Metric spaces with point character equal to their size

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In memory of Honza Pelant's 60th birthday.

Abstract. In this paper we consider the point character of metric spaces. This parameter which is a uniform version of dimension, was introduced in the context of uniform spaces in the late seventies by Jan Pelant, *Cardinal reflections and point-character of uniformities*, Seminar Uniform Spaces (Prague, 1973–1974), Math. Inst. Czech. Acad. Sci., Prague, 1975, pp. 149–158. Here we prove for each cardinal κ, the existence of a metric space of cardinality and point character κ. Since the point character can never exceed the cardinality of a metric space this gives the construction of metric spaces with "largest possible" point character. The existence of such spaces was already proved using GCH in Rödl V., *Small spaces with large point character*, European J. Combin. **8** (1987), no. 1, 55–58. The goal of this note is to remove this assumption.

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1. Introduction

Let (X, d) be a metric space. An *open cover* $\mathcal{U}$ of (X, d) is a family of open subsets of X with $X = \bigcup \mathcal{U}$. An open cover of the space X is called *locally finite* if every point of the set X has a neighborhood which intersects only finitely many sets in the cover. Given two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ *refines* $\mathcal{U}$ and write $\mathcal{V} \prec \mathcal{U}$ if every set in $\mathcal{V}$ is contained in some set in $\mathcal{U}$. A.H. Stone proved the following.

Theorem 1 (See [4]). *Any open cover of a metric space has a locally finite open refinement.*

In general, a topological space having the property that every open cover admits a locally finite refinement is called *paracompact*. Stone's Theorem states that metric spaces (and thus metrizable spaces) are paracompact. A natural question raised by this result is whether the uniform version of Stone Theorem is valid. To make this statement precise we need to recall a few more definitions.

An open cover $\mathcal{U}$ is called ε*-uniform* if for every $x \in X$ there is a $U \in \mathcal{U}$ which contains the ε-ball $B_\varepsilon(x) = \{y : d(x, y) < \varepsilon\}$. If there exists ε such that $\mathcal{U}$ is ε-uniform, we say that $\mathcal{U}$ is *uniform*.

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In [5] (see also [6]), A.H. Stone asked whether it is true that in a metric space, every uniform cover has a locally finite uniform refinement. A space X having the property that any uniform cover of X has a locally finite refinement will be said to have the *Stone Uniform Property*. It was proved by E.V. Schepin [12] and J. Pelant [7] that $l_\infty(\kappa)$, for sufficiently large κ, does not have this property. Subsequently, the analogous statement was proved for $l_1(\kappa)$ in [10]. These results motivated further interest into determining "how far" a metric space can be from satisfying the Stone Uniform Property, which leads to the concept of the *point character* of a metric space.

2. Point character

For a family $\mathcal{V}$ of sets, we define $\operatorname{ord}(\mathcal{V}) = \sup\,\{|\mathcal{D}|^+ : \mathcal{D} \subset \mathcal{V},\ \bigcap \mathcal{D} \neq \emptyset\}$. For instance, a locally finite cover $\mathcal{U}$ is a cover satisfying $\operatorname{ord}(\mathcal{U}) \leq \omega_0$. Note that it is equivalent to define $\operatorname{ord}(\mathcal{V}) = \min\,\{\beta :\ \forall x \in X, |\{V \in \mathcal{V} : x \in V\}| < \beta\}$.

Definition 1 (Point Character)**.** Let (X, d) be a metric space. The *point character* $\operatorname{pc}(X, d)$ (or $\operatorname{pc}(X)$ if there is no confusion) of (X, d) is the least cardinal β such that each uniform cover $\mathcal{U}$ of X has a uniform refinement $\mathcal{V}$ with $\operatorname{ord}(\mathcal{V}) \leq \beta$.

