Approximation and shape preserving properties of the nonlinear Bleimann-Butzer-Hahn operators of max-product kind

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Abstract. Starting from the study of the *Shepard nonlinear operator of max-prod type* in (Bede, Nobuhara et al., 2006, 2008), in the book (Gal, 2008), Open Problem 5.5.4, pp. 324–326, the *Bleimann-Butzer-Hahn max-prod type operator* is introduced and the question of the approximation order by this operator is raised. In this paper firstly we obtain an upper estimate of the approximation error of the form $\omega_1(f;(1+x)^{\frac{3}{2}}\sqrt{x/n})$. A consequence of this result is that for each compact subinterval $[0,a]$, with arbitrary $a>0$, the order of uniform approximation by the Bleimann-Butzer-Hahn operator is less than $\mathcal{O}(1/\sqrt{n})$. Then, one proves by a counterexample that in a sense, for arbitrary f this order of uniform approximation cannot be improved. Also, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order $\omega_1(f;(x+1)^2/n)$ is obtained. Shape preserving properties are also investigated.

Keywords: nonlinear Bleimann-Butzer-Hahn operator of max-product kind, degree of approximation, shape preserving properties

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1. Introduction

In the recent papers [4], [5], [1] and the monograph [8], the study of nonlinear approximation operators of max-product kind was proposed. New techniques for the study of these problems were proposed in [2] and [3]. These new methods allow estimates with explicit constants and produce a counterexample showing that the order of approximation cannot be improved. Also, a statistical approach in approximation by max-product operators was given in [7].

Starting from the study of the *Shepard nonlinear operator of max-prod type* in [4], [5], in the Open Problem 5.5.4, pp. 324–326 of the recent monograph [8], the following *nonlinear Bleimann-Butzer-Hahn operator of max-prod type* is introduced

$$H_n^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{n}\binom{n}{k}x^k f\left(\frac{k}{n+1-k}\right)}{\bigvee_{k=0}^{n}\binom{n}{k}x^k},\tag{1}$$

and the order of approximation by this operator is raised. In this paper we answer this question and the order of approximation $\omega_1(f;(1+x)^{\frac{3}{2}}\sqrt{x/n})$ on $[0,\infty)$ is obtained. A consequence of this result is that for each compact subinterval $[0,a]$, with arbitrary $a>0$, the order of uniform approximation by the Bleimann-Butzer-Hahn operator is less than $\mathcal{O}(1/\sqrt{n})$. Then, one proves by a counterexample that in a sense, for arbitrary f this order of uniform approximation cannot be improved. Also, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order $\omega_1(f;(x+1)^2/n)$ on $[0,\infty)$ is obtained. This allows us to put in evidence large classes of functions (e.g. bounded, nondecreasing concave polygonal lines on $[0,\infty)$) for which the order of approximation given by the max-product Bleimann-Butzer-Hahn operator, can be essentially better than the order given by the linear Bleimann-Butzer-Hahn operator, introduced and studied in [6], [9].

Section 2 presents some general results on nonlinear operators, in Section 3 we prove several auxiliary lemmas, while Section 4 contains the main approximation results. Section 5 is devoted to the study of shape preserving properties.

2. Preliminaries

For the proof of the main result we need some general considerations on the so-called nonlinear operators of max-prod kind. Over the set of positive reals, $\mathbb{R}_+$, we consider the operations $\vee$ (maximum) and $\cdot$, product. Then $(\mathbb{R}_+,\vee,\cdot)$ has a semiring structure and we call it *Max-Product algebra*.

Let $I\subset\mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I)=\{f:I\to\mathbb{R}_+; f \text{ continuous and bounded on } I\}.$$

The general form of $L_n: CB_+(I)\to CB_+(I)$ (called here a discrete max-product type approximation operator) studied in the paper will be

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_n(x,x_i)\cdot f(x_i),$$

or

$$L_n(f)(x)=\bigvee_{i=0}^{\infty}K_n(x,x_i)\cdot f(x_i),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_n(\cdot,x_i)\in CB_+(I)$ and $x_i\in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy a pseudo-linearity condition of the form

$$L_n(\alpha\cdot f\vee\beta\cdot g)(x)=\alpha\cdot L_n(f)(x)\vee\beta\cdot L_n(g)(x),\ \forall\alpha,\beta\in\mathbb{R}_+, f,g:I\to\mathbb{R}_+.$$

In this section we present some general results on these kinds of operators which will be useful later in the study of the Bleimann-Butzer-Hahn max-product kind operator considered in Introduction.

Lemma 2.1 ([1]). *Let $I \subset \mathbb{R}$ be a bounded or unbounded interval,*

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties:

(i) *if $f, g \in CB_+(I)$ satisfy $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in \mathbb{N}$;*
(ii) *$L_n(f + g) \leq L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.*

Then for all $f, g \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f - g|)(x).$$

Proof: Since it is very simple, we reproduce here the proof from [1]. Let $f, g \in CB_+(I)$. We have $f = f - g + g \leq |f - g| + g$, which by the conditions (i)–(ii) successively implies $L_n(f)(x) \leq L_n(|f - g|)(x) + L_n(g)(x)$, that is $L_n(f)(x) - L_n(g)(x) \leq L_n(|f - g|)(x)$.

Writing now $g = g - f + f \leq |f - g| + f$ and applying the above reasonings, it follows $L_n(g)(x) - L_n(f)(x) \leq L_n(|f - g|)(x)$, which combined with the above inequality gives $|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f - g|)(x)$. □

Remarks. 1) It is easy to see that the Bleimann-Butzer-Hahn max-product operator satisfies the conditions in Lemma 2.1(i), (ii). In fact, instead of (i) it satisfies the stronger condition

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x), \ f, g \in CB_+(I).$$

Indeed, taking in the above equality $f \leq g$, $f, g \in CB_+(I)$, it easily follows $L_n(f)(x) \leq L_n(g)(x)$.

2) In addition, it is immediate that the Bleimann-Butzer-Hahn max-product operator is positive homogenous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \geq 0$.

Corollary 2.2 ([1]). *Let $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the conditions* (i)–(ii) *in Lemma* 2.1 *and in addition being positive homogenous. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have*

$$|f(x) - L_n(x)| \leq \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega_1(f; \delta)_I + f(x) \cdot |L_n(e_0)(x) - 1|,$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t - x|$ for all $t \in I$, $x \in I$, $\omega_1(f; \delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x - y| \leq \delta\}$ and if I is unbounded then we suppose that there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \bigcup \{+\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.

