Uniform non-squareness and property (β) of Besicovitch-Orlicz spaces of almost periodic functions with Orlicz norm

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Abstract. We characterize the uniform non-squareness and the property (β) of Besicovitch-Orlicz spaces of almost periodic functions equipped with Orlicz norm.

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1. Introduction

A Banach space $(E, \|\cdot\|)$ is said to have the *fixed point property* (f.p.p.) if for any nonempty, closed, bounded and convex subset A of E, any non expansive mapping $P : A \to A$ (satisfying $\|P(x) - P(y)\| \leq \|x - y\|$ for any $x, y \in A$) has a fixed point $z \in A$ (that is, $P(z) = z$).

The fixed point property is a fundamental tool in applied mathematics. It is then important to obtain some practical characterizations of this property in general Banach spaces as well as in some usual function spaces.

Metric properties of Banach spaces are very useful in this direction. In [4], R.C. James introduced the notion of uniform non-squareness, namely, a Banach space $(E, \|\cdot\|)$ is said to be *uniformly non-square* if there is a $\delta > 0$ such that for any x, y in the unit sphere of E we have

$$\|x + y\| \leq 2(1 - \delta) \text{ or } \|x - y\| \leq 2(1 - \delta).$$

It was proved that uniformly non-square Banach spaces are reflexive. Recently, a new interest on this property was motivated by its connection with the fixed point property. It was proved that uniformly non-square Banach spaces have the fixed point property (see [7]).

For the same purpose, S. Rolewicz ([14]) was the first who introduced the metric property called (β)-property. In [8] this notion was reformulated in a more convenient form: A Banach space $(E, \|\cdot\|)$ is said to have the *property* (β) if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in B(E)$ and each sequence

$\{x_n\} \subset B(E)$ with $\operatorname{sep}(\{x_n\}) \geq \varepsilon$, there is an index k for which

$$\left\| \frac{1}{2}(x + x_k) \right\| \leq 1 - \delta,$$

where $\operatorname{sep}(\{x_n\}) = \inf\{\|x_n - x_m\|, n \neq m\}$.

The importance of the property (β) is related to the following assertions:

If a Banach space E has the property (β) then it is reflexive and both E and E^* have the fixed point property.

All these properties and their characterizations were subject to investigations in the class of Orlicz spaces (see [9] and [3]).

Here, we consider such a characterization in the widest class of Besicovitch-Orlicz spaces of almost periodic functions.

2. Preliminaries

2.1 Orlicz functions. A function $\phi : \mathbb{R} \to \mathbb{R}^+$ is said to be an *Orlicz function* if it is even, convex, $\phi(0) = 0$, $\phi(u) > 0$ iff $u \neq 0$ and $\lim_{u \to 0} \frac{\phi(u)}{u} = 0$, $\lim_{u \to \infty} \frac{\phi(u)}{u} = +\infty$.

An Orlicz function admits a derivative ϕ' everywhere except perhaps on a countable set of points. It satisfies $\phi'(0) = 0$, $\phi'(|u|) > 0$ whenever $u \neq 0$ and $\lim_{|u| \to +\infty} \phi'(|u|) = +\infty$, so that it is increasing to infinity (see [1]).

For every Orlicz function ϕ we define the complementary function ψ by the formula

$$\psi(y) = \sup\{x|y| - \phi(x), x \geq 0\}, \quad y \in \mathbb{R}.$$

The complementary function ψ is also an Orlicz function.

The pair (ϕ, ψ) satisfies the Young inequality

$$xy \leq \phi(x) + \psi(y),\ x \in \mathbb{R},\ \ y \in \mathbb{R}.$$

We say that an Orlicz function ϕ satisfies the Δ_2-*condition* if there exist a constant $K > 2$ and $u_0 \geq 0$ for which

$$\phi(2u) \leq K\phi(u), \quad u \geq u_0.$$

In this case we write $\phi \in \triangle_2$. If $\phi \in \triangle_2$ and $\psi \in \triangle_2$ we write $\phi \in \triangle_2 \cap \nabla_2$ (see [1]).

