Neochromatica

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Abstract. We create and discuss several modifications to traditional graph coloring. In particular, we classify various notions of coloring in a proper hierarchy. We concentrate on grid graphs whose colorings can be represented by natural number entries in arrays with various restrictions.

Keywords: graph coloring, paths, conflict-free

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1. Introduction and related work

A *vertex coloring* of a graph $G = (V, E)$ is a (coloring) function $C\colon V \to \mathbb{N}^+$ such that adjacent vertices are colored with different colors. Formally:

$$(\forall e \in E)(e = \{v, v'\} \wedge v \neq v' \to C(v) \neq C(v')).$$

A *hypergraph* $H = (V, E)$ is a generalization of a graph for which *hyperedges* can be subsets of V of arbitrary size. Vertex coloring in hypergraphs can be defined in many ways, so that restricting the definition to simple graphs coincides with traditional graph coloring. On one extreme, it is only required that the vertices of each hyperedge are not all colored with the same color. Formally:

$$(\forall e \in E)(\exists v \in e)(\exists v' \in e)(|e| = 1 \vee C(v) \neq C(v')).$$

On the other extreme, it is required that the vertices of each hyperedge are all colored with different colors. Formally:

$$(\forall e \in E)(\forall v \in e)(\forall v' \in e)(v \neq v' \to C(v) \neq C(v')).$$

In between these two extremes, there is another possible generalization: A vertex coloring C of hypergraph H is called *conflict-free* if the vertices of each hyperedge are colored in such a way that there is a vertex whose color is unique. Formally:

$$(\forall e \in E)(\exists v \in e)(\forall v' \in e)(v' \neq v \to C(v') \neq C(v)).$$

Conflict-free coloring can model frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: *base stations* and *mobile agents*. Base stations have fixed positions and make up the backbone of the network; they are modeled by vertices in V. Mobile agents are the clients of the network, served by base stations, as follows: Every base station has a fixed frequency; this fact is modeled by the coloring C, i.e., colors represent frequencies. When agents want to establish a link with a base station they tune to the base station's frequency. Since agents are mobile, they can be in the range of many different base stations. The range of communication of every agent is modeled by a hyperedge $e \in E$, which is the set of base stations that are able to communicate with the agent. To avoid interference, the system must assign frequencies to base stations in the following way: For any range, there must be a base station in the range with a frequency that is not reused by some other base station in the range. This requirement is fulfilled by the conflict-free property. One can of course solve the problem by assigning n different frequencies to the n base stations. However, using many frequencies is expensive, and therefore, a scheme that reuses frequencies, wherever possible, is preferable.

The study of conflict-free colorings originated in the work of Even et al. [11]. In addition to the practical motivation described above, this new coloring model has drawn much attention of researchers through its purely theoretical interest and such colorings have been the focus of several recent papers (see, e.g., [21], [13], [7], [3], [5]).

A *chain* or *path* is the graph $P_n = (\{1, \ldots, n\}, \{\{i, i+1\} \mid 1 \leq i < n\})$. For $n \geq 3$, a *ring* or *cycle* is the graph C_n defined as a P_n with the additional edge $\{n, 1\}$. A grid graph G_m is the cartesian product of two paths, $P_m \times P_m$.

2. Conflict-free coloring with respect to paths of a graph

Given is a graph G, with vertex set $V(G)$ and edge set $E(G)$. The aim is to color the vertices of the graph so that for each path p in the graph, there is a vertex v in p whose color is different than the color of any other vertex in p. This coloring is called *conflict-free* (CF) coloring of graph G with respect to paths. It is a minimization problem, i.e., the goal is to find such a coloring with as few colors as possible. Formally:

Definition 1. A *k-CF-coloring* is a function $C\colon V(G) \to \{1, \ldots, k\}$ such that:

$$(\forall \text{path } p \in G)(\exists v \in p)(\forall v' \in p)(v' \neq v \to C(v') \neq C(v)).$$

The *conflict-free chromatic number* of a graph G, denoted by $\chi_{\mathrm{cf}}(G)$, is the minimum k for which G has a k-CF-coloring.

Since the above coloring involves sets of vertices included in a path, one can ask the same question in terms of hypergraphs.