Proof: The proof is identical with that for positive linear operators and because of its simplicity we reproduce it below. Indeed, from the identity

$$L_n(f)(x) - f(x) = [L_n(f)(x) - f(x) \cdot L_n(e_0)(x)] + f(x)[L_n(e_0)(x) - 1],$$

it follows (by the positive homogeneity and by Lemma 2.1)

$$\begin{aligned}|f(x) - L_n(f)(x)| &\leq |L_n(f(x))(x) - L_n(f(t))(x)| + |f(x)| \cdot |L_n(e_0)(x) - 1| \\ &\leq L_n(|f(t) - f(x)|)(x) + |f(x)| \cdot |L_n(e_0)(x) - 1|.\end{aligned}$$

Now, since for all $t, x \in I$ we have

$$|f(t) - f(x)| \leq \omega_1(f; |t - x|)_I \leq \left[\frac{1}{\delta}|t - x| + 1\right] \omega_1(f; \delta)_I,$$

replacing above we immediately obtain the estimate in the statement. □

An immediate consequence of Corollary 2.2 is the following.

Corollary 2.3 ([1]). *Suppose that in addition to the conditions in Corollary 2.2, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have*

$$|f(x) - L_n(x)| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega_1(f; \delta)_I.$$

The nonlinear max-product Bleimann-Butzer-Hahn operator satisfies the following useful result.

Lemma 2.4. *For any arbitrary bounded function $f : [0, \infty) \to [0, \infty)$, the max-product operator $H_n^{(M)}(f)(x)$ is positive, bounded, continuous on $[0, \infty)$ and satisfies $H_n^{(M)}(f)(0) = f(0)$.*

PROOF: The positivity of $H_n^{(M)}(f)(x)$ is immediate. Also, if $f(x) \leq K$ for all $x \in [0, \infty)$ then it is immediate that $H_n^{(M)}(f)(x) \leq K$, for all $x \in [0, \infty)$.

Since $s_{n,k}(x) > 0$ for all $x \in (0, \infty)$, $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$, it follows that the denominator $\bigvee_{k=0}^{n} s_{n,k}(x) > 0$ for all $x \in (0, \infty)$ and $n \in \mathbb{N}$.

The continuity on $[0, \infty)$ of the numerator is immediate since the numerator is the maximum of a finite number of continuous functions. Therefore, as a first conclusion we get the continuity of $H_n^{(M)}(f)(x)$ on $(0, \infty)$.

To prove now the continuity of $H_n^{(M)}(f)(x)$ at $x = 0$, we observe that $s_{n,k}(0) = 0$ for all $k \in \{1, 2, \ldots, n\}$ and $s_{n,k}(0) = 1$ for $k = 0$, which implies that $\bigvee_{k=0}^{n} s_{n,k}(x) = 1$ in the case of $x = 0$. The fact that $H_n^{(M)}(f)(x)$ coincides with $f(x)$ at $x = 0$ immediately follows from the above considerations, which proves the lemma. □

Remark. From the above considerations, it is clear that $H_n^{(M)}(f)(x)$ satisfies all the conditions in Lemma 2.1, Corollary 2.2 and Corollary 2.3 for $I = [0, \infty)$.

3. Auxiliary results

We consider the following *nonlinear Bleimann-Butzer-Hahn operator of max-product type*

$$H_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n+1-k}\right)}{\bigvee_{k=0}^{n} \binom{n}{k} x^k}. \tag{2}$$

Remark. Since $H_n^{(M)}(f)(0) - f(0) = 0$ for all n, notice that in the notations, proofs and statements of the all approximation results, that is in Lemmas 3.1–3.3, Theorem 4.1, Lemma 4.2, Corollaries 4.4, 4.5, in fact we always may suppose that $x > 0$.

For each $k \in \{0, 1, 2, \ldots, n\}$, $j \in \{0, 1, 2, \ldots, n-1\}$ and $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$, or $j = n$ and $x \in [n, \infty)$, let us denote $s_{n,k}(x) = \binom{n}{k} x^k$,

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)\left|\frac{k}{n+1-k} - x\right|}{s_{n,j}(x)}, \ m_{k,n,j}(x) = \frac{s_{n,k}(x)}{s_{n,j}(x)}.$$

It is clear that if $k \geq j+1$ then

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)(\frac{k}{n+1-k} - x)}{s_{n,j}(x)}$$

and if $k \leq j$ then

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)(x - \frac{k}{n+1-k})}{s_{n,j}(x)}.$$

Lemma 3.1. *For all $k \in \{0, 1, 2, \ldots, n\}$, $j \in \{0, 1, 2, \ldots, n-1\}$ and $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$ or $j = n$ and $x \in [n, \infty)$ we have*

$$m_{k,n,j}(x) \leq 1.$$

Proof: We have two cases: 1) $k \geq j$ and 2) $k \leq j$.

Case 1). Take $k \geq j$, $j \in \{0, \ldots, n-1\}$. Since clearly the function $h(x) = \frac{1}{x}$ is nonincreasing on $[\frac{j}{n-j+1}, \frac{j+1}{n-j}]$, it follows that

$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} = \frac{k+1}{n-k} \cdot \frac{1}{x} \geq \frac{k+1}{n-k} \cdot \frac{n-j}{j+1} = \frac{k+1}{j+1} \cdot \frac{n-j}{n-k} \geq 1$$

which implies $m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq m_{j+2,n,j}(x) \geq \cdots \geq m_{n,n,j}(x)$.

Case 2). Take $k \leq j$, $j \in \{0, 1, \ldots, n\}$. Then we get

$$\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} = \frac{n-k+1}{k} x \geq \frac{n-k+1}{k} \cdot \frac{j}{n-j+1} = \frac{n-k+1}{n-j+1} \cdot \frac{j}{k} \geq 1$$

which implies

$$m_{j,n,j}(x) \geq m_{j-1,n,j}(x) \geq m_{j-2,n,j}(x) \geq \cdots \geq m_{0,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$, the conclusion of the lemma is immediate. □

Lemma 3.2. (i) *Let* $j \in \{0, 1, 2, \ldots, n-1\}$ *and* $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$. *If* $k \in \{j+1, \ldots, n-1\}$ *is such that* $k - \sqrt{k+1} \geq j$, *then* $M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$.

(ii) *Let* $j \in \{1, 2, \ldots, n-1\}$ *and* $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$. *If* $k \in \{1, \ldots, j\}$ *is such that* $k + \sqrt{k} \leq j$, *then* $M_{k,n,j}(x) \geq M_{k-1,n,j}(x)$.