Note that in the definition of the point character, we are interested in how many times a point of the space is covered, while Theorem 1 ensures a neighborhood of each $x \in X$ which is intersected by only a few members of a cover. In fact these two points of view are equivalent in terms of uniform covers:

Proposition 1. *Given a metric space X, the following properties are equivalent:*

1. *For every uniform cover $\mathcal{W}$ of X there exists a uniform cover $\mathcal{U}$, $\mathcal{U} \prec \mathcal{W}$, such that every $x \in X$ belongs to less than α elements of $\mathcal{U}$.*
2. *For every uniform cover $\mathcal{W}$ of X, there exists $\delta > 0$ and a δ-uniform cover $\mathcal{V}$, $\mathcal{V} \prec \mathcal{W}$, such that for every $x \in X$, the ball $B_\delta(x)$ meets less than α elements of $\mathcal{V}$.*

Proof: Property 2 clearly implies 1. To prove the opposite implication, consider $\mathcal{W}$ a given uniform cover. By Property 1, there exists $\mathcal{U} = \{U_i : i \in I\} \prec \mathcal{W}$ which is ε-uniform for some $\varepsilon > 0$ and such that every $x \in X$ belongs to less than α elements of $\mathcal{U}$. We will prove the existence of a $\varepsilon/2$-uniform cover $\mathcal{V}$ satisfying Property 2.

For each $x \in X$, there is $i(x) \in I$ such that $B_\varepsilon(x) \subseteq U_{i(x)}$. Set $V_i = \bigcup\{B_{\varepsilon/2}(x) : i(x) = i\}$. Since for every $x \in X$, $B_{\varepsilon/2}(x) \subseteq V_{i(x)}$, the family $\mathcal{V} = \{V_i : i \in I\}$ is an $\varepsilon/2$-uniform cover refining $\mathcal{U}$.

Let $y \in X$ be an arbitrary point. We claim that $B_{\varepsilon/2}(y)$ meets less than α elements of $\mathcal{V}$. Indeed, if $B_{\varepsilon/2}(y) \bigcap V_i \neq \emptyset$, then there are z_i, x_i with $i(x_i) = i$, $z_i \in B_{\varepsilon/2}(y) \bigcap B_{\varepsilon/2}(x_i)$. Therefore, $d(y, x_i) < \varepsilon$ and so $y \in B_\varepsilon(x_i) \subseteq U_i$. Since y is in less than α of the U_i's, $B_{\varepsilon/2}(y)$ meets less than α elements of $\mathcal{V}$, concluding the proof. □

For any Euclidean space $\mathbb{R}^n$ we have $\operatorname{pc}(\mathbb{R}^n) = n + 2$. Consequently the point character provides a suitable generalization of the notion of dimension for the "infinite dimensional case". Note also that a space having the Uniform Stone Property satisfies $\mathrm{pc}(X) \leq \omega_0$, so that any space with uncountable point character is a counter example to Stone's question.

Several examples of spaces with large point character have been given. It was proved by E.V. Schepin [12] and J. Pelant [7] that $l_\infty(\kappa)$, for sufficiently large κ, satisfies $\mathrm{pc}(l_\infty(\kappa)) \geq \kappa$. Subsequently, in [9], the analogous statement was proved for l_p spaces. More precisely, it is shown that, if α is a limit ordinal then $\mathrm{pc}(l_p(\omega_\alpha)) \geq \omega_\alpha$, for any $p \geq 1$.

A result concerning uniform spaces proved in [8] by J. Pelant implies that the point character of a metric space cannot be larger than its cardinality. In [11], V. Rödl gave an example of a space for which $\mathrm{pc}(X) = |X|$. The construction is based on infinite graphs considered by Erdös, Galvin and Hajnal [2], [3]. The proof given in [11] assumes the Generalized Continuum Hypothesis (GCH). The main result of this paper is the elimination of the need of GCH, thus proving the following:

Theorem 2. (i) *For every infinite cardinal* κ, *there exists a metric space* X *satisfying* $\mathrm{pc}(X) = |X| = \kappa$.
(ii) *For every metric space* X, $\mathrm{pc}(X) \leq |X|$.