Definition 2. Given a graph $G = (V, E)$, let:

(a) paths(G) be the set of paths of G,

(b) vert(p) be the set of vertices of path p,

(c) H_G be the hypergraph:

$$H_G = (V, \{\text{vert}(p) \mid p \in \text{paths}(G)\}).$$

Fact 3. *A conflict-free coloring of graph G with respect to paths is a conflict-free coloring of H_G and vice versa.*

3. Relation of conflict-free coloring with other problems

3.1 Ordered coloring. A closely related problem to CF coloring with respect to paths is *ordered coloring* [16] or *vertex ranking* [14]. Ordered coloring is conflict-free coloring with an additional constraint: the unique color in each path must also be the maximum color in the path (where colors are from the set $\{1, \ldots, k\}$).

Definition 4. A *unique maximum* (UM) coloring is a CF coloring in which the maximum color in every path p is unique in path p.

We remark that the aforementioned definition is not what is typically given in the literature [16]. Instead the following definition is more typical:

Definition 5. An *ordered* k-coloring of a graph G is a function $C\colon V(G) \to \{1, \ldots, k\}$ such that for every pair of distinct vertices v, v', and every path p from v to v', if $C(v) = C(v')$, there is an internal vertex v'' of p such that $C(v) < C(v'')$. The *ordered chromatic number* of a graph G, denoted by $\chi_{\mathrm{o}}(G)$, is the minimum k for which G has an ordered k-coloring.

We prove that the two definitions are equivalent:

Proposition 6. *C is a UM coloring if and only if C is an ordered coloring.*

Proof: If C is a UM coloring, then for any two same-color vertices v, v', every (v, v')-path p has a unique maximum color, greater than $C(v)$, which occurs in some internal vertex of p.

If C is an ordered coloring, then consider any path p in G. The maximum color in p has to occur exactly in one vertex. If it occurs in two vertices v, v' of p then there is a (v, v')-path contained in p which has an internal vertex with a greater color; a contradiction to the maximality of $C(v)$ in p. □

Corollary 7. *Every ordered coloring is also a CF coloring. Thus $\chi_{\mathrm{cf}}(G) \leq \chi_{\mathrm{o}}(G)$.*

In ordered colorings, an even stronger property is true:

Proposition 8. *In any ordered coloring C of G, in every connected subset S of vertices of G, the maximum color occurring in S, i.e., $\max\{C(v) \mid v \in S\}$, is unique in S.*

Proof: By contradiction; if there are two different vertices x, y in S with the maximum color, then there is a (x, y)-path in S, for which there is no internal vertex with higher color. □

If we relax the requirement that in every connected subset the maximum color is unique so that there is just a unique color (not necessary the maximum in the connected subset), we get the notion of a *centered coloring* (and the corresponding *centered chromatic number*), which was introduced in [19]. In [19] it was proved that for every graph the centered chromatic number equals the ordered chromatic number, i.e., the conflict-free and unique maximum chromatic numbers with respect to connected subsets equal each other. However, an analogous statement is not true for the conflict-free and unique maximum chromatic numbers with respect to paths: The smallest graph G for which $\chi_{\text{cf}}(G) < \chi_{\text{o}}(G)$ consists of a triangle, K_3, a complement of a triangle, $\overline{K_3}$, and a matching of three edges where each edge has one vertex in K_3 and the other in $\overline{K_3}$. It is not difficult to prove that $\chi_{\text{cf}}(G) = 3$ whereas $\chi_{\text{o}}(G) = 4$ (see [6]).

