z) = (x * y, z)$. Finally, we observe that the action of G preserves the product.

The Monster simple group was first constructed as a group of automorphisms of a 196883-dimensional algebra [10], [11]. We call this algebra $\mathcal{B}_0$. This construction was achieved by piecing together representations of several finite groups and choosing a suitable algebra structure. We point out in the next example that $\mathcal{B}_0$ can be described as a contraction (given existence of the Monster).

Example 3.1. If G is the Monster simple group, order about 10^{54}, and Ω is the conjugacy class of involutions called $2A$, then $|\Omega|$ is about 10^{20}. There is a G-submodule A_0 of the permutation module $\mathbb{Q}\Omega$ so that $\dim(A_0) = 196883$ and that A_0 has dimension 196883. The contraction A_0 is isomorphic to $\mathcal{B}_0$.

After the original construction of $\mathcal{B}_0$ and the Monster, a 196884 dimensional algebra with unit $\mathcal{B}$ was proposed (possibly first in [8]). It contains a copy of $\mathcal{B}_0$ as a subspace and its full automorphism group is the Monster. The algebra $\mathcal{B}$ has become more widely used than the original $\mathcal{B}_0$. One nice feature of $\mathcal{B}$ is that it has a 300-dimensional subalgebra which is the Jordan algebra of degree 24 symmetric matrices. See consequence (4) below.

There is literature on commutative nonassociative algebras A which are related to permutation representations of finite groups, including sporadic groups (or central extension of such). In some cases, there are results on $\mathrm{Aut}(A)$. See later sections in this article; also [26], [28], [21].

Along came the theory of VOAs (vertex operator algebras) in the mid 1980s.

An abbreviated definition of VOA.

$(V, Y, \mathbf{1}, \omega)$ is a VOA means:

- $V = \bigoplus_{n\in\mathbb{Z}} V_n$, each V_n is a finite dimensional complex vector space;
- $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ (Y is the "vertex operator"), so for $a \in V$, $Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, for $a_n \in \mathrm{End}(V)$.
- $\mathbf{1}$ is a *vacuum element* and ω is a *Virasoro element*.

There are many axioms including a kind of Jacobi identity (involving power series in several variables).

The axioms imply that $\mathbf{1} \in V_0$ and $\omega \in V_2$. Therefore, $\dim(V_0) \geq 1$ and $\dim(V_2) \geq 1$. Any dimension is possible for $\dim(V_1)$.

See a full treatment and historical remarks in [8]. The survey [9] is useful.

Consequences.

We focus on a few points involving finite dimensional algebras.

(1) For each k, we have a product on V, $a, b \mapsto a_k(b)$. We call it the k^{th} product. It takes $V_i \times V_j \to V_{i+j-k-1}$. If $i = j = k+1$, then (V_{k+1}, k^{th}) is a finite dimensional algebra.

(2) If V is *CFT type* ($V_n = 0$ for $n \leq -1$, $V_0 = \mathbb{C}\mathbf{1}$), then $(V_1, 0^{th})$ is a Lie algebra. (The abbreviation CFT means "conformal field theory".)

(3) If V is an *OZVOA* (=CFT type and $V_1 = 0$), then $(V_2, 1^{st})$ is *commutative*. (Here we may get associative algebras, classical Jordan algebras, $\mathcal{B}$ and many others).

(4) If G is any group of automorphism of a VOA V, then the set V^G of fixed points is also a VOA. (By definition, an automorphism of a VOA preserves $\mathbf{1}$ and ω and so preserves each V_n. Therefore, $V^G = \bigoplus_n V_n^G$.)

4. Lattice type VOAs and their degree 2 summands

Suppose that L is an even lattice, i.e. a free abelian group in Euclidean space so that $(x, y) \in \mathbb{Z}$ and $(x, x) \in 2\mathbb{Z}$ for all $x, y \in L$.

There is a standard way to make a *lattice VOA* from L. As a linear space it looks like $V_L = \mathbb{S} \otimes \mathbb{C}[L]$, where $\mathbb{C}[L]$ is the group algebra of L (basis e^α, $\alpha \in L$) and where $\mathbb{S}$ is the symmetric algebra on the vector space $(H \otimes t^{-1}) \oplus (H \otimes t^{-2}) \oplus (H \otimes t^{-3}) \oplus \cdots$. For details, see [8]. A *lattice type VOA* is the fixed point subVOA V_L^G, where V_L is a lattice VOA and G is a finite subgroup of $\mathrm{Aut}(V_L)$.

Grading on V is based on $\deg(e^\alpha) = \frac{1}{2}(\alpha, \alpha)$, $\deg(H \otimes t^{-m}) = m$.