As mentioned above, the second part of Theorem 2 follows from a more general result of [8]. In order to give a self contained exposition, following a similar idea, we now present a simple proof of this fact.

Proof of Theorem 2, Part (ii): Let X be a metric space and $d(X)$ the smallest cardinality of a dense subset of X. We will in fact proof a slightly stronger statement, namely that

$$\mathrm{pc}(X) \leq d(X).$$

Let $\varepsilon > 0$ and let $\mathcal{U}$ be an ε-uniform cover of the metric space X. Set $\kappa = d(X)$ and let $\{d_\alpha : \alpha < \kappa\}$ be a dense subset of X. For each $\alpha < \kappa$ we consider the following open set:

$$V_\alpha = B_\varepsilon(d_\alpha) \setminus \overline{\bigcup_{\beta<\alpha} B_{\varepsilon/5}(d_\beta)}.$$

Since $\mathcal{U}$ is ε-uniform, for every $\alpha \in \kappa$, there exists $U \in \mathcal{U}$ such that

$$V_\alpha \subseteq B_\varepsilon(d_\alpha) \subseteq U.$$

Consequently the family $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ refines the cover $\mathcal{U}$. We will show that $\mathcal{V}$ is an $\varepsilon/4$-uniform cover of X with no point contained in κ elements of $\mathcal{V}$.

Claim 1. *The set* $\mathcal{V}$ *is an* $\varepsilon/4$-*uniform cover of* X.

Proof: Fix $x \in X$ and let $\alpha \leq \kappa$ be minimal with $d(x, d_\alpha) < \varepsilon/2$. We will show that $B_{\varepsilon/4}(x) \subseteq V_\alpha$. Indeed, for every $z \in B_{\varepsilon/4}(x)$ we have $d(d_\alpha, z) <$

$d(d_\alpha, x) + d(x, z) < \varepsilon/2 + \varepsilon/4 < \varepsilon$, proving that $z \in B_\varepsilon(d_\alpha)$. For any $\beta < \alpha$, we have $z \notin B_{\varepsilon/4}(d_\beta)$ as otherwise

$$d(x, d_\beta) \leq d(x, z) + d(z, d_\beta) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

contradicting the choice of α. This implies that $z \notin \bigcup_{\beta<\alpha} B_{\varepsilon/4}(d_\beta)$, hence $z \notin \overline{\bigcup_{\beta<\alpha} B_{\varepsilon/5}(d_\beta)}$. □

Claim 2. *For each $\beta < \alpha < \kappa$, $B_{\varepsilon/5}(d_\beta) \bigcap V_\alpha = \emptyset$.*

PROOF: This is a consequence of the construction of V_α. □

Claim 3. *For each $x \in X$ there are less than κ of the sets V_α containing x.*

PROOF: Let $x \in X$ and $\beta < \kappa$ be such that $d(x, d_\beta) < \varepsilon/5$. Then $x \in B_{\varepsilon/5}(d_\beta)$ and by the previous claim, $x \notin V_\alpha$ for any $\alpha > \beta$. □

Summarizing, for any ε-uniform cover $\mathcal{U}$ of X, we constructed an $\varepsilon/4$-uniform refinement $\mathcal{V}$ satisfying $\mathrm{ord}(\mathcal{V}) \leq \kappa = d(X)$. Thus $\mathrm{pc}(X) \leq d(X)$, concluding the proof of Part (ii) of Theorem 2. □

The proof of the first part follows closely the proof given in [11]. Instead of using the existence of graphs with large chromatic number and arbitrary girth, we give an explicit construction of appropriate graphs. This will allow us to eliminate GCH. The proof is based on a combinatorial lemma which we state and prove in Section 4.

Note that it is sufficient to prove the first part of Theorem 2 for successor cardinals only. Indeed by contradiction, suppose θ is the least limit cardinal for which the statement fails. Let then (X_i, d_i), for each $i \in \theta$, be a metric space satisfying $\mathrm{pc}(X_i) = |X_i|$. Set $X = \bigcup_{i\in\theta} X_i$ and define a metric d on X by setting $d(x, y) = d_i(x, y)$ if x, y belongs to the same X_i for some i, and $d(x, y) = \infty$ otherwise. It is clear that $\mathrm{pc}(X) = \sup\{\mathrm{pc}(X_i) : i \in \theta\} = \sup\{|X_i| : i \in \theta\} = \theta = |X|$.