Proof: (i) We observe that

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} = \frac{k+1}{n-k} \cdot \frac{1}{x} \cdot \frac{\frac{k}{n+1-k} - x}{\frac{k+1}{n-k} - x}.$$

Since the function $g(x) = \frac{1}{x} \cdot \frac{\frac{k}{n+1-k} - x}{\frac{k+1}{n-k} - x}$ is nonincreasing, it follows that $g(x) \geq g(\frac{j+1}{n-j})$ for all $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$ and we have

$$\begin{aligned}\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} &\geq \frac{k+1}{n-k} \cdot \frac{n-j}{j+1} \cdot \frac{\frac{k}{n+1-k} - \frac{j+1}{n-j}}{\frac{k+1}{n-k} - \frac{j+1}{n-j}} \\ &= \frac{k+1}{j+1} \cdot \frac{\frac{n-j}{n+1-k}k - (j+1)}{k+1 - \frac{n-k}{n-j}(j+1)}.\end{aligned}$$

Let $h(n) = \frac{\frac{n-j}{n+1-k}k - (j+1)}{k+1 - \frac{n-k}{n-j}(j+1)}$. Then $h'(n) = \frac{-1}{k-j} \frac{(j-k+1)^2}{(n-k+1)^2} < 0$ so h is nonincreasing and we have

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq \lim_{n\to\infty} \frac{k+1}{j+1} \cdot \frac{\frac{n-j}{n+1-k}k - (j+1)}{k+1 - \frac{n+1-k}{n-j}(j+1)} = \frac{(k+1)(k-j-1)}{(j+1)(k-j)}.$$

Then, since the condition $k - \sqrt{k+1} \geq j$ implies $(k+1)(k-j-1) \geq (j+1)(k-j)$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq 1.$$

(ii) We have

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} = \frac{n-k+1}{k} \cdot x \cdot \frac{x - \frac{k}{n+1-k}}{x - \frac{k-1}{n+2-k}}.$$

Since the function $g_1(x) = x \cdot \frac{x - \frac{k}{n+1-k}}{x - \frac{k-1}{n+2-k}}$ is nondecreasing, it follows that $g_1(x) \geq g_1(\frac{j}{n-j+1}) = \frac{j}{n-j+1} \cdot \frac{\frac{j}{n-j+1} - \frac{k}{n+1-k}}{\frac{j}{n-j+1} - \frac{k-1}{n+2-k}}$ for all $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$. We have

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq \frac{n-k+1}{k} \cdot \frac{j}{n-j+1} \cdot \frac{\frac{j}{n-j+1} - \frac{k}{n+1-k}}{\frac{j}{n-j+1} - \frac{k-1}{n+2-k}}$$
$$= \frac{j}{k} \cdot \frac{\frac{n-k+1}{n-j+1}j - k}{j - \frac{n-j+1}{n-k+2}(k-1)}.$$

Let $h_1(n) = \frac{\frac{n-k+1}{n-j+1}j - k}{j - \frac{n-j+1}{n-k+2}(k-1)}$. Then $h_1'(n) = -\frac{j-k}{(n-j+1)^2} < 0$ and we obtain

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq \lim_{n \to \infty} \frac{j}{k} \cdot \frac{\frac{n-k+1}{n-j+1}j - k}{j - \frac{n-j+1}{n-k+2}(k-1)} = \frac{j(j-k)}{k(j-k+1)}.$$

Then, since the condition $k + \sqrt{k} \leq j$ implies $j(j-k) \geq k(j+1-k)$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq 1,$$

which proves the lemma. □

Also, a key result in the proof of the main result is the following.

Lemma 3.3. *Denoting* $s_{n,k}(x) = \binom{n}{k} x^k$, *we have*

$$\bigvee_{k=0}^{n} s_{n,k}(x) = s_{n,j}(x), \quad \textit{for all } \ x \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right], \ j = 0, 1, \ldots, n-1$$

and

$$\bigvee_{k=0}^{n} s_{n,k}(x) = s_{n,n}(x), \quad \textit{if } \ x \in [n, \infty).$$

PROOF: It is immediate from the proof of Lemma 3.1. □

4. Approximation properties

If $H_n^{(M)}(f)(x)$ represents the *Bleimann-Butzer-Hahn operator of max-product kind* defined in Introduction, then the main result is the following.

Theorem 4.1. *Let* $f : [0, \infty) \to \mathbb{R}_+$ *be continuous. Then for any* $n + 1 \geq \max\{1 + 2x, 16x(1 + x)\}$ *we have the estimate*

(3) $$|H_n^{(M)}(f)(x) - f(x)| \leq 5\omega_1 \left(f, \frac{(1+x)^{\frac{3}{2}}\sqrt{x}}{\sqrt{n+1}}\right), \ x \in [0, \infty),$$

where

$$\omega_1(f,\delta)=\sup\{|f(x)-f(y)|; x,y\in[0,\infty), |x-y|\le\delta\}.$$

PROOF: It is easy to check that the max-product Bleimann-Butzer-Hahn operator fulfills the conditions in Corollary 2.3 for $I=[0,\infty)$ and we have

$$|H_n^{(M)}(f)(x)-f(x)|\le\left(1+\frac{1}{\delta_n}H_n^{(M)}(\varphi_x)(x)\right)\omega_1(f,\delta_n), \tag{4}$$

where $\varphi_x(t)=|t-x|$ and $\omega_1(f,\delta_n)$ is the modulus of continuity on $[0,\infty)$. It is enough to estimate

$$E_n(x):=H_n^{(M)}(\varphi_x)(x)=\frac{\bigvee_{k=0}^n s_{n,k}(x)\left|\frac{k}{n+1-k}-x\right|}{\bigvee_{k=0}^n s_{n,k}(x)},\ x\in[0,\infty).$$

Let $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$, where $j=0,1,\dots,n-1$ is fixed, arbitrary. By Lemma 3.3 we easily obtain

$$E_n(x)=\bigvee_{k=0}^n M_{k,n,j}(x),\ x\in\left[\frac{j}{n-j+1},\frac{j+1}{n-j}\right].$$

In all what follows we may suppose that $j\in\{1,2,\dots,n-1\}$, because for $j=0$ we get $E_n(x)\le e\sqrt{x}\frac{1}{\sqrt{n}}$, for all $x\in[0,\frac{1}{n}]$. Indeed, in this case we obtain $M_{k,n,0}(x)=\binom{n}{k}x^k|\frac{k}{n+1-k}-x|$, which for $k=0$ gives $M_{k,n,0}(x)=x=\sqrt{x}\cdot\sqrt{x}\le\sqrt{x}\cdot\frac{1}{\sqrt{n}}$. Also, for any $k\ge1$ we have

$$\begin{aligned}M_{k,n,0}(x)&=\binom{n}{k}x^k\left(\frac{k}{n+1-k}-x\right)\le\binom{n}{k}x^k\cdot\frac{k}{n+1-k}\\&=\binom{n}{k-1}x^k=\binom{n}{k-1}x^{k-1}\cdot x\le(1+x)^n\cdot x\\&\le\left(1+\frac{1}{n}\right)^n\cdot x\le e\sqrt{x}\cdot\frac{1}{\sqrt{n}}.\end{aligned}$$

So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j=1,\dots,n-1$ is fixed, $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ and $k=0,1,\dots,n$.

