A note on Lipschitz isomorphisms in Hilbert spaces

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Abstract. We show that the following well-known open problems on existence of Lipschitz isomorphisms between subsets of Hilbert spaces are equivalent: Are balls isomorphic to spheres? Is the whole space isomorphic to the half space?

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The topological classification of subsets of Banach spaces has been the subject of much investigation. In 1931 O.H. Keller [3] proved that the infinite dimensional compact convex subsets of Hilbert space are mutually homeomorphic and all homogeneous, and in 1950 V. Klee [4] proved that in an infinite dimensional separable Hilbert space balls and spheres are homeomorphic. By now, this classification is fairly complete as documented in the authoritative text [2]. However, as can be seen from [1] and from a number of research papers, the problem of uniform or Lipschitz classification of Banach spaces is still far from being understood. Even worse is the situation when it comes to the classification of subsets of Banach spaces. It is not even known whether an infinite dimensional separable Hilbert space is Lipschitz homeomorphic to its half space. Similarly, it is not known whether in such a space balls and spheres are Lipschitz homeomorphic. We have only the highly interesting result of R. Nahum [5] that balls in Hilbert space are Lipschitz homeomorphic to spheres if and only if the balls are Lipschitz homogeneous.

In this modest contribution we show that the two problems mentioned above are in fact equivalent. The main ingredient of our argument is the perhaps somewhat surprising Lemma 1.1 saying that the space inversion, although not Lipschitz itself, under fairly general assumptions conjugates Lipschitz maps to Lipschitz maps.

Recall that a Lipschitz homeomorphism of a metric space M onto a metric space N is a bijection $f\colon M \to N$ which is Lipschitz and has a Lipschitz inverse. If such an f exists, M and N are said to be Lipschitz homeomorphic. Instead of Lipschitz homeomorphism one often says Lipschitz isomorphism (which we will also use) or Lipschitz equivalence (which we will try to avoid).

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1. Inversion lemma

Recall that the inversion T on a Hilbert space H is the map of $H \setminus \{0\}$ onto itself defined by

$$T\mathbf{x} = \frac{\mathbf{x}}{\|\mathbf{x}\|^2}\,.$$

Notice that $TT\mathbf{x} = \mathbf{x}$, $\|T\mathbf{x}\| = \frac{1}{\|\mathbf{x}\|}$ and that T is Lipschitz on the complement of every ball $B(0,r)$. More precisely, we have that for $\|\mathbf{x}\| \geq \|\mathbf{y}\| \geq r > 0$,

$$(\star)\qquad \begin{aligned} \|T\mathbf{y} - T\mathbf{x}\| &= \left\| \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}\|^2} + \frac{(\|\mathbf{x}\|^2 - \|\mathbf{y}\|^2)\mathbf{x}}{\|\mathbf{y}\|^2\|\mathbf{x}\|^2} \right\| \\ &\leq \frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{y}\|^2} + \frac{|\|\mathbf{x}\| - \|\mathbf{y}\||(\|\mathbf{x}\| + \|\mathbf{y}\|)\|\mathbf{x}\|}{\|\mathbf{y}\|^2\|\mathbf{x}\|^2} \\ &\leq \frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{y}\|^2} + \frac{2\|\mathbf{x}-\mathbf{y}\|\|\mathbf{x}\|^2}{\|\mathbf{y}\|^2\|\mathbf{x}\|^2} \\ &\leq \frac{3}{r^2}\|\mathbf{y}-\mathbf{x}\|. \end{aligned}$$

Lemma 1.1. *Suppose that $h\colon A \subset H \setminus \{0\} \to H$ is Lipschitz and for some $c > 0$ satisfies $\|h(\mathbf{x})\| \geq c\|\mathbf{x}\|$. Then $ThT : TA \to H$ is Lipschitz.*

Proof: Suppose that $\mathbf{x}, \mathbf{y} \in A$ and $\|\mathbf{x}\| \geq \|\mathbf{y}\|$.
Consider first the case when $\|\mathbf{x}\| > 2\|\mathbf{y}\|$. Then

$$\|\mathbf{y}\| + \|\mathbf{x}\| \leq \frac{3}{2}\|\mathbf{x}\| \leq 3\|\mathbf{y}-\mathbf{x}\|$$

and hence

$$\begin{aligned} \|ThT\mathbf{y} - ThT\mathbf{x}\| \leq \|ThT\mathbf{y}\| + \|ThT\mathbf{x}\| &= \frac{1}{\|hT\mathbf{y}\|} + \frac{1}{\|hT\mathbf{x}\|} \\ &\leq \frac{1}{c}\Big(\frac{1}{\|T\mathbf{y}\|} + \frac{1}{\|T\mathbf{x}\|}\Big) = \frac{1}{c}(\|\mathbf{y}\| + \|\mathbf{x}\|) \\ &\leq \frac{3}{c}\|\mathbf{y}-\mathbf{x}\|. \end{aligned}$$