Both χ_{cf} and χ_{o} are monotone with respect to subgraphs:

Proposition 9. *If $X \subseteq Y$, then $\chi_{\text{cf}}(X) \leq \chi_{\text{cf}}(Y)$ and $\chi_{\text{o}}(X) \leq \chi_{\text{o}}(Y)$.*

Proof: Graph X contains a subset of the paths of Y, so the restriction of an optimal coloring of $V(Y)$ to $V(X)$ is a CF-coloring for X. $\square$

Additionally, the ordered chromatic number is monotone with respect to minors. A graph X is a *minor* of Y, denoted as $X \preccurlyeq Y$, if there is a subgraph G of Y, and a sequence $G_0, \dots, G_k$, with $G_0 = G$ and $G_k = X$, such that $G_i = G_{i-1}/e_{i-1}$, where $e_{i-1} \in E(G_{i-1})$ (i.e., edge e_{i-1} is *contracted* in G_{i-1}), for $i \in \{1, \dots, k\}$. We give a self-contained proof of χ_{o}'s monotonicity under minors for completeness, although one can also be deduced from results in [19]:

Proposition 10. *If $X \preccurlyeq Y$, then $\chi_{\text{o}}(X) \leq \chi_{\text{o}}(Y)$.*

Proof: If G is a subgraph of Y, and C an ordered coloring of Y, the restriction of C to $V(G)$ is an ordered coloring of G. Given G and an ordered coloring C of G, then the following is an ordered coloring of G/xy: For v different from x, y, use the same color as in C. For the vertex v_{xy} that arises from the contraction of edge xy, use $\max\{C(x), C(y)\}$. For every path p of G/xy, either $v_{xy} \notin p$, in which case p is also a path in G, and thus it contains a maximum unique color, or $p = p_1 v_{xy} p_2$ (with p_1, p_2 possibly empty paths), in which case the unique color of p occurs either in v_{xy} or in some vertex of p_1 or p_2, because there is a path in G containing p_1, p_2 and at least one of x, y. $\square$

3.2 Squarefree colorings. We obtain another related problem by looking at colorings of paths as strings. We impose the following restriction: Every coloring of a path, when viewed as a string, shall not contain a repetition. Formally, a string $w \in (\mathbb{N}^+)^*$ is called *squarefree* if there is no substring of w of the form $x^2 = xx$, where x is a nonempty string. Given a coloring C of the vertices of a graph, for every path $p = v_1 \dots v_\ell$, we define the color string of p to be $C(v_1) \dots C(v_\ell)$.

Definition 11. A coloring $C\colon V(G) \to \{1, \dots, k\}$ is a *squarefree* k-coloring if the color string of every path in G is squarefree.

Corollary 12. *Every CF-coloring is squarefree and thus $\chi_{\text{sf}}(G) \leq \chi_{\text{cf}}(G)$.*

We have the following hierarchical relation between colorings:

$$\mathcal{C} \supset \mathcal{SF} \supset \mathcal{CF} \supset \mathcal{OC}$$

where $\mathcal{C}$ is the class of 'traditional' vertex colorings of graphs. The above is a proper hierarchy as can be exhibited by the following colorings of the chain P_8:

12121212	traditional but not squarefree;
12312131	squarefree but not conflict-free;
41421431	conflict-free but not ordered;
12131214	ordered.

In terms of chromatic numbers:

Proposition 13. *For every graph G, $\chi(G) \leq \chi_{\mathrm{sf}}(G) \leq \chi_{\mathrm{cf}}(G) \leq \chi_{\mathrm{o}}(G)$.*

For many graphs, squarefree coloring requires substantially fewer colors than CF coloring and ordered coloring: For example, a seminal result by Thue shows that 3 colors suffice to color any chain [25]. As we will see, for chains, both ordered coloring and CF coloring require $\Omega(\log n)$ colors. Traditional chain coloring can be done, of course, with at most 2 colors. A coloring of a chain can be seen as a string, over an alphabet of possible colors. The proof of Thue relies on squarefreeness preserving morphisms. The following is a squarefreeness preserving morphism on the three letter alphabet $\{a, b, c\}$: $a \mapsto abcab$, $b \mapsto acabcb$, $c \mapsto acbcacb$. Starting with the word a, and by repetitive applications, the above morphism gives arbitrarily long squarefree words on three letters: $abcabacabcbacbcacbabcabacabcb\ldots$

More recently, in [8] it was proved that every ring can be squarefree colored with 3 colors, except a few of them (C_5, C_7, C_{10}, C_{14}, C_{17}) that require 4 colors. As we will see, for rings, both ordered coloring and CF coloring require $\Omega(\log n)$ colors. Traditional ring coloring can be done, of course, with at most 3 colors. The above squarefree result for rings can be also interpreted as follows: For every ring, there is a *subdivision* of it which is squarefree colorable using 3 colors. Recently, in [22], it was proved that every graph has a subdivision which is squarefree colorable using 3 colors, which is a striking generalization of Thue's result.