In order to prove (3) we distinguish the following cases:

Case 1). Let $k\in\{j+1,\dots,n\}$ with $j\in\{0,1,\dots,n-1\}$.

Subcase a). Suppose that $k-\sqrt{k+1}\le j$. Taking into account that $\frac{n+1}{n-j+1}\le 1+x$ we have

$$\begin{aligned}M_{k,n,j}(x)&=m_{k,n,j}(x)(\frac{k}{n+1-k}-x)\le\frac{k}{n+1-k}-\frac{j}{n-j+1}\\&\le\frac{(n+1)\sqrt{k+1}}{(n-k+1)(n-j+1)}\le\frac{(n+1)\sqrt{k+1}}{(n-j+1-\sqrt{k+1})(n-j+1)}\\&\le(1+x)\frac{\sqrt{k+1}}{\left(n-j+1-\sqrt{k+1}\right)}.\end{aligned}$$

We observe that $k-\sqrt{k+1}\le j$ gives $k+1\le 4j$. Indeed, if we suppose $k+1>4j$ we get $4j-1-2\sqrt{j}<k-\sqrt{k+1}\le j$ which implies $3j-1<2\sqrt{j}$ and this is false if $j\ge 1$. Also, since $\frac{j}{n-j+1}\le x$ we have $j\le\frac{(n+1)x}{1+x}$ and for $n+1>j+2\sqrt{j}$ we get

$$\begin{aligned}M_{k,n,j}(x)&\le(1+x)\frac{2\sqrt{j}}{\left(n-j+1-2\sqrt{j}\right)}\\&\le 2\,(1+x)^{\frac{3}{2}}\,\sqrt{x}\frac{\sqrt{(n+1)}}{n+1-2\sqrt{(n+1)x(1+x)}}.\end{aligned}$$

If $n+1\ge 16x(1+x)$ then we observe that $\frac{\sqrt{n+1}}{n+1-2\sqrt{x(1+x)}\sqrt{n+1}}\le\frac{2}{\sqrt{n+1}}$. Also, the same condition ensures $n+1>j+2\sqrt{j}$. Finally we obtain $M_{k,n,j}(x)\le 4(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}$ for any $n+1\ge 4x(1+x)$.

Subcase b). Suppose now that $k-\sqrt{k+1}>j$. Since the function $f(x)=x-\sqrt{x+1}$ is nondecreasing on the interval $[0,\infty)$ it follows that there exists $\overline{k}\in\{0,1,\dots,n\}$, of maximum value, such that $\overline{k}-\sqrt{\overline{k}+1}\le j$. Then for $k_1=\overline{k}+1$ we get $k_1-\sqrt{k_1+1}>j$. Also, we have $k_1\ge j+1$. Indeed, this is a consequence of the fact that f is nondecreasing and because is easy to see that $f(j)<j$. In addition $k_1\le 4j$ and similar to subcase a) we obtain

$$M_{\overline{k}+1,n,j}(x)=m_{\overline{k}+1,n,j}(x)\left(\frac{\bar{k}+1}{n-\bar{k}}-x\right)\le 4(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}.$$

By Lemma 3.2(i), it follows that $M_{\overline{k}+1,n,j}(x)\ge M_{\overline{k}+2,n,j}(x)\ge\cdots\ge M_{n,n,j}(x)$. We thus obtain $M_{k,n,j}(x)\le 4(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}$ for any $k\in\{\overline{k}+1,\overline{k}+2,\dots n\}$, $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$.

Case 2). Let $k\in\{0,1,\dots,j\}$ with $j\in\{0,1,\dots,n-1\}$.

Subcase a). Suppose that $k+\sqrt{k}>j$. Then we obtain

$$\begin{aligned}
M_{k,n,j}(x) &= m_{k,n,j}(x)\left(x-\frac{k}{n+1-k}\right) \le \frac{j+1}{n-j}-\frac{k}{n+1-k} \\
&\le \frac{(n+2)(j-k)}{(n-k+1)\,(n-j)} \le \frac{(n+2)\sqrt{k}}{\left(n-j+1+\sqrt{k}\right)(n-j)} \\
&\le \frac{(n+2)\sqrt{j}}{\left(n-j+1+\sqrt{j}\right)(n-j)}\,.
\end{aligned}$$

Since $\frac{j}{n-j+1}\le x$ we have $j\le\frac{(n+1)x}{1+x}$. Taking these inequalities into account we get

$$\begin{aligned}
M_{k,n,j}(x) &\le \frac{(n+2)\sqrt{\frac{(n+1)x}{1+x}}}{\left(\frac{(n+1)}{1+x}+\sqrt{\frac{(n+1)x}{1+x}}\right)\left(n-\frac{(n+1)x}{1+x}\right)} \\
&= (1+x)^{\frac{3}{2}}\sqrt{x}\frac{(n+2)\sqrt{n+1}}{\left(n+1+\sqrt{(n+1)x(1+x)}\right)(n-x)} \\
&= (1+x)^{\frac{3}{2}}\sqrt{x}\frac{(n+2)}{\left(\sqrt{n+1}+\sqrt{x(1+x)}\right)(n-x)}\,.
\end{aligned}$$

We observe that

$$\frac{(n+2)}{\left(\sqrt{n+1}+\sqrt{x(1+x)}\right)(n-x)} \le \frac{n+2}{\sqrt{n+1}(n-x)} \le \frac{3\sqrt{n+1}}{2(n-x)}\,.$$