It remains to prove the first part of Theorem 2 for all successor cardinal κ^+. In order to keep the exposition self-contained, we first present the elements of the proof from [11] which we will use.

3. Preliminary results

3.1 An equivalent definition of the point character. Let (X, d) be a metric space and let $U \subset X$. The *diameter* of U is defined by $\mathrm{diam}(U) = \sup\{d(x, y) : x, y \in U\}$. A cover $\mathcal{U}$ of X is *bounded* if there exists $b > 0$ such that $\mathrm{diam}\, U < b$ for all $U \in \mathcal{U}$. We find convenient to use a variant to the definition of the point character which we now give.

Proposition 2. *Let (X, d) be a metric space. The point character $\mathrm{pc}(X)$ is the least infinite cardinal α such that for every $b > 0$, there exists a b-bounded uniform cover $\mathcal{U}$ with no point of X in α sets of $\mathcal{U}$.*

Proof of Proposition 2: Let us write $\mathrm{pc}^*(X) = \min\{\beta : \forall b > 0, \exists \mathcal{U}$, b-bounded uniform cover with $\mathrm{ord}(\mathcal{U}) < \beta\}$. With this notation, our goal is to show that $\mathrm{pc}(X) = \mathrm{pc}^*(X)$. We will first show that $\mathrm{pc}^*(X) \leq \mathrm{pc}(X)$ and then that $\mathrm{pc}(X) \leq \mathrm{pc}^*(X)$.

Suppose $\mathrm{pc}(X) \leq \alpha$. Fix $b > 0$ and let $\mathcal{U}$ be a b-bounded uniform cover of X. Let $\mathcal{V}$ be a uniform refinement of $\mathcal{U}$ satisfying $\mathrm{ord}(\mathcal{V}) \leq \alpha$. Then $\mathcal{V}$ is also b-bounded and every point of X belongs to less than α members of $\mathcal{V}$, proving $\mathrm{pc}^*(X) \leq \alpha$.

Conversely, suppose $\mathrm{pc}^*(X) \leq \alpha$ and let $\mathcal{U}$ be an arbitrary ε-uniform cover of X. Choose $b = \varepsilon$. Since $\mathrm{pc}^*(X) \leq \alpha$, there exists an ε-bounded uniform cover $\mathcal{V}$ with every point of X in less than α members of $\mathcal{V}$. We will observe that $\mathcal{V}$ refines $\mathcal{U}$ and thus $\mathrm{pc}(X) \leq \alpha$. Indeed, let $V \in \mathcal{V}$ and $v \in V$. Since $\mathrm{diam}(V) \leq \varepsilon$, $V \subseteq B(v, \varepsilon)$. On the other hand, since $\mathcal{U}$ is ε-uniform we also have $B(v, \varepsilon) \subseteq U$ for some $U \in \mathcal{U}$. Hence $V \subseteq B(v, \varepsilon) \subseteq U$. In other words, the cover $\mathcal{V}$ refines $\mathcal{U}$ and thus $\mathrm{pc}(X) \leq \alpha$. □

Let X be a metric space. According to Proposition 2, $\mathrm{pc}(X) > \alpha$ if there exists $b > 0$ such that every b-bounded uniform cover of $\mathcal{U}$ covers a point of X at least α times. We will use this fact in the proof of Proposition 3.