If $\psi \in \Delta_2$ then for every $0 < \gamma < 1$ there exists $0 < \beta < 1$ such that

$$\phi(\gamma u) \leq \gamma\beta\phi(u) \quad \forall\, u \geq u_0 \ \text{ (see [2] and [5]).} \tag{2.1}$$

An Orlicz function ϕ is said to be

(1) *strictly convex* if $\phi(\frac{u+v}{2}) < \frac{\phi(u)+\phi(v)}{2}$ for all $u, v \in \mathbb{R}$, with $u \neq v$,

(2) *uniformly convex* on $[d,+\infty[$ if for any $\varepsilon>0$ there exists $\delta(\varepsilon)\in]0,1[$ such that the inequality

$$\phi\left(\frac{u+v}{2}\right)\le(1-\delta(\varepsilon))\left(\frac{\phi(u)+\phi(v)}{2}\right)$$

holds true for all $u,v\in[d,+\infty[$ satisfying $|u-v|\ge\varepsilon\max(|u|,|v|)$ (see [1]).

2.2 The Besicovitch-Orlicz space of almost periodic functions. Let $M(\mathbb{R})$ be the set of all real Lebesgue measurable functions and μ the Lebesgue measure on $\mathbb{R}$. Let ϕ be an Orlicz function.

The functional $\rho_{B^\phi}: M(\mathbb{R})\to[0,+\infty]$,

$$\rho_{B^\phi}(f)=\overline{\lim_{T\to+\infty}}\frac{1}{2T}\int_{-T}^{+T}\phi(|f(t)|)\,d\mu,$$

is a pseudomodular (see [10]).
The associated modular space

$$\begin{aligned}B^\phi(\mathbb{R}) &= \left\{f\in M(\mathbb{R}),\lim_{\alpha\to0}\rho_{B^\phi}(\alpha f)=0\right\}\\ &= \{f\in M(\mathbb{R}),\rho_{B^\phi}(\lambda f)<+\infty\ \text{ for some }\ \lambda>0\}\end{aligned}$$

is called the *Besicovitch-Orlicz space*. This space is endowed with the pseudonorm

$$\|f\|_{B^\phi}=\inf\left\{k>0,\rho_{B^\phi}\left(\frac{f}{k}\right)\le1\right\},\ f\in B^\phi(\mathbb{R}),$$

called the *Luxemburg norm* (see [10]).

Let now $\mathcal{A}$ be the linear set of all generalized trigonometric polynomials, i.e.

$$\mathcal{A}=\left\{P(t)=\sum_{j=1}^n\alpha_j\exp(i\lambda_jt),\ \lambda_j\in\mathbb{R},\ \alpha_j\in\mathbb{C},\ j\in\mathbb{N}\right\}.$$

The *Besicovitch-Orlicz space of almost periodic functions* denoted by $B^\phi a.p.(\mathbb{R})$ (resp. $\widetilde{B}^\phi a.p.(\mathbb{R})$) is the closure of $\mathcal{A}$ in $B^\phi(\mathbb{R})$ with respect to the pseudonorm $\|\cdot\|_{B^\phi}$ (resp. to the modular ρ_{B^ϕ}); more exactly:

$$\begin{aligned}B^\phi a.p.(\mathbb{R}) &= \left\{f\in B^\phi(\mathbb{R}):\exists(P_n)_{n\ge1}\subset\mathcal{A}\text{ s.t. }\lim_{n\to\infty}\|f-P_n\|_{B^\phi}=0\right\}\\ &= \left\{f\in B^\phi(\mathbb{R}):\exists(P_n)_{n\ge1}\subset\mathcal{A}\text{ s.t. }\forall\,k>0,\lim_{n\to\infty}\rho_{B^\phi}(k(f-P_n))=0\right\},\\ \widetilde{B}^\phi a.p.(\mathbb{R}) &= \left\{f\in B^\phi(\mathbb{R}):\exists(P_n)_{n\ge1}\subset\mathcal{A}\text{ s.t. }\exists k>0,\lim_{n\to\infty}\rho_{B^\phi}(k(f-P_n))=0\right\}.\end{aligned}$$

It is known that $B^\phi a.p.(\mathbb{R})=\widetilde{B}^\phi a.p.(\mathbb{R})$ iff $\phi\in\Delta_2$ (see [10]).

From [10] we know that if $f \in B^{\phi}a.p.(\mathbb{R})$ then the limit in the expression of $\rho_{B^{\phi}}(f)$ exists (and is finite), i.e.:

$$\rho_{B^{\phi}}(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} \phi(|f(t)|)\, d\mu, \quad f \in B^{\phi}a.p.(\mathbb{R}).$$

This fact is very useful in our computations.

Beside the Luxemburg pseudonorm defined on $B^{\phi}a.p.(\mathbb{R})$, we may define an Orlicz pseudonorm in the following way

$$\|f\|^{o}_{B^{\phi}} = \sup\left\{M(fg),\ g \in B^{\psi}a.p.(\mathbb{R}),\ \rho_{B^{\psi}}(g) \le 1\right\},$$

where $M(f) = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{+T} |f(t)|\, d\mu$.