In particular, if we take an isometry of the lattice L, it can be lifted to an automorphism of V_L (see [8] and the appendix of [20]). The -1 isometry lifts, and the set of fixed points of the lift is denoted V_L^+. Lifts of the -1 isometry are not unique, but any two are conjugate [20].

Note. Formulas for multiplying the standard basis elements of $(V_L)_2$ are given in several places, including [8] and [16]. Examples: $S^2(H \otimes t^{-1})$ is isomorphic to the Jordan algebra of symmetric matrices with product $A \circ B = \frac{1}{2}(AB + BA)$; also, $(e^\alpha + e^{-\alpha}) * (e^\beta + e^{-\beta}) = \pm(e^{\alpha+\beta} + e^{-\alpha-\beta})$ when α, β are norm 4 vectors such that $(\alpha, \beta) = -2$.

Some care is needed for the "sign function" (though it may be arranged to be identically 1 when $(L, L) \leq 2\mathbb{Z}$, e.g. for the 156-dimensional example described later).

5. The 196884-dimensional algebra $\mathcal{B}$

The algebra $\mathcal{B}$ occurs as the degree 2 piece of $V^\natural$ the Moonshine VOA of Frenkel-Lepowsky-Meurman. This VOA has graded dimension

$$1 + 196884q^2 + 21493760q^3 + 864299970q^4 + 20245856256q^5 + \cdots ,$$

which is the elliptic modular function + constant, with degrees shifted. See [8].

Theorem 5.1 ([17]). *There is no nonzero homogeneous polynomial identity for $\mathcal{B}$ of degree less than or equal to* 5.

Question. A nontrivial homogeneous polynomial identity exists, by finite dimensionality, but can it be of practical use if its degree is high?

There is a lot of work on subalgebras of $\mathcal{B}$ generated by idempotents [25], [4], [24], [26]. A striking result is the following, which proved a conjecture of Meyer-Neutsch [25].

Theorem 5.2 (Miyamoto [24]). *A maximal set of pairwise orthogonal idempotents in $\mathcal{B}$ has cardinality* 48. *Therefore,* 48 *is the maximum dimension of an associative semisimple subalgebra.*

Miyamoto's theory connects idempotents in $\mathcal{B}$ to so-called Virasoro elements in a VOA (these elements generate a subVOA of very special form, a highest weight module for the Virasoro Lie algebra). *The proof of this finite dimensional result uses infinite dimensional techniques!*

6. A 156-dimensional algebra

This example is based on an 8-dimensional lattice $M \cong \sqrt{2}E_8$ [16]. Define H to be the ambient complex vector space $\mathbb{C} \otimes M$.

We take the lattice VOA V_M and its subVOA V_M^+. This is lattice type and its degree 1 term is 0. Its degree 2-part looks, as a linear space, like $S^2(H) \oplus \bigoplus_{\{\alpha,-\alpha\}} \mathbb{C}(e^\alpha + e^{-\alpha})$, where we sum over pairs $\alpha, -\alpha$ of norm 4 vectors in M (there are 120 such pairs). The dimension of H is 8 and the dimension of $S^2(H)$ is $\binom{9}{2} = 36$. So $(V_M^+)_2$ has dimension 120+36=156. Its automorphism group is the finite orthogonal group $O^+(10, 2)$.

This algebra is a subalgebra of $\mathcal{B}$. In $\mathrm{Aut}(\mathcal{B})$, isomorphic to the Monster, the stabilizer of such a subalgebra is a complicated group of the form $2^{10+16}.\Omega^+(10, 2)$. This subgroup induces the group $\Omega^+(10, 2)$ on the subalgebra, an index 2 subgroup of its full automorphism group.

The VOA V_M^+ may also be realized as a subVOA of V_L, the lattice VOA based on the E_8 lattice. It is isomorphic to the fixed points of an elementary abelian group of order 32 in the automorphism group of V_L, isomorphic to $E_8(\mathbb{C})$. More specifically, there are two conjugacy classes of involutions in the Lie group $E_8(\mathbb{C})$, called $2A$ and $2B$. There exists an elementary abelian group of order 2^r which is $2B$-pure (i.e., all involutions are in the class $2B$) if and only if $r \leq 5$, and for each such r, there is a single conjugacy class of such groups. See [1], [18], [19].

7. A 27-dimensional algebra

Here is an example coming from fixed points of a group of *odd order* [21].

Take the root lattice $L = E_6$ and form V_L. There is a group of automorphism, E, of order 3^3 so that $(V_L^E)_1 = 0$ and $(V_L^E)_2$ has dimension 27 and is commutative. It has $3^3{:}\,\mathrm{SL}(3,3)$ in its automorphism group. It fixes 1 and has an irreducible 26-dimensional complement.