3.2 Graphs and point character. We recall some standard definition from graph theory. A graph is a couple (V, E), where $V = V(G)$ is the set of *vertices* and $E = E(G)$ is a subset of the set of unordered pairs of V. The elements of E are called the *edges* of the graph. A sequence of vertices x_i, $1 \leq i \leq n$, such that for every $1 \leq i \leq n$, $\{x_i, x_{i+1}\}$ is an edge of G is called a *path* of length n. Given two vertices x and y of a graph G, the *distance* between x and y is the smallest integer n, if it exists, such that there is a path of length n from x to y. We then write $d_G(x, y) = n$ and $d_G(x, y) = \infty$ if no such path exist. The *neighborhood* of a vertex $x \in V$ is the set $N(x) = \{y \in V : \{x, y\} \in E\}$. The elements of $N(x)$ are called the *neighbors* of x.

A *vertex coloring* of G is a map $c : V \to C$, where C is any set. The elements of C are called *colors*. A vertex coloring c of a graph G will be called *n-bounded* if any pair of vertices x and y satisfying $d_G(x, y) \geq n$ are colored differently by c. More formally:

Definition 2. Let $n \in \mathbb{N}$ and let $G = (V, E)$ be a graph. A vertex coloring $c : V \to C$ is n-bounded if for every pair of vertices x and y,

$$d_G(x, y) \geq n \Longrightarrow c(x) \neq c(y).$$

We now give the definition of the point character of a graph, resembling the definition of the point character of a metric space as in Proposition 2:

Definition 3. Let G be a graph, $n \in \mathbb{N}$ and κ a cardinal. We will say the n^{th} point character of G is bigger than κ and we write $\mathrm{pc}_n(G) > \kappa$ if for every n-bounded vertex coloring $c : V \to C$ there exists a vertex x_0 of G with $|c(N(x_0))| \geq \kappa$.

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of connected graphs on disjoint sets of vertices. We define a metric space (X,d) by setting $X=\bigcup_{n\in\mathbb{N}}V(G_n)$ and, for any $x,y\in X$, $d(x,y)=\frac{1}{n}d_{G_n}(x,y)$ if $x,y\ \in V(G_n)$ for some n, and $d(x,y)=\infty$ if no such n exists.

The following result was proved in [11].

Proposition 3. *Let κ be a cardinal. Suppose $(G_n)_{n\in\mathbb{N}}$ is a sequence of connected graphs on disjoint sets of vertices such that $\mathrm{pc}_n(G_n)>\kappa$ holds. Then the point character of the space (X,d) is bigger than κ.*

Proof: Let $\mathcal{U}$ be a 1-bounded ϵ-uniform covering of X. For every $x\in X$, choose $U_x\in\mathcal{U}$ with the property that $B(x,\epsilon)\subset\mathcal{U}_x$. This defines a mapping $\varphi:X\to\mathcal{U}$. Choose n_0 sufficiently large so the $1/n_0<\epsilon$. If $x,y\in V(G_{n_0})$,

$$d_{G_{n_0}}(x,y)\geq n_0\Longrightarrow\varphi(x)\neq\varphi(y).$$

Consequently, there exists a vertex $x_0\in V(G_{n_0})$ such that $|\varphi(N(x_0))|\geq\kappa$. Since $d_{G_{n_0}}(x_0,y)\leq 1$ implies $d(x_0,y)<\epsilon$, x_0 is contained in at least κ members of $\mathcal{U}$. $\square$

With the above proposition in mind and in order to prove Theorem 2, Part (i), it is sufficient to show the existence of a graph G_n satisfying $|V(G_n)|=\kappa^+$ and $\mathrm{pc}_n(G_n)>\kappa$, for every $n\in\mathbb{N}$ and infinite cardinal κ. Indeed, under this assumption, the space X as defined above has cardinality $|\bigcup_{n\in\mathbb{N}}V(G_n)|=\kappa^+$ and satisfies $\mathrm{pc}(X)>\kappa$. As mentioned in the introduction, the point character of a metric space does not exceed its cardinality, and thus $|X|=\mathrm{pc}(X)=\kappa^+$. The next section is devoted to the combinatorial lemma which implies the existence of graphs $(G_n)_{n\in\mathbb{N}}$ with $\mathrm{pc}_n(G_n)>\kappa$.