3.3 Cubefree and other colorings. Another related class of colorings consists of *cubefree* colorings, where color strings of paths can not contain a x^3 substring, for x non-empty. It is known ([25] and implicit in [23]) that 2 colors suffice to color any chain. A cubefreeness preserving morphism, on a two letter alphabet $\{a, b\}$, is quite simple: $a \mapsto ba$, $\quad b \mapsto ba$. A cubefree word starts like: $abbabaabbaab$. Cubefree colorings can also be put in the above hierarchy over squarefree colorings but they are not comparable with traditional colorings. Squarefree, cubefree, and related colorings have been studied extensively for strings (i.e., for the chain graph in our setting). A good starting point for the interested reader is the book by Allouche and Shallit [1]. Both squarefree and cubefree colorings are special cases of nonrepetitive colorings [2], [12]. The relationship between nonrepetitive colorings and other notions, including ordered colorings, was recently investigated in [20].

4. Conflict-free coloring some families of graphs

4.1 Chain. Conflict-free coloring of a chain is better known as conflict-free coloring with respect to intervals [7]. For completeness, we give a proof that $\chi_{\mathrm{cf}}(P_n) = 1 + \lfloor \lg n \rfloor$. The method yields a CF coloring which is also an ordered coloring and is highly symmetric, which we call *recursively palindromic* coloring.

4.1.1 A lower bound for $\chi_{\mathbf{cf}}(P_n)$. Observe that in any conflict-free coloring of P_n there is a uniquely colored vertex v and thus every path that contains v has the conflict-free property. Graph $G - v$ consists of at least two chains, one of which has at least $\lfloor n/2 \rfloor$ vertices. Therefore we have the following recurrence for $\chi_{\mathrm{cf}}(P_n)$: $\chi_{\mathrm{cf}}(P_1) = 1$ and $\chi_{\mathrm{cf}}(P_n) \geq 1 + \chi_{\mathrm{cf}}(P_{\lfloor n/2 \rfloor})$, which easily implies $\chi_{\mathrm{cf}}(P_n) \geq 1 + \lfloor \lg n \rfloor$.

4.1.2 An optimal conflict-free coloring of P_n. An optimal coloring is acquired by taking the first n terms of the sequence C^k with k such that $n \leq 2^k - 1$, defined recursively as follows: $C^1 = (1)$, and for $k > 1$, $C^k = C^{k-1} \circ (k) \circ C^{k-1}$, where $\circ$ is the concatenation operation for sequences of symbols. Color i is used only if $n \geq 2^{i-1}$, so in fact $1 + \lfloor \lg n \rfloor$ colors are used by the coloring. It is not difficult to prove that the coloring is conflict-free.

4.2 Ring. To conflict-free color a ring, we use the conflict-free coloring of a chain. We pick an arbitrary vertex v and color it with a unique color (not to be reused anywhere else in the coloring). The remaining vertices form a chain that we color with the method for chains described in Section 4.1. This method colors C_n, a ring of n vertices, with $2 + \lfloor \lg(n-1) \rfloor$ colors. For example, if $n = 8$, the coloring is 41213121, where '4' is the first unique color used for v.

Claim 14. *A conflict-free coloring of C_n requires $2 + \lfloor \lg (n-1) \rfloor$ colors.*

Proof: Assume for the sake of contradiction that you can color with $1+\lfloor \lg (n-1) \rfloor$ colors. Remove a uniquely colored vertex in the ring. The remaining $n-1$ vertices use $\lfloor \lg(n-1) \rfloor$ colors and constitute a CF coloring of P_{n-1}, which is impossible since at least $1 + \lfloor \lg(n-1) \rfloor$ colors are required to CF color P_{n-1}. □

4.3 Tree. For a *tree* graph, we use the idea of a 1/2-separator [15], [17], [10]. A 1/2-separator is a vertex which, when removed, leaves connected components whose size is bounded by $n/2$. The method to color a tree is as follows: Find a 1/2-separator, color it with a unique color. Then color recursively the connected components, after the removal of the 1/2-separator. Thus, $\chi_{\mathrm{cf}}(T) \leq 1+\lfloor \lg n \rfloor$ for a tree with n vertices. See also [16]. If a maximum color is used for every separator, the above coloring is an ordered coloring. Moreover, one can find optimal ordered colorings of trees [14], [24].