Let $\mathcal{P}(\mathbb{R})$ be the family of all subsets of $\mathbb{R}$ and $\Sigma(\mathbb{R})$ the Σ-algebra of Lebesgue measurable sets. We define the set function

$$\overline{\mu}(A) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{+T} \chi_A(t)\, dt = \overline{\lim_{T\to\infty}} \frac{1}{2T}\mu(A \cap [-T, +T]),$$

where χ_A denotes the characteristic function of $A \in \Sigma(\mathbb{R})$.

Some geometrical properties of these spaces are considered in [10], [11], [12] and [13], where it is shown that the space $(B^{\phi}a.p.(\mathbb{R}), \|\cdot\|^{o}_{B^{\Phi}})$ is uniformly convex if and only if ϕ is uniformly convex and it is of Δ_2-type (see [13]). It is reflexive if and only if $\phi \in \Delta_2 \cap \nabla_2$ (see [11]).

3. Auxiliary results

We give here some technical results that will be helpful in the proof of our main theorem.

Lemma 1 ([10]). *Let $\{a_n\}_{n\ge1}$, $a_n > 0$, be a sequence of real numbers. With every $n \ge 1$, we associate a measurable set $A_n \subset [0, 1]$ in such a way that:*

(a) $A_i \cap A_j = \emptyset$ *if* $i \ne j$ *and* $\bigcup_{n\ge1} A_n \subset [0, \alpha]$, *where* $0 < \alpha < 1$;
(b) $\sum_{n\ge1} \phi(a_n)\mu(A_n) < +\infty$.

Consider the function $f = \sum_{n\ge1} a_n\chi_{A_n}$ on $[0, 1]$ and let $\widetilde{f}$ be the periodic extension of f to the whole $\mathbb{R}$, with period $\tau = 1$. Then $\widetilde{f} \in \widetilde{B}^{\phi}a.p.(\mathbb{R})$.

Proposition 1 ([11]). *Let $f \in B^{\phi}a.p.(\mathbb{R})$, $\|f\|_{B^{\phi}} \ne 0$. Then,*

(1) $\|f\|^{o}_{B^{\phi}} = \inf_{k>0} \frac{1}{k}[1 + \rho_{B^{\phi}}(kf)]$. *Moreover, the set* $K(f) = \{k > 0 : \|f\|^{o}_{B^{\phi}} = \frac{1}{k}[1 + \rho_{B^{\phi}}(kf)]\}$ *is nonempty.*
(2) $\|f\|_{B^{\phi}} \le \|f\|^{o}_{B^{\phi}} \le 2\|f\|_{B^{\phi}}$.

These two norms are equivalent, nevertheless, the corresponding geometric properties between them are different.

Lemma 2. (1) *If $\phi \in \Delta_2$ then*

$$\overline{k} = \inf\left\{k \in K(f) : \|f\|^0_{B^\phi} = 1, f \in B^\phi a.p.(\mathbb{R})\right\} > 1.$$

(2) *If ψ, the conjugate function of ϕ, is of Δ_2-type, then the set*

$$Q = \left\{k \in K(f), a \leq \|f\|^o_{B^\phi} \leq b,\ f \in B^\phi a.p.(\mathbb{R})\right\}$$

is bounded for each $b \geq a > 0$.

Proof: The arguments are exactly the same as those used in the Orlicz space case (see [1]), so we omit it. □

Lemma 3. [12] *Let ϕ satisfy the Δ_2-condition, and $L^\phi([0,1])$ be the classical Orlicz space of functions defined on $[0,1]$. Then the mapping defined by:*

$$\begin{array}{rccc} I: & (L^\phi([0,1]), \|\cdot\|_\phi) & \rightarrow & B^\phi a.p.(\mathbb{R}) \\ & f & \mapsto & \widetilde{f} \end{array}$$

(where $\widetilde{f}$ is the periodic extension of f), is an isometry for the respective modulars and also an isometry for the respective norms.

4. Results

Now, we can state our main results concerning the characterization of uniform non-squareness and property (β) in the Besicovitch-Orlicz space of almost periodic functions.