4. The combinatorial lemma

The construction of the graphs G_n is based on the following:

Lemma 1. *Assume that $1\leq n<\omega$ and $f:[\kappa^+]^n\to\kappa^+$ is a function such that if $x_1<x_2<\cdots<x_n<\kappa^+$ then*

$$|\{f(y_1,\dots,y_n):x_1<y_1<x_2<y_2<\cdots<x_n<y_n\}|<\kappa.$$

Then there exist $x_1<x_2<\cdots<x_n<y_1<y_2<\cdots<y_n$ with

$$f(x_1,\dots,x_n)=f(y_1,\dots,y_n).$$

We prove the following stronger statement.

Theorem 3. *Assume that $1\leq n<\omega$ and the function $F:[\kappa^+]^n\to[\kappa^+]^{<\kappa}$ be such that for all $\{x_1,x_2,\cdots,x_n\}\in[\kappa^+]^n$*

(i) *$F(x_1,x_2,\cdots,x_n)\neq 0$, and*
(ii) *$|\bigcup_{\{y_1,\dots,y_n\}}\{F(y_1,\dots,y_n):x_1<y_1<x_2<y_2<\cdots<x_n<y_n\}|<\kappa.$*

Then there is a ξ such that for every $\alpha < \kappa^+$ there are

$$\alpha < y_1 < y_2 < \cdots < y_n < \kappa^+$$

with $\xi \in F(y_1, \ldots, y_n)$.

Proof of Lemma 1: Indeed, given f as in Lemma 1, set

$$F(x_1, \ldots, x_n) = \{f(y_1, \ldots, y_n) : x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n < \kappa^+\}$$

for $x_1 < x_2 < \cdots < x_n < \kappa^+$. Then (ii) holds for F, so there is ξ as in Theorem 3. Apply the statement with $\alpha = 0$ and get $x_1 < \cdots < x_n$ with $f(x_1, x_2, \ldots, x_n) = \xi$, and then apply the statement with $\alpha = x_n$ and get $x_n < y_1 < y_2 < \cdots < y_n$ with $f(y_1, y_2, \ldots, y_n) = \xi$.

Proof of Theorem 3: We prove the statement by induction on n.

Assume first that $n = 1$. Then F is a function from κ^+ such that $F(\alpha)$ is always a nonempty subset of κ^+ with $|F(\alpha)| < \kappa$ and

$$\left|\bigcup\{F(\alpha) : \alpha < \kappa^+\}\right| < \kappa.$$

Obviously, some value must be taken κ^+ times.

Assume that we proved the result for n and F is a function, satisfying (i) and (ii), on the $(n+1)$-tuples of κ^+.

Let $0 \notin Y \subseteq \kappa^+$ be a subset of cardinality κ^+ with no consecutive elements. Notice that F restricted to Y still satisfies (i) and (ii).

What we have gained is that whenever $y_1 < y_2 < \cdots < y_n$ are in Y then

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n < x_{n+1}$$

where $x_1 = 0$, $x_{i+1} = y_i + 1$ $(1 \le i \le n)$, and so, by (ii), we get

$$(*) \qquad \left|\bigcup\{F(y_1, y_2, \ldots, y_n, y) : y_n < y \in Y\}\right| < \kappa.$$

We now identify Y with κ^+, i.e., assume that both (ii) and $(*)$ hold for F.

Set

$$A(y_1, \ldots, y_n) = \bigcup\{F(y_1, \ldots, y_n, y) : y_n < y < \kappa^+\}.$$

For $y_1 < y_2 < \cdots < y_n < \kappa^+$ define the set $F^*(y_1, \ldots, y_n)$ as follows. $\xi \in F^*(y_1, y_2, \ldots, y_n)$ if there are arbitrarily large $y < \kappa^+$ with $\xi \in F(y_1, y_2, \ldots, y_n, y)$.

Claim 4. $F^*(y_1, y_2, \ldots, y_n) \neq \emptyset$ $(y_1 < y_2 < \cdots < y_n < \kappa^+)$.