4.4 Grid. A grid of size $m \times m$, i.e, with $n = m^2$ vertices can be colored with an ordered coloring with at most $4m$ colors: The idea is to use unique maximum colors for the row closest to the middle and column closest to the middle (that is less than $2m$ colors), and then color recursively in the 4 subgrids with size at most $\lfloor m/2 \rfloor \times \lfloor m/2 \rfloor$ each. A slight variation gives a coloring with at most $3m$ color: Use m unique maximum colors for the row closest to the middle, and then use about $m/2$ more unique colors for the part of the middle column over the middle row, and the same $m/2$ colors for the middle column under the middle row; then use recursion in the 4 subgrids with size at most $\lfloor m/2 \rfloor \times \lfloor m/2 \rfloor$ each. The above coloring is good enough even if we add one edge in every internal face of the standard drawing of G_m to make every internal face triangular (we get a triangular grid), or even if we add two edges in every face. This indicates that $3m$ is not optimal and, in fact, it has been improved in [4]. For a general planar graph G, using separator theorems [18], [9], it can be proved that $\chi_o(G) \leq 3(\sqrt{6}+2)\sqrt{n} \approx 13.3485\sqrt{n}$ (see [16]). As we have seen, the previous result can be far from optimal for well structured planar graphs like the grid. There is also a lower bound of $\chi_o(G_m) \geq m$ (also from [16]). We give another proof of the basic lower bound of $\chi_o(G_m) \geq m$, based on a minor graphs argument:

Proposition 15. *If G_m is the $m \times m$ grid, $\chi_o(G_m) \geq m$.*

Proof: By induction. Base: For $m = 1$, it is true, as $\chi_o(K_1) = 1$. For the inductive step, consider a Hamilton path p of G_m, with $m > 1$. If G_m is ordered colored, then there is a vertex v with a unique color in p (and thus in G). So, for some v, $\chi_o(G_m) = 1 + \chi_o(G_m - v)$. However, for every v, $G_{m-1} \preccurlyeq G_m - v$ (easy proof). Therefore, from Proposition 10, $\chi_o(G) \geq 1 + \chi_o(G_{m-1})$ and from the inductive hypothesis, $\chi_o(G) \geq 1 + m - 1 = m$. $\square$

In order to improve the upper bound of $3m$, we need to find more intricate separators, that will be colored with unique colors. The idea is to use separators along diagonals in the grid. We will also need to find efficient colorings of some subgraphs that are left after we remove diagonal-like separators. Such a subgraph of the grid is the rhombus R_x, shown in the left part of Figure 1; it has height and width x. For example, it is proved in [4] that $\chi_o(R_x) \leq 3x/2$ (see the right part of Figure 1 for the separation method) and that $\chi_o(G_m) \leq 18m/7 \approx 2.5714m$ as a consequence of a partition of the grid with the help of separators shown in Figure 2. A lower bound of $4m/3 \approx 1.333m$ is also proved in [4].

5. Conflict-free coloring in arrays

5.1 Arrays and meander paths. Consider the $m \times m$ grid with a conflict-free coloring with respect to paths. The paths defined in the graph are simple, in the sense that they are not self-intersecting, and in the standard drawing of a graph (see Figure 3) they always go along the horizontal or the vertical direction. Thus they look like *meanders*.