Also, if $n\ge 1+2x$ then we have $\frac{\sqrt{n+1}}{n-x}\le\frac{2}{\sqrt{n+1}}$. Finally we obtain

$$M_{k,n,j}(x)\le 3(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}\,.$$

Subcase b). Suppose now that $k+\sqrt{k}\le j$. Let $\widetilde{k}\in\{0,1,\ldots,j\}$ be the minimum value such that $\widetilde{k}+\sqrt{\widetilde{k}}>j$. Then $k_2=\widetilde{k}-1$ satisfies $k_2+\sqrt{k_2}\le j$ and

$$\begin{aligned}
M_{\widetilde{k}-1,n,j}(x) &= m_{k,n,j}(x)(x-\frac{\tilde{k}-1}{n-\tilde{k}+2}) \\
&\le \frac{j+1}{n-j}-\frac{\tilde{k}-1}{n-\tilde{k}+2}\le 3(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}\,.
\end{aligned}$$

By Lemma 3.2(ii) it follows that $M_{\widetilde{k}-1,n,j}(x)\ge M_{\widetilde{k}-2,n,j}(x)\ge\cdots\ge M_{0,n,j}(x)$. We thus obtain $M_{k,n,j}(x)\le 4(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}$ for any $k\in\{0,1,\ldots,j\}$ and

$x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$. In conclusion, collecting all the estimates in the above cases and subcases, we have

$$M_{k,n,j}(x) \leq 4(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}$$

for $n+1 \geq \max\{1+2x, 16x(1+x)\}$.

By taking $\delta_n = (1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n+1}}$ we easily get (3), which completes the proof. $\square$

Remark. It is clear that on each compact subinterval $[0,a]$, with arbitrary $a>0$, the order of approximation in Theorem 4.1 is $\mathcal{O}(1/\sqrt{n})$. In what follows, we will prove that this order cannot be improved. Indeed, for $n \in \mathbb{N}$ sufficiently large, let us denote $j(n,a) := j = [\frac{na}{a+1}]$, $k(n,a) := k = j + [\frac{\sqrt{n}}{a+1}]$ and $x(n,a) := x = \frac{j}{n-j+1}$. It is easy to check that $x(n,a) \leq a$ and $\lim_{n\to\infty} x(n,a) = a$. Then by simple calculation we get

$$\begin{aligned}
&M_{k,n,j}(x)\cdot\sqrt{n+1}\\
&= \frac{s_{n,k}(x)\left|\frac{k}{n+1-k}-x\right|}{s_{n,j}(x)}\cdot\sqrt{n+1} = \frac{j!}{k!}\frac{(n-j)!}{(n-k)!}x^{k-j}\left(\frac{k}{n+1-k}-x\right)\cdot\sqrt{n+1}\\
&= \frac{j!}{\left(j+[\frac{\sqrt{n}}{a+1}]\right)!}\,\frac{(n-j)!}{\left(n-j-[\frac{\sqrt{n}}{a+1}]\right)!}\left(\frac{j}{n-j+1}\right)^{[\frac{\sqrt{n}}{a+1}]}\frac{(n+1)^{\frac{3}{2}}(k-j)}{(n-k+1)\,(n-j+1)}\\
&= \frac{\left(n-j-[\frac{\sqrt{n}}{a+1}]+1\right)\dots(n-j)}{(j+1)(j+2)\dots\left(j+[\frac{\sqrt{n}}{a+1}]\right)}\left(\frac{j}{n-j+1}\right)^{[\frac{\sqrt{n}}{a+1}]}\frac{(n+1)^{\frac{3}{2}}(k-j)}{(n-k+1)\,(n-j+1)}.
\end{aligned}$$

It is easy to prove that if $0<a\leq b$ then $\frac{a}{b} \leq \frac{a+1}{b+1}$. Because for n sufficiently large we have $n-j-[\frac{\sqrt{n}}{a+1}]+1 \leq j+1$, it immediately follows that

$$\frac{\left(n-j-[\frac{\sqrt{n}}{a+1}]+1\right)\dots(n-j)}{(j+1)(j+2)\dots\left(j+[\frac{\sqrt{n}}{a+1}]\right)} \geq \left(\frac{n-j-[\frac{\sqrt{n}}{a+1}]+1}{j+1}\right)^{[\frac{\sqrt{n}}{a+1}]}$$

which implies

$$\begin{aligned}
&M_{k,n,j}(x)\cdot\sqrt{n+1}\\
&\geq \left(\frac{n-j-[\frac{\sqrt{n}}{a+1}]+1}{j+1}\right)^{[\frac{\sqrt{n}}{a+1}]}\left(\frac{j}{n-j+1}\right)^{[\frac{\sqrt{n}}{a+1}]}\frac{(n+1)^{\frac{3}{2}}(k-j)}{(n-k+1)(n-j+1)}.
\end{aligned}$$

We have

$$\lim_{n\to\infty}\left(\frac{n-j-[\frac{\sqrt{n}}{a+1}]+1}{j+1}\right)^{[\frac{\sqrt{n}}{a+1}]}\left(\frac{j}{n-j+1}\right)^{[\frac{\sqrt{n}}{a+1}]}$$
$$=\lim_{n\to\infty}\left(\frac{n-\sqrt{n}+a+1}{an+a+1}\right)^{\frac{\sqrt{n}}{a+1}}\left(\frac{na}{n+a+1}\right)^{\frac{\sqrt{n}}{a+1}}$$
$$=\lim_{n\to\infty}\left(\frac{an-a\sqrt{n}+a^2+a}{an+a+1}\right)^{\frac{\sqrt{n}}{a+1}}\left(\frac{an}{an+a^2+a}\right)^{\frac{\sqrt{n}}{a+1}}$$
$$=e^{-\frac{1}{a+1}}$$

and

$$\lim_{n\to\infty}\frac{(n+1)^{\frac{3}{2}}(k-j)}{(n-k+1)(n-j+1)}=(a+1)^{\frac{3}{2}}.$$

It follows that there exists $n_0\in\mathbb{N}$ such that

$$M_{k,n,j}(x)\geq(a+1)e^{-\frac{1}{a+1}}\frac{1}{\sqrt{n+1}}$$

for any $n\geq n_0$ which implies the desired conclusion.

In what follows we will prove that for some subclasses of functions f, the order of approximation $\omega_1(f;(1+x)^{\frac{3}{2}}\sqrt{x}/\sqrt{n})$ in Theorem 4.1 can essentially be improved to $\omega_1(f;(1+x)^2/n)$.