At first, one may guess that the algebra $(V_L^E)_2$ is the famous exceptional Jordan algebra. The automorphism group of the exceptional Jordan algebra is the group $F_4(\mathbb{C})$, which does in fact contain a subgroup isomorphic to $3^3{:}\,\mathrm{SL}(3,3)$.

The automorphism group of any finite dimensional algebra is a (possibly disconnected) algebraic group. In particular, such an automorphism group could be finite. A search shows that there are several finite groups with an irreducible degree 26 representation and which contain $3^3{:}\,\mathrm{SL}(3,3)$ as a subgroup. Some of these groups leave invariant a commutative algebra structure on a 26-dimensional representation. For example, one is the finite group $\mathrm{PGL}(4,3)$.

The analysis in [21] shows that the algebra $(V_L^E)_2$ does not satisfy the Jordan identity, so is not the exceptional 27-dimensional Jordan algebra. Finally, it turns out that its automorphism group is the finite group $3^3{:}\,\mathrm{GL}(3,3)$ of affine transformations on $\mathbb{F}_3^3$. We are not aware of other occurrences of the algebra $(V_L^E)_2$. It would be interesting to know about the other 27-dimensional algebras mentioned above.

8. Possible directions

(1) Study some VOAs V with small dimensional degree 2 term to determine identities, connections between the algebra product on V_2 and automorphisms (e.g., idempotents and involutions).

(2) Assume that we are in characteristic 0. An algebra of dimension 196883 which supports the monster as a group of automorphisms is unique up to isomorphism. An algebra of dimension 196884 which as a unit is not uniquely determined by the property of having the monster as a group of automorphisms, but is uniquely determined if a naturally defined 300-dimensional subalgebra is isomorphic to the Jordan algebra of symmetric degree 24 matrices (such a subalgebra is the set of fixed points of the extraspecial group $O_2(C)$ of order 2^{25}, where C is the centralizer of a $2B$-involution).

If the algebra $\mathcal{B}$ has a uniqueness result, purely as an algebra, (without the assumption that it has the monster as automorphism group), there could be important applications to VOA theory, in particular to the open uniqueness problem for the moonshine VOA. One could try to approach this by proving uniqueness results for subalgebras of $\mathcal{B}$.

(3) A context for many commutative (not necessarily associative) algebras is in VOA theory, as the degree 2 piece of some VOA [2], [22], [23]. Can one take a commutative algebra, A, and create a VOA V so that $(V_2, 1st) \cong A$? This is open, and hard. A partial answer is announced in [27]. It may help to assume that there is an associative form (i.e., $(x * y, z) = (x, y * z)$ for all x, y, z).

(4) Is the study of identities on a finite dimensional commutative algebra the right way to go for developing a theory of finite simple groups? Is there a good alternative?

(5) Take a root lattice, L. The automorphism group of the VOA V_L is the adjoint group of type L extended by graph automorphisms [5]. If S is a subgroup of $G := \mathrm{Aut}(V_L)$, then V^S (the subVOA of fixed points) inherits an action of the quotient group $N_G(S)/S$ as automorphisms.

More studies of such VOAs V_L^S would be fascinating. In this article, we have mentioned examples at rank 6 (for the E_6 lattice) and rank 8 (for the E_8 lattice). Examples of this in rank 1 are already quite interesting and nontrivial [6], [7]. For the case where S is a finite group, there is an enormous amount of information available (see the survey [18], [19] to get started on examples). In case $V_1^S = 0$, the finite dimensional algebra $(V_2^S, 1^{st})$ is *commutative* and typically nonassociative, so falls into the general category we considered earlier.

(6) The author has done some work on loops and their relations to structure theory of finite groups, nonassociative algebras and group cohomology [12], [14], [13], [15].

References

[1] Cohen A.M., Griess R.L., Jr., *On finite simple subgroups of the complex Lie group of type E_8*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 367–405, Proc. Sympos. Pure Math., 47, Part 2, Amer. Math. Soc., Providence, RI, 1987.

[2] Ashihara T., Miyamoto M., *Deformation of central charges, vertex operator algebras whose Griess algebras are Jordan algebras*, J. Algebra **321** (2009), no. 6, 1593–1599.

[3] Cohen A.M., Wales D.B., *Finite subgroups of $F_4(C)$ and $E_6(C)$*, Proc. London Math. Soc. (3) **74** (1997), no. 1, 105–150.

[4] Conway J.H., *A simple construction for the Fischer-Griess monster group*, Invent. Math. **79** (1985), no. 3, 513–540.

[5] Dong C., Nagatomo K., *Automorphism groups and twisted modules for lattice vertex operator algebras*, in Recent Developments in Quantum Affine Algebras and Related Topics, Contemp. Math., 248, pp. 117–133, Amer. Math. Soc., Providence, RI, 1999.