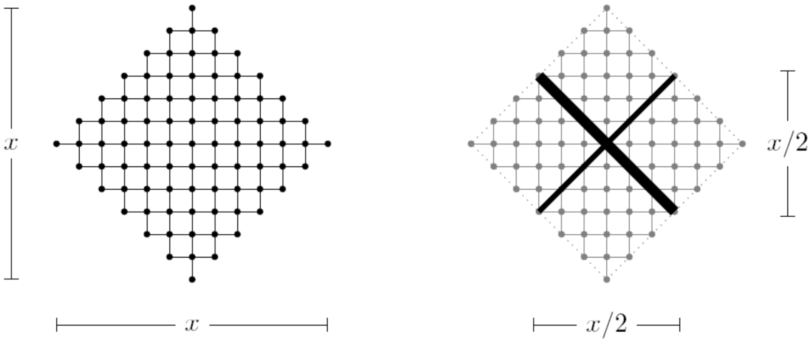


FIGURE 1. The rhombus subgraph R_x and its separation

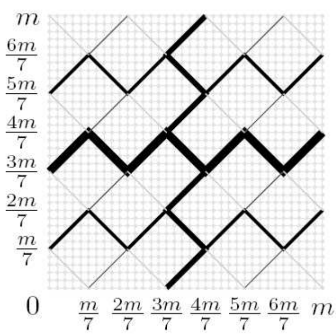
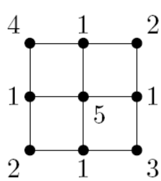


FIGURE 2. An $18m/7$ upper bound



4	1	2
1	5	1
2	1	3

FIGURE 3. A conflict-free coloring of the 3×3 grid with respect to paths and of the 3×3 array with respect to meander paths

Instead of placing colors on a grid drawn like the one in the left part of Figure 3, it is more convenient to fill the colors in a two-dimensional array, of size m in each dimension, as in the right part of Figure 3.

5.2 Arrays, subarrays, and thin subarrays. We are going to relax some of the constraints of conflict-free coloring with respect to meander paths (that forces linear use of colors with respect to m), in order to achieve logarithmic colorings with respect to m. We relax constraints in the following two ways:

- In every *subarray*, there must be a unique color.
- In every *thin* subarray (i.e., a subarray which has length 1 in one of the two dimensions), there must be a unique color.

We are going to extensively use the conflict-free coloring of the chain, given in Section 4.1. If points are numbered 1 through n, from left to right on a chain, then the i-th point's color is denoted by $C(i)$, i.e., $C(1) = 1$, $C(2) = 2$, $C(3) = 1$, $C(4) = 3$, and so on. We mention some results without proof details.

Proposition 16. *There is a conflict-free coloring with respect to subarrays of the $m \times m$ array with asymptotically $2 \lg m$ colors.*

PROOF: Each entry in the 2-dimensional array is encoded by a pair (i, j), where i is the row, and j is the column of the entry. The entry (i_1, i_2) is colored as $C(i_1, i_2) = C(i_1) + C(i_2) - 1$. □

Proposition 17. *There is a conflict-free coloring with respect to thin subarrays of the $m \times m$ array with asymptotically $\lg m$ colors.*

PROOF: Color (i_1, i_2) with $C(i_1, i_2) = (C(i_1)+C(i_2)-1) \bmod_1 \lceil \lg(m+1) \rceil$, where $\bmod_1$ is the modulo operator, but returning $\lceil \lg(m + 1) \rceil$ instead of 0 (i.e., its minimum output value is 1). □

5.3 Multidimensional arrays. One can generalize the previous results to *multidimensional* grids or arrays. A grid in d dimensions, in which each side has length m, contains m^d vertices. A multidimensional grid can also be viewed as a multidimensional array. One can conflict-free color with respect to subarrays, or with respect to thin subarrays (subarrays which have length different than one in at most one dimension). Each point (or cell) of the grid (or array) is denoted by its d coordinates: $(i_1, \ldots, i_d)$. Each coordinate ranges from 1 to m. We mention some results without proof details.