Theorem 1. *The space $B^\phi a.p.(\mathbb{R})$ equipped with the Orlicz norm is uniformly non-square if and only if both the functions ϕ and its Young's conjugate ψ satisfy the Δ_2-condition.*

Proof: **Sufficiency**. Let f_1, f_2 be two functions in the unit sphere of $(B^\phi a.p(\mathbb{R}), \|\cdot\|^o_{B^\phi})$ and let $k_i \in K(f_i)$, i.e.,

$$1 = \|f_i\|^o_{B^\phi} = \frac{1}{k_i}\left(1 + \rho_{B^\phi}(k_i f_i)\right) \quad i = 1, 2.$$

Define

$$\begin{aligned} a &= \inf\left\{k \in K(f) : \|f\|^o_{B^\phi} = 1, f \in B^\phi a.p(\mathbb{R})\right\}, \\ b &= \sup\left\{k \in K(f) : \|f\|^o_{B^\phi} = 1, f \in B^\phi a.p(\mathbb{R})\right\}. \end{aligned}$$

By Lemma 2, $1 < a \leq b < +\infty$.

From the convexity of ϕ we have

$$\phi\left(\frac{k_1 k_2}{k_1 + k_2}(u_1 + u_2)\right) \leq \frac{k_2}{k_1 + k_2}\phi(k_1 u_1) + \frac{k_1}{k_1 + k_2}\phi(k_2 u_2). \tag{4.1}$$

Note that since ϕ is even, we may assume without loss of generality that $u_1 \geq 0$, $u_2 \geq 0$.

Take $u_0 = \frac{1}{b}\phi^{-1}\left[\frac{b}{2}(1-\frac{1}{a})\right]$, by (2.1) there exists $0 < \delta < 1$ such that

$$\phi\left(\frac{bu}{a+b}\right) \leq (1-\delta)\frac{b}{a+b}\phi(u), \quad |u| \geq u_0. \tag{4.2}$$

Using the fact that the function $u \mapsto \frac{\phi(u)}{u}$ is increasing, we obtain

$$\phi(lu) \leq l(1-\delta)\phi(u) \quad \forall\, l \in \left[\frac{a}{a+b}, \frac{b}{a+b}\right] \text{ and } |u| \geq u_0.$$

Assume that $\max(u_1, u_2) \geq u_0$. Then, since $\frac{k_i}{k_1+k_2} \in \left[\frac{a}{a+b}, \frac{b}{a+b}\right]$, $i = 1, 2$, we obtain

$$\begin{aligned} \phi\left(\frac{k_1k_2}{k_1+k_2}(u_1-u_2)\right) &= \phi\left(\frac{k_2}{k_1+k_2}k_1u_1 - \frac{k_1}{k_1+k_2}k_2u_2\right) \\ &\leq \max\left\{\phi\left(\frac{k_2}{k_1+k_2}k_1u_1\right), \phi\left(\frac{k_1}{k_1+k_2}k_2u_2\right)\right\} \\ &\leq \phi\left(\frac{k_2}{k_1+k_2}k_1u_1\right) + \phi\left(\frac{k_1}{k_1+k_2}k_2u_2\right). \end{aligned}$$

It follows that

$$\phi\left(\frac{k_1k_2}{k_1+k_2}(u_1-u_2)\right) \leq (1-\delta)\left[\frac{k_2}{k_1+k_2}\phi(k_1u_1) + \frac{k_1}{k_1+k_2}\phi(k_2u_2)\right]. \tag{4.3}$$

Combining (4.1) and (4.3) we can write

$$\begin{aligned} &\frac{k_1+k_2}{k_1k_2}\left[\phi\left(\frac{k_1k_2}{k_1+k_2}(u_1+u_2)\right) + \phi\left(\frac{k_1k_2}{k_1+k_2}(u_1-u_2)\right)\right] \\ &\leq (2-\delta)\left[\frac{1}{k_1}\phi(k_1u_1) + \frac{1}{k_2}\phi(k_2u_2)\right]. \end{aligned} \tag{4.4}$$

If $\max(u_1, u_2) < u_0$, the convexity of ϕ gives

$$\begin{aligned} &\frac{k_1+k_2}{k_1k_2}\left[\phi\left(\frac{k_1k_2}{k_1+k_2}(u_1+u_2)\right) + \phi\left(\frac{k_1k_2}{k_1+k_2}(u_1-u_2)\right)\right] \\ &\leq 2\left[\frac{1}{k_1}\phi(k_1u_1) + \frac{1}{k_2}\phi(k_2u_2)\right]. \end{aligned} \tag{4.5}$$