[6] Dong C., Griess R.L., Jr., *Rank one lattice type vertex operator algebras and their automorphism groups*, J. Algebra **208** (1998), 262–275. q-alg/9710017

[7] Dong C., Griess, R.L., Jr., Ryba A.J.E., *Rank one lattice type vertex operator algebras and their automorphism groups*, II: E-series, J. Algebra **217** (1999), 701–710.

[8] Frenkel I., Lepowsky J., Meurman, A., *Vertex Operator Algebras and the Monster*, Pure and Applied Math., 134, Academic Press, Boston, 1988.

[9] Gebert R.W., *Introduction to vertex algebras, Borcherds algebras and the Monster Lie algebra*, Internat. J. Modern Phys. A **8** (1993), no. 31, 5441–5503.

[10] Griess R.L., Jr., *A construction of* F_1 *as automorphisms of a* 196,883*-dimensional algebra*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), 686–691.

[11] Griess R.L., Jr., *The friendly giant*, Invent. Math. **69** (1982), 1–102.

[12] Griess R.L., Jr., *Sporadic groups, code loops and nonvanishing cohomology*, J. Pure Appl. Algebra **44** (1987), 191–214.

[13] Griess R.L., Jr., *Code loops and a large finite group containing triality for* D_4, Proc. Atti del Convegno Internazionale di Teoria Dei Gruppi e Geometria Combinatoria (Firenze, October 1986), Serie II, 19, 1988, pp. 79–98.

[14] Griess R.L., Jr., *A Moufang loop, the exceptional Jordan algebra and a cubic form in* 27 *variables*, J. Algebra **131** (1990), no. 1, 281–293.

[15] Griess R.L., Jr., *Codes, Loops and p-Locals*, Groups, Difference Sets and the Monster (Columbus, OH, 1993), pp. 369–375, de Gruyter, Berlin, 1996.

[16] Griess R.L., Jr., *A vertex operator algebra related to* E_8 *with automorphism group* $O^+(10,2)$, The Monster and Lie Algebras (Columbus, OH 1996), ed. J. Ferrar, K. Harada, de Gruyter, Berlin, 1998.

[17] Griess R.L., Jr., *The monster and its nonassociative algebra*, in Proceedings of the Montreal Conference on Finite Groups, Contemporary Mathematics, 45, 121-157, 1985, American Mathematical Society, Providence, RI.

[18] Griess R.L., Jr., Ryba A.J.E., *Finite simple groups which projectively embed in an exceptional Lie group are classified!*, Bull. Amer. Math. Soc. **36** (1999), no. 1, 75–93.

[19] Griess R.L., Jr., Ryba A.J.E., *Quasisimple finite subgroups of exceptional algebraic groups*, Journal of Group Theory, 2002, 1–39.

[20] Griess R.L., Jr., Höhn G., *Virasoro frames and their stabilizers for the* E_8 *lattice type vertex operator algebra*, J. Reine Angew. Math. **561** (2003), 1–37.

[21] Griess R.L., Jr., *GNAVOA, I. Studies in groups, nonassociative algebras and vertex operator algbras*, Vertex Operator Algebras in Mathematics and Physics (Toronto, 2000), pp. 71–88, Fields Ins. Commun., 39, Amer. Math. Soc., Providence, 2003.

[22] Lam C.H., *Construction of vertex operator algebras from commutative associative algebras*, Comm. Algebra **24** (1996), no. 14, 4339–4360.

[23] Lam C.H., *On VOA associated with special Jordan algebras*, Comm. Algebra **27** (1999), no. 4, 1665–1681.

[24] Miyamoto M., *Griess algebras and conformal vectors in vertex operator algebras*, J. Algebra **179** (1996), no. 2, 523–548.

[25] Meyer W., Neutsch W., *Associative subalgebras of the Griess algebra*, J. Algebra **158** (1993), no. 1, 1–17.

[26] Norton S., *The Monster Algebra: some new formulae*, appeared in AMS Contemp. Math. 193 "Moonshine, the Monster and Related Topics" (eds. Chongying Dong and Geoffrey Mason), 1996, pp. 297–306 (Conference at Mount Holyoke, 1994).

[27] Roitman M., *On Griess algebras*, Symmetry, Integrability and Geometry, Methods and Applications, SIGMA 4 (2008); `http://www.emis.de/journals/SIGMA/2008/057/`

[28] Smith S.D., *Nonassociative commutative algebras for triple covers of* 3*-transposition groups*, Michigan Math. J. **24** (1977), 273–287.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA

E-mail: rlg@umich.edu

(*Received* October 22, 2009, *revised* January 22, 2010)