Proposition 18. *There is a conflict-free coloring with respect to subarrays of the $m \times m \times \cdots \times m$ d-dimensional array with asymptotically $d \lg m$ colors.*

PROOF: The point $(i_1, \ldots, i_d)$ of the d-dimensional grid is colored as follows:

$$C(i_1, \ldots, i_k) = \sum_{k=1}^{d} C(i_k) - (d - 1).$$

□

Proposition 19. *There is a conflict-free coloring with respect to thin subarrays of the $m \times m \times \cdots \times m$ d-dimensional array with asymptotically $\lg m$ colors.*

PROOF: The point $(i_1, \ldots, i_d)$ of the d-dimensional grid is colored as follows:

$$C(i_1, \ldots, i_k) = \left(\sum_{k=1}^{d} C(i_k) - (d-1) \right) \bmod_1 \lceil \lg(m+1) \rceil$$

where $\bmod_1$ is the modulo operator, but returning $\lceil \lg(m+1) \rceil$ instead of 0 (i.e., its minimum output value is 1). □

It is interesting that the above coloring with respect to thin subarrays uses asymptotically only $\lg m$ colors, i.e., the number of colors used does not depend on the dimension d. Another interesting fact is that the coloring is very far from satisfying the unique maximum property. It is an open problem, whether one can use $O(\log m)$ colors with this additional stronger constraint.

5.4 Conflict-free coloring with respect to first minor submatrices. Given is a matrix A (i.e., a two dimensional array) with r rows and c columns. For every row i and every column j of A, a submatrix M_{ij}, called a *first minor submatrix*, is defined by removing the elements of row i and the elements of column c. Determinants of first minor submatrices are used in the Laplace expansion of the determinant of a square matrix A.

We denote by $\chi_{\mathrm{cf}}(H_{r,c})$ the minimum number of colors required to conflict-free color the $r \times c$ matrix with respect to first minor submatrices. Because of symmetry, we have $\chi_{\mathrm{cf}}(H_{r,c}) = \chi_{\mathrm{cf}}(H_{c,r})$. We denote by $\chi_{\mathrm{um}}(H_{r,c})$ the minimum number of colors required to conflict-free color the $r \times c$ matrix, with the additional constraint that the unique color is the maximum color. Again, because of symmetry, $\chi_{\mathrm{um}}(H_{r,c}) = \chi_{\mathrm{um}}(H_{c,r})$.

One can conflict-free color a $r \times c$ matrix with respect to all first minor submatrices by using a constant number of colors. In fact, four colors suffice, even if we require the stronger property of unique maximum color.

Proposition 20. *For all r, c, $\chi_{\mathrm{cf}}(H_{r,c}) \le \chi_{\mathrm{um}}(H_{r,c}) \le 4$.*

PROOF: Color all entries of the matrix with 1, except a 2×2 submatrix which is colored as $\left(\begin{smallmatrix} 3 & 2 \\ 2 & 4 \end{smallmatrix}\right)$. Every first minor is conflict-free colored with the unique maximum property, because it either (a) contains one of 3 or 4, or (b) if it contains no 3 and 4, then it contains exactly one 2. □

The above result is tight for both $\chi_{\mathrm{cf}}(H_{r,c})$ and $\chi_{\mathrm{um}}(H_{r,c})$, except for some small values of r, c. For example, $\chi_{\mathrm{cf}}(H_{2,2}) = \chi_{\mathrm{cf}}(H_{2,2}) = 1$, $\chi_{\mathrm{um}}(H_{2,c}) = 3$ for $c \ge 2$, $\chi_{\mathrm{cf}}(H_{3,c}) = 3$ for $c \ge 3$. Moreover, for some small values of r, c, the two chromatic numbers differ, e.g., $\chi_{\mathrm{um}}(H_{2,4}) = 3$, whereas $\chi_{\mathrm{cf}}(H_{2,4}) = 2$.

6. Open problems and future research

One could study the conflict-free coloring problem in an online setting; for relevant results, see [7], [5]. The most important open problem in the online setting for chains is narrowing the gap between lower and upper bound in the deterministic online model: $\Omega(\log n)$, and $O(\log^2 n)$, respectively, which are a logarithmic factor apart.

Another open problem is finding the exact ordered and conflict-free chromatic number of the $m \times m$ grid, improving the lower and upper bounds of [4].

Finally, it would be nice to develop a better understanding of the relationship between conflict-free and ordered colorings. We have seen that the two respective chromatic numbers, χ_{cf} and χ_{o}, are not always equal, but how far can they be? There are some initial results in that direction in [6]. One particularly interesting open problem is whether the conflict-free chromatic number with respect to paths is monotone under taking minors, like the ordered chromatic number.

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