For this purpose, for any $j\in\{0,1,\ldots,n-1\}$, $k\in\{0,1,\ldots,n\}$ let us define the functions $f_{k,n,j}:[\frac{j}{n-j+1},\frac{j+1}{n-j}]\to\mathbb{R}$, and $f_{k,n,n}:[n,\infty)\to\mathbb{R}$

$$f_{k,n,j}(x)=m_{k,n,j}(x)f\left(\frac{k}{n+1-k}\right)=\frac{s_{n,k}(x)}{s_{n,j}(x)}f\left(\frac{k}{n+1-k}\right)$$
$$=\frac{j!(n-j)!}{k!(n-k)!}\cdot x^{k-j}f\left(\frac{k}{n+1-k}\right).$$

Then it is clear that for any $j\in\{0,1,\ldots,n-1\}$ and $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $j=n$ and $x\in[n,\infty)$ we can write

$$H_n^{(M)}(f)(x)=\bigvee_{k=0}^{n}f_{k,n,j}(x).$$

Also, we need the following auxiliary lemmas.

Lemma 4.2. *Let* $f:[0,\infty)\to[0,\infty)$ *be such that*

$$H_n^{(M)}(f)(x)=\max\{f_{j,n,j}(x),f_{j+1,n,j}(x)\}\quad\textit{for all}\quad x\in\left[\frac{j}{n-j+1},\frac{j+1}{n-j}\right].$$

Then

$$\left|H_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{(1+x)^2}{n}\right), \quad \text{for all } x \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right],$$

where $n \geq 2x$ *and* $\omega_1(f;\delta) = \max\{|f(x) - f(y)|; x, y \in [0,\infty), |x-y| \leq \delta\} < \infty$.

Proof: We distinguish two cases:

Case (i). Let $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$ be fixed such that $H_n^{(M)}(f)(x) = f_{j,n,j}(x)$. By simple calculation we get $0 \leq x - \frac{j}{n+1-j} \leq \frac{j+1}{n-j} - \frac{j}{n+1-j} = \frac{n+1}{(n-j)(n-j+1)}$. Since $j \leq \frac{(n+1)x}{1+x}$ we have $\frac{n+1}{(n-j)(n-j+1)} \leq \frac{n+1}{(n-\frac{(n+1)x}{1+x})(n-\frac{(n+1)x}{1+x}+1)} = (1+x)^2\frac{1}{(n-x)}$. Since $f_{j,n,j}(x) = f(\frac{j}{n+1-j})$, it follows that

$$\left|H_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{(1+x)^2}{n-x}\right).$$

If $n \geq 2x$ we have $\frac{(1+x)^2}{n-x} \leq 2\frac{(1+x)^2}{n}$ and we obtain

$$\left|H_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{(1+x)^2}{n}\right).$$

Case (ii). Let $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$ be such that $H_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$. We have two subcases:

(ii$_a$) $H_n^{(M)}(f)(x) \leq f(x)$, when evidently $f_{j,n,j}(x) \leq f_{j+1,n,j}(x) \leq f(x)$ and we immediately get

$$\begin{aligned}\left|H_n^{(M)}(f)(x) - f(x)\right| &= |f_{j+1,n,j}(x) - f(x)| \\ &= f(x) - f_{j+1,n,j}(x) \leq f(x) - f_{j,n,j}(x) \\ &\leq \omega_1\left(f; \frac{(1+x)^2}{n-x}\right) \leq 2\omega_1\left(f; \frac{(1+x)^2}{n}\right).\end{aligned}$$

(ii$_b$) $H_n^{(M)}(f)(x) > f(x)$, when

$$\begin{aligned}\left|H_n^{(M)}(f)(x) - f(x)\right| &= f_{j+1,n,j}(x) - f(x) \\ &= m_{j+1,n,j}(x) f(\frac{j+1}{n-j}) - f(x) \leq f(\frac{j+1}{n-j}) - f(x).\end{aligned}$$

Because $0 \leq \frac{j+1}{n-j} - x \leq (1+x)^2\frac{1}{(n-x)}$ it follows

$$f(\frac{j+1}{n-j}) - f(x) \leq \omega_1\left(f; \frac{(1+x)^2}{n-x}\right) \leq 2\omega_1\left(f; \frac{(1+x)^2}{n}\right),$$

for $n \geq 2x$, which proves the lemma. □

Lemma 4.3. *If a function $f : [0,\infty) \to [0,\infty)$ is concave, then the function $g : (0,\infty) \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing.*

PROOF: Let $x, y \in (0,\infty)$ be with $x \le y$. Then

$$f(x) = f\left(\frac{x}{y}y + \frac{y-x}{y}0\right) \ge \frac{x}{y}f(y) + \frac{y-x}{y}f(0) \ge \frac{x}{y}f(y),$$

which implies $\frac{f(x)}{x} \ge \frac{f(y)}{y}$. □

Corollary 4.4. *If $f : [0,\infty) \to [0,\infty)$ is bounded, nondecreasing and such that the function $g : [0,\infty) \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing, then*

$$\left|H_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{(1+x)^2}{n}\right), \quad \textit{for all } \; x \in [0,\infty), n \ge 2x.$$

PROOF: Since f is nondecreasing it follows (see the proof of Theorem 5.3 in the next section)

$$H_n^{(M)}(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x), \quad \text{for all } \; x \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right].$$

Let $x \in [0,\infty)$ and $j \in \{0,1,\ldots,n-1\}$ such that $x \in [\frac{j}{n-j+1}, \frac{j+1}{n-j}]$. Let $k \in \{1,\ldots,n\}$ be with $k \ge j+1$. Then

$$f_{k+1,n,j}(x) = \frac{j!(n-j)!}{(k+1)!(n-k-1)!} \cdot x^{k-j+1} f\left(\frac{k+1}{n-k}\right).$$

Since $g(x)$ is nonincreasing we get $\frac{f(\frac{k+1}{n-k})}{\frac{k+1}{n-k}} \le \frac{f(\frac{k}{n-k+1})}{\frac{k}{n-k+1}}$ that is $f(\frac{k+1}{n-k}) \le \frac{k+1}{n-k}\frac{n-k+1}{k}f(\frac{k}{n-k+1})$. From $x \le \frac{j+1}{n-j}$ it follows

$$\begin{aligned} f_{k+1,n,j}(x) &\le \frac{j!(n-j)!}{k!(n-k)!}\frac{j+1}{n-j}x^{k-j}\frac{n-k+1}{k}f\left(\frac{k}{n-k+1}\right) \\ &= f_{k,n,j}(x)\frac{j+1}{n-j}\frac{n-k+1}{k} \le f_{k,n,j}(x). \end{aligned}$$