Put $E_i = \{t \in \mathbb{R} : |f_i(t)| \geq u_0\}$ and denote by E_i^c the complement of E_i. Then, there exists $T_0 > 0$ such that for each $T \geq T_0$,

$$
\begin{aligned}
\frac{1}{k_i}\frac{1}{2T}\int\limits_{[-T,T]\cap E_i^c} \phi\left(k_i\left|f_i(t)\right|\right)\,dt &\leq \frac{1}{k_i}\frac{1}{2T}\phi\left(k_i u_0\right)\mu\left([-T,+T]\cap E_i^c\right) \\
&\leq \frac{1}{k_i}\frac{1}{2T}\frac{b}{2}\left(1-\frac{1}{a}\right)\ \mu\left([-T,+T]\cap E_i^c\right) \\
&\leq \frac{1}{2}\left(1-\frac{1}{a}\right).
\end{aligned}
\tag{4.6}
$$

We then deduce that

$$
\begin{aligned}
&\frac{k_1+k_2}{k_1k_2}\frac{1}{2T}\int\limits_{-T}^{T}\left[\phi\left(\frac{k_1k_2}{k_1+k_2}\left|(f_1(t)+f_2(t))\right|\right)\right. \\
&\quad\left.+\phi\left(\frac{k_1k_2}{k_1+k_2}\left|(f_1(t)-f_2(t))\right|\right)\right]\,dt \\
&=\frac{k_1+k_2}{k_1k_2}\frac{1}{2T}\int\limits_{[-T,T]\cap(E_1\cup E_2)}\phi\left(\frac{k_1k_2}{k_1+k_2}\left|(f_1(t)+f_2(t))\right|\right) \\
&\quad+\phi\left(\frac{k_1k_2}{k_1+k_2}\left|(f_1(t)-f_2(t))\right|\right)\,dt \\
&\quad+\frac{k_1+k_2}{k_1k_2}\frac{1}{2T}\int\limits_{[-T,T]\cap(E_1\cup E_2)^c}\left[\phi\left(\frac{k_1k_2}{k_1+k_2}\left|(f_1(t)+f_2(t))\right|\right)\right. \\
&\quad\left.+\phi\left(\frac{k_1k_2}{k_1+k_2}\left|(f_1(t)-f_2(t))\right|\right)\right]\,dt \\
&\leq(2-\delta)\frac{1}{2T}\int\limits_{[-T,T]\cap(E_1\cup E_2)}\frac{1}{k_1}\phi\left(\left|k_1f_1(t)\right|\right)+\frac{1}{k_2}\phi\left(\left|k_2f_2(t)\right|\right)\,dt \\
&\quad+2\frac{1}{2T}\int\limits_{[-T,T]\cap(E_1\cup E_2)^c}\left(\frac{1}{k_1}\phi\left(\left|k_1f_1(t)\right|\right)+\frac{1}{k_2}\phi\left(\left|k_2f_2(t)\right|\right)\right)\,dt \\
&\leq 2\frac{1}{2T}\int\limits_{[-T,T]}\left(\frac{1}{k_1}\phi\left(\left|k_1f_1(t)\right|\right)+\frac{1}{k_2}\phi\left(\left|k_2f_2(t)\right|\right)\right)\,dt \\
&\quad-\delta\frac{1}{2T}\int\limits_{[-T,T]\cap(E_1\cup E_2)}\left(\frac{1}{k_1}\phi\left(\left|k_1f_1(t)\right|\right)+\frac{1}{k_2}\phi\left(\left|k_2f_2(t)\right|\right)\right) \\
&\leq(2-\delta)\frac{1}{2T}\int\limits_{[-T,T]}\left(\frac{1}{k_1}\phi\left(\left|k_1f_1(t)\right|\right)+\frac{1}{k_2}\phi\left(\left|k_2f_2(t)\right|\right)\right)\,dt
\end{aligned}
$$

$$+\delta\frac{1}{2T}\int\limits_{[-T,T]\cap(E_1\cup E_2)^c}\left(\frac{1}{k_1}\phi\left(|k_1f_1(t)|\right)+\frac{1}{k_2}\phi\left(|k_2f_2(t)|\right)\right)$$
$$\times(2-\delta)\frac{1}{2T}\int\limits_{[-T,T]}\left(\frac{1}{k_1}\phi\left(|k_1f_1(t)|\right)+\frac{1}{k_2}\phi\left(|k_2f_2(t)|\right)\right)dt+\delta\left(1-\frac{1}{a}\right).$$