Proof: For each y with $y_n < y < \kappa^+$, $F(y_1, \ldots, y_n, y)$ is a nonempty subset of $A(y_1, y_2, \ldots, y_n)$ with $|A(y_1, y_2, \ldots, y_n)| < \kappa$. As we choose κ^+ times a nonempty subset of some set of cardinality less than κ, some element of $A(y_1, y_2, \ldots, y_n)$ must be chosen κ^+ times. $\square$

Notice that, as $|A(y_1, y_2, \ldots, y_n)| < \kappa$, we always have $|F^*(y_1, y_2, \ldots, y_n)| < \kappa$.

Claim 5. *If* $x_1 < x_2 < \cdots < x_n < \kappa^+$, *then*

$$\left|\bigcup\{F^*(y_1, y_2, \ldots, y_n) : x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n\}\right| < \kappa.$$

PROOF: Assume that this does not hold. Then we can choose a set A,

$$A \subseteq \bigcup\{F^*(y_1, y_2, \ldots, y_n) : x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n\}$$

with $|A| = \kappa$.

For each $\xi \in A$ pick $y_1^\xi < y_2^\xi < \cdots < y_n^\xi$ with

$$x_1 < y_1^\xi < x_2 < y_2^\xi < \cdots < x_n < y_n^\xi$$

such that $\xi \in F^*(y_1^\xi, \ldots, y_n^\xi)$.

Choose x_{n+1} with $x_{n+1} > \sup\{y_n^\xi : \xi \in A\}$ and for each $\xi \in A$ choose y_{n+1}^ξ with $y_{n+1}^\xi > x_{n+1}$ such that

$$\xi \in F(y_1^\xi, y_2^\xi, \ldots, y_n^\xi, y_{n+1}^\xi).$$

Then A, a set of cardinality κ, is a subset of

$$\bigcup\{F(y_1, y_2, \ldots, y_{n+1}) : x_1 < y_1 < \cdots < x_{n+1} < y_{n+1}\},$$

which by (ii) has cardinality less than κ, a contradiction. □

From Claim 5 we can conclude the proof of Theorem 2: if ξ is such that there are arbitrarily large $y_1 < y_2 < \cdots < y_n < \kappa^+$ with $\xi \in F^*(y_1, \ldots, y_n)$, then there are arbitrarily large $y_1 < y_2 < \cdots < y_n < y_{n+1}$ with $\xi \in F(y_1, \ldots, y_n, y_{n+1})$. □

5. The graphs G_n

Let κ^+ be an infinite cardinal and let $n \in \mathbb{N}$. We define $V(G_n) = [\kappa^+]^n$. Given $v = (v_1, v_2, \cdots, v_n)$ and $w = (w_1, w_2, \cdots, w_n)$ both in $[\kappa^+]^n$, we set

$$\{v, w\} \in E(G_n) \iff v_1 < w_1 < v_2 < \cdots < v_n < w_n.$$

We will show that $\mathrm{pc}_n(G_n) > \kappa$.

Indeed, let $\varphi : V(G_n) \to C$ be an n-bounded coloring. We need to show that there exists $v \in V(G)$ such that $|\varphi(N(v))| \geq \kappa$. Suppose such a v does not exist. Then for every $(v_1, v_2, \cdots, v_n) \in V(G_n)$ we have $|\{\varphi(w_1, w_2, \cdots, w_n) : v_1 < w_1 < \cdots < v_n < w_n\}| < \kappa$. According to Lemma 1, this implies that there exists $v = (v_1, v_2, \cdots, v_n)$ and $w = (w_1, w_2, \cdots, w_n) \in V(G_n)$ such that $\varphi(v) = \varphi(w)$ and $v_1 < v_2 < \cdots < v_n < w_1 < \cdots < w_n$, contradicting the assumption that φ is n-bounded.

Consequently, the graph G_n satisfies $|V(G_n)| = \kappa^+$ and $\mathrm{pc}_n(G_n) > \kappa$, concluding the proof of Theorem 2.

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