Thus we obtain

$$f_{j+1,n,j}(x) \ge f_{j+2,n,j}(x) \ge \cdots \ge f_{n,n,j}(x)$$

that is

$$H_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \quad \text{for all } \; x \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right],$$

and from Lemma 4.2 we obtain $|H_n^{(M)}(f)(x) - f(x)| \le 2\omega_1(f; \frac{(1+x)^2}{n})$. □

Corollary 4.5. *Let $f : [0,\infty) \to [0,\infty)$ be a bounded, nondecreasing concave function. Then*

$$\left|H_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{(1+x)^2}{n}\right), \quad \textit{for all} \ \ x \in [0,\infty), n \ge 2x.$$

PROOF: The proof is immediate by Lemma 4.3 and Corollary 4.4. □

Remarks. 1) If we suppose, for example, that in addition to the hypothesis in Corollary 4.5, $f : [0,\infty) \to [0,\infty)$ is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \le M|x-y|$, for all $x, y \in [0,\infty)$, then it follows that the order of uniform approximation on $[0,\infty)$ by $H_n^{(M)}(f)(x)$ is $2(1+x)^2\frac{1}{n}$, which is essentially better than the order $4(1+x)^{\frac{3}{2}}\sqrt{x}\frac{1}{\sqrt{n}}$ obtained from Theorem 4.1 for f Lipschitz on $[0,\infty)$.

2) Let us recall here also, that for the linear Bleimann-Butzer-Hahn operator given by

$$H_n(f)(x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n}\binom{n}{k}x^k f\left(\frac{k}{n+1-k}\right),$$

we have the estimate (see [9])

$$|H_n(f)(x) - f(x)| \le C\omega_2(f; (1+x)\frac{\sqrt{x}}{\sqrt{n}}) + x(1+x)^2\|f\|/n, n \in \mathbb{N}, x \in [0,\infty),$$

where $\|f\| = \sup\{|f(x)|; x \in [0,\infty)\}$ and $\omega_2(f;\delta)$ is the second order modulus of smoothness on $[0,\infty)$ given by

$$\omega_2(f;\delta) = \sup\{\sup\{|f(x+h)) - 2f(x) + f(x-h)|; x \pm h \in [0,\infty)\}, h \in [0,\delta]\}.$$

Now, if f is, for example, a nondecreasing concave polygonal line on $[0,\infty)$, constant on an interval $[a,\infty)$, then by simple reasonings we get that $\omega_2(f;\delta) \sim \delta$ for $\delta \le 1$, which shows that the order of approximation obtained in this case by the linear Bleimann-Butzer-Hahn operator is exactly $\frac{(1+x)\sqrt{x}}{\sqrt{n}}$. On the other hand, since such of function f obviously is a Lipschitz function on $[0,\infty)$ (as having bounded all the derivative numbers), we get by Corollary 4.5 that the order of approximation by the max-product Bleimann-Butzer-Hahn operator is less than $\frac{(1+x)^2}{n}$, which is essentially better than $\frac{(1+x)\sqrt{x}}{\sqrt{n}}$ on any compact subinterval of $[0,\infty)$. In a similar manner, by Corollary 4.4 we can produce many subclasses of functions for which the order of approximation given by the max-product Bleimann-Butzer-Hahn operator is essentially better than the order of approximation given by the linear Bleimann-Butzer-Hahn operator, on any compact subinterval of $[0,\infty)$. Intuitively, the max-product Bleimann-Butzer-Hahn operator has better approximation properties than its linear counterpart, for non-differentiable functions in a finite number of points (with the graphs having some "corners"), as for example for functions defined as a maximum of a finite number of continuous functions on $[0,\infty)$.

3) Since it is clear that a bounded nonincreasing concave function on $[0,\infty)$ necessarily reduces to a constant function, the approximation of such functions is not of interest.

5. Shape preserving properties

In this section we will present some shape preserving properties. First we have the following simple result.

Remark. Note that because of the continuity of $H_n^{(M)}(f)(x)$ on $[0,\infty)$, it will suffice to prove the shape properties of $H_n^{(M)}(f)(x)$ on $(0,\infty)$ only. As a consequence, in the notations and proofs below we always may suppose that $x>0$.

As in Section 4, for any $j\in\{0,1,\ldots,n-1\}$, $k\in\{0,1,\ldots,n\}$ let us define the functions $f_{k,n,j}:[\frac{j}{n-j+1},\frac{j+1}{n-j}]\to\mathbb{R}$, $f_{k,n,n}:[n,\infty)\to\mathbb{R}$

$$f_{k,n,j}(x)=m_{k,n,j}(x)f\left(\frac{k}{n+1-k}\right)=\frac{s_{n,k}(x)}{s_{n,j}(x)}f\left(\frac{k}{n+1-k}\right)$$
$$=\frac{j!(n-j)!}{k!(n-k)!}\cdot x^{k-j}f\left(\frac{k}{n+1-k}\right).$$

For any $j\in\{0,1,\ldots,n-1\}$ and $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $j=n$ and $x\in[n,\infty)$ we can write

$$H_n^{(M)}(f)(x)=\bigvee_{k=0}^{n}f_{k,n,j}(x).$$

Lemma 5.1. *If $f:[0,\infty)\to\mathbb{R}_+$ is a nondecreasing function then $f_{k,n,j}(x)\geq f_{k-1,n,j}(x)$ for any $j\in\{0,1,\ldots,n\}$ and $k\in\{1,2,\ldots,n\}$, with $k\leq j$ and $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $x\in[n,\infty]$ for $j=n$.*

Proof: Because $k\leq j$, by the proof of Lemma 3.1, case 2), it follows that $m_{k,n,j}(x)\geq m_{k-1,n,j}(x)$. From the monotonicity of f we have $f(\frac{k}{n+1-k})\geq f(\frac{k-1}{n+2-k})$, so we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n+1-k}\right)\geq m_{k-1,n,j}(x)f\left(\frac{k-1}{n+2-k}\right),$$

which proves the lemma. □

Corollary 5.2. *If $f:[0,\infty)\to\mathbb{R}_+$ is nonincreasing then $f_{k,n,j}(x)\geq f_{k+1,n,j}(x)$ for any $j\in\{0,1,\ldots,n\}$ and $k\in\{0,1,\ldots,n-1\}$ with $k\geq j$ and $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $x\in[n,\infty]$ for $j=n$.*

Proof: Because $k\geq j+1$, by the proof of Lemma 3.1, case 1), it follows that $m_{k,n,j}(x)\geq m_{k+1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n+1-k}\right)\geq$

$f\left(\frac{k+1}{n-k}\right)$. Thus we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n+1-k}\right) \geq m_{k+1,n,j}(x)f\left(\frac{k+1}{n-k}\right),$$

which proves the corollary. $\square$

Theorem 5.3. *If $f:[0,\infty)\to\mathbb{R}_+$ is nondecreasing and bounded on $[0,\infty)$ then $H_n^{(M)}(f)$ is nondecreasing (and bounded).*

Proof: Because $H_n^{(M)}(f)$ is continuous (and bounded) on $[0,\infty)$, it suffices to prove that on each subinterval of the form $[\frac{j}{n-j+1},\frac{j+1}{n-j}]$, with $j\in\{0,1,\ldots,n-1\}$, *or* $[n,\infty)$ for $j=n$, $H_n^{(M)}(f)$ is nondecreasing.