Then, letting T tend to infinity and using (4.6), it follows that

$$\begin{aligned}&\frac{k_1+k_2}{k_1k_2}\left[\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1+f_2\right)\right)+\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1-f_2\right)\right)\right]\\&\leq(2-\delta)\left[\frac{1}{k_1}\rho_{B^\phi}\left(k_1f_1\right)+\frac{1}{k_2}\rho_{B^\phi}\phi\left(k_2f_2\right)\right]+\delta\left(1-\frac{1}{a}\right).\end{aligned}$$

Thus, we have the following inequalities:

$$\begin{aligned}&\|f_1+f_2\|^o_{B^\phi}+\|f_1-f_2\|^o_{B^\phi}\\
&\leq\frac{k_1+k_2}{k_1k_2}\left[1+\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1+f_2\right)\right)\right]\\
&\quad+\frac{k_1+k_2}{k_1k_2}\left[1+\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1-f_2\right)\right)\right]\\
&\leq2\frac{k_1+k_2}{k_1k_2}+\frac{k_1+k_2}{k_1k_2}\left[\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1+f_2\right)\right)+\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1-f_2\right)\right)\right]\\
&\leq2\left(\frac{1}{k_1}+\frac{1}{k_2}\right)+\frac{k_1+k_2}{k_1k_2}\left[\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1+f_2\right)\right)+\rho_{B^\phi}\left(\frac{k_1k_2}{k_1+k_2}\left(f_1-f_2\right)\right)\right]\\
&\leq2\left(\frac{1}{k_1}+\frac{1}{k_2}\right)+(2-\delta)\left[\frac{1}{k_1}\rho_{B^\phi}\left(k_1f_1\right)+\frac{1}{k_2}\rho_{B^\phi}\phi\left(k_2f_2\right)\right]+\delta\left(1-\frac{1}{a}\right)\\
&\leq2\left(\frac{1}{k_1}+\frac{1}{k_2}\right)+(2-\delta)\left[\frac{1}{k_1}\rho_{B^\phi}\left(k_1f_1\right)+\frac{1}{k_2}\rho_{B^\phi}\phi\left(k_2f_2\right)\right]+\delta\left(1-\frac{1}{a}\right)\\
&\leq\frac{2}{k_1}\left[1+\rho_{B^\phi}\left(k_1f_1\right)\right]+\frac{2}{k_2}\left[1+\rho_{B^\phi}\left(k_2f_2\right)\right]\\
&\quad-\delta\left[\frac{1}{k_1}\rho_{B^\phi}\left(k_1f_1\right)+\frac{1}{k_2}\rho_{B^\phi}\left(k_2f_2\right)\right]+\delta\left(1-\frac{1}{a}\right)\\
&\leq2\|f_1\|^o_{B^\phi}+2\|f_2\|^o_{B^\phi}-\delta\left[\frac{1}{k_1}\rho_{B^\phi}\left(k_1f_1\right)+\frac{1}{k_2}\rho_{B^\phi}\left(k_2f_2\right)\right]+\delta\left(1-\frac{1}{a}\right)\\
&\leq4-\delta\left(1-\frac{1}{a}\right).\end{aligned}$$

Finally,

$$\min\{\|f_1+f_2\|^o_{B^\phi}, \|f_1-f_2\|^o_{B^\phi}\} \le 2\left(1-\frac{\delta}{4}\left(1-\frac{1}{a}\right)\right)$$

and so $B^\phi a.p.(\mathbb{R})$ is uniformly non-square.

Necessity. If we suppose that $B^\phi a.p.(\mathbb{R})$ is uniformly non-square, then it is reflexive. Consequently, $\phi \in \Delta_2 \cap \nabla_2$. □

Theorem 2. *Let ϕ be an Orlicz function. Then the space $B^\phi a.p.(\mathbb{R})$ with Orlicz norm has the property (β) if and only if ϕ is uniformly convex and satisfies the Δ_2-condition.*

PROOF: **Sufficiency**. If ϕ is uniformly convex and satisfies the condition-Δ_2 then by Theorem 1 in [13], $B^\phi a.p.(\mathbb{R})$ is uniformly convex and then it has the property (β).

Necessity. Suppose that $B^\phi a.p.(\mathbb{R})$ has the property (β). It follows that it is reflexive and then (see [11]) $\phi \in \Delta_2 \cap \nabla_2$.

We show that ϕ is uniformly convex. Note that, considering the canonical isometry of Lemma 3, we deduce that $L^\phi[0,1]$ has the property (β) and so ϕ is uniformly convex (see [15]). □

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