So let $j\in\{0,1,\ldots,n-1\}$ and $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $x\in[n,\infty)$ for $j=n$. Because f is nondecreasing, from Lemma 5.1 it follows that

$$f_{j,n,j}(x)\geq f_{j-1,n,j}(x)\geq f_{j-2,n,j}(x)\geq\cdots\geq f_{0,n,j}(x).$$

But then it is immediate that

$$H_n^{(M)}(f)(x)=\bigvee_{k\geq j} f_{k,n,j}(x),$$

for all $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $x\in[n,\infty)$ for $j=n$. Clearly that for $k\geq j$ the functions $f_{k,n,j}$ are nondecreasing and since $H_n^{(M)}(f)$ is defined as supremum of nondecreasing functions, it follows that it is nondecreasing. $\square$

Corollary 5.4. *If $f:[0,\infty)\to\mathbb{R}_+$ is nonincreasing then $H_n^{(M)}(f)$ is nonincreasing.*

Proof: By hypothesis, f implicitly is bounded on $[0,\infty)$. Because $H_n^{(M)}(f)$ is continuous and bounded on $[0,\infty)$, it suffices to prove that on each subinterval of the form $[\frac{j}{n-j+1},\frac{j+1}{n-j}]$, with $j\in\{0,1,\ldots,n-1\}$, *or* $[n,\infty)$ for $j=n$, $H_n^{(M)}(f)$ is nonincreasing.

So let $j\in\{0,1,\ldots,n-1\}$ and $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $x\in[n,\infty)$ for $j=n$. Because f is nonincreasing, from Corollary 5.2 it follows that

$$f_{j,n,j}(x)\geq f_{j+1,n,j}(x)\geq f_{j+2,n,j}(x)\geq\ldots f_{n,n,j}(x).$$

But then it is immediate that

$$H_n^{(M)}(f)(x)=\bigvee_{k\geq 0}^{j} f_{k,n,j}(x),$$

for all $x\in[\frac{j}{n-j+1},\frac{j+1}{n-j}]$ or $x\in[n,\infty)$ for $j=n$. Clearly that for $k\leq j$ the function $f_{k,n,j}$ is nonincreasing and since $H_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. $\square$

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

Definition 5.5. Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$. One says that f is quasi-convex on $[0, \infty)$ if it satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \quad \text{for all } \ x, y \in [0, \infty) \ \text{ and } \ \lambda \in [0, 1]$$

(see e.g. the book [8, p. 4, (iv)]).

Remark. By [10], the continuous function f is quasi-convex on the bounded interval $[0, \infty)$, equivalently means that there exists a point $c \in [0, \infty)$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$.

The class of quasi-convex functions includes the both classes of nondecreasing functions and of nonincreasing functions (obtained from the class of quasi-convex functions by taking $c = 0$ and $c = \infty$, respectively). Also, it obviously includes the class of convex functions on $[0, \infty)$.

Corollary 5.6. *If $f : [0, \infty) \to \mathbb{R}_+$ is continuous and quasi-convex on $[0, \infty)$ then for all $n \in \mathbb{N}$, $H_n^{(M)}(f)$ is quasi-convex on $[0, \infty)$.*

Proof: If f is nonincreasing (or nondecreasing) on $[0, \infty)$ (that is the point $c = \infty$ (or $c = 0$) in the above Remark) then by the Corollary 5.4 (or Theorem 5.3, respectively) it follows that for all $n \in \mathbb{N}$, $H_n^{(M)}(f)$ is nonincreasing (or nondecreasing) on $[0, \infty)$.

Suppose now that there exists $c \in (0, \infty)$, such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$. Define the functions $F, G : [0, \infty) \to \mathbb{R}_+$ by $F(x) = f(x)$ for all $x \in [0, c]$, $F(x) = f(c)$ for all $x \in [c, \infty)$ and $G(x) = f(c)$ for all $x \in [0, c]$, $G(x) = f(x)$ for all $x \in [c, \infty)$.

It is clear that F is nonincreasing and continuous on $[0, \infty)$, G is nondecreasing and continuous on $[0, \infty)$ and that $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0, \infty)$.

But it is easy to show (see also Remark 1 after the proof of Lemma 2.1) that

$$H_n^{(M)}(f)(x) = \max\{H_n^{(M)}(F)(x), H_n^{(M)}(G)(x)\}, \quad \text{for all } \ x \in [0, \infty),$$

where by the Corollary 5.4 and Theorem 5.3, $H_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0, \infty)$ and $H_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0, \infty)$. We have two cases: 1) $H_n^{(M)}(F)(x)$ and $H_n^{(M)}(G)(x)$ do not intersect each other; 2) $H_n^{(M)}(F)(x)$ and $H_n^{(M)}(G)(x)$ intersect each other.

Case 1). We have $\max\{H_n^{(M)}(F)(x), H_n^{(M)}(G)(x)\} = H_n^{(M)}(F)(x)$ for all $x \in [0, \infty)$ or $\max\{H_n^{(M)}(F)(x), H_n^{(M)}(G)(x)\} = H_n^{(M)}(G)(x)$ for all $x \in [0, \infty)$, which obviously proves that $H_n^{(M)}(f)(x)$ is quasi-convex on $[0, \infty)$.

Case 2). In this case it is clear that there exists a point $c' \in [0, \infty)$ such that $H_n^{(M)}(f)(x)$ is nonincreasing on $[0, c']$ and nondecreasing on $[c', \infty)$, which by the considerations in the above remark implies that $H_n^{(M)}(f)(x)$ is quasiconvex on $[0, \infty)$ and proves the corollary. $\square$

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