It remains to consider the case when $\|\mathbf{x}\| \leq 2\|\mathbf{y}\|$. Then we use $(\star)$ together with $\|hT\mathbf{y}\| \geq c\|T\mathbf{y}\| \geq \frac{c}{\|\mathbf{x}\|}$ and $\|hT\mathbf{x}\| \geq c\|T\mathbf{x}\| = \frac{c}{\|\mathbf{x}\|}$ to infer that

$$\begin{aligned} \|ThT\mathbf{y} - ThT\mathbf{x}\| &\leq \frac{3\|\mathbf{x}\|^2}{c^2}\|hT\mathbf{y} - hT\mathbf{x}\| \leq \frac{3\|\mathbf{x}\|^2 \mathrm{Lip}(h)}{c^2}\|T\mathbf{y} - T\mathbf{x}\| \\ &\leq \frac{3\|\mathbf{x}\|^2 \mathrm{Lip}(h)}{c^2}\frac{3}{\|\mathbf{y}\|^2}\|\mathbf{y}-\mathbf{x}\| \\ &\leq \frac{36\mathrm{Lip}(h)}{c^2}\|\mathbf{y}-\mathbf{x}\|. \end{aligned}$$

□

We will also need the following simple mapping properties of the inversion.

Lemma 1.2. *Let* $\mathbf{e}$ *be a unit vector in* H *and* $t > 0$*. Then* T *maps the affine space* $V_t = \{\mathbf{x} \in H : \langle \mathbf{x}, \mathbf{e}\rangle = t\}$ *onto* $S_t = \{\mathbf{y} \in H \backslash \{0\} : \|\mathbf{y} - \frac{1}{2t}\mathbf{e}\| = \frac{1}{2t}\mathbf{e}\}$ *and the closed half space* $H_t = \{\mathbf{x} \in H : \langle \mathbf{x}, \mathbf{e}\rangle \geq t\}$ *onto* $B_t = \{\mathbf{y} \in H \setminus \{0\} : \|\mathbf{y} - \frac{1}{2t}\mathbf{e}\| \leq \frac{1}{2t}\mathbf{e}\}$.

Proof: Writing $\mathbf{x} \in V_t$ as $\mathbf{x} = t\mathbf{e} + \mathbf{z}$, we find that $T\mathbf{x} = \frac{1}{t^2 + \|\mathbf{z}\|^2}\mathbf{x}$. From this $\|T\mathbf{x} - \frac{1}{2t}\mathbf{e}\| = \frac{1}{2t}$ so that T maps V_t into S_t. Since T is a bijection of the open half space $H^+ = \{\mathbf{x} \in H : \langle \mathbf{x}, \mathbf{e}\rangle > 0\}$ onto itself, and both V_t and S_t form a disjoint decomposition of H^+, it follows that T maps V_t onto S_t. Finally, H_t is the union of V_s over $s \geq t$, hence its image is the union of S_s over $s \geq t$, which is B_t. $\square$

2. The space and half space

Let H be a separable Hilbert space. Since any two balls in H are linearly isomorphic, and so are any two spheres or any two closed half spaces, we may, and will, use particular balls, spheres or half-spaces when treating the general case.

As the use of inversion will lead to incomplete spaces we provide a simple lemma extending Lipschitz homeomorphisms to limit points of their domain.

Lemma 2.1. *Let* X *and* Y *be complete metric spaces and* $f : X \setminus \{x_0\} \to Y \setminus \{y_0\}$ *be a Lipschitz homeomorphism with* x_0 *and* y_0 *each non-isolated. Then* f *extends to a Lipschitz homeomorphism* $f : X \to Y$ *with* $f(x_0) = y_0$.

Proof: Let a sequence $(x_n)_{n=1}^{\infty} \subset X \setminus \{x_0\}$ converge to x_0. Since f is Lipschitz, $y_n = f(x_n)$ is a Cauchy sequence in $Y \setminus \{y_0\}$ that converges to some point z in Y. But $z \in Y \setminus \{y_0\}$ would imply, by continuity of f^{-1}, that x_n converges to $f^{-1}(z) \neq x_0$. Hence y_n has limit y_0 and the extension $f(x_0) = y_0$ is continuous. Moreover, the Lipschitz constant of the extension is equal to the Lipschitz constant of f. Applying the same argument to f^{-1} we see that the extended f is indeed a Lipschitz homeomorphism of X onto Y. $\square$

Theorem 2.2. *The unit ball and the unit sphere of an infinite dimensional separable Hilbert space* H *are Lipschitz isomorphic if and only if the space* H *is Lipschitz isomorphic to its closed half space.*

Proof: Choose a unit vector $\mathbf{e} \in H$ and recall the notation V_t, S_t, H_t and B_t from Lemma 1.2. We fix some $t > 0$.

Let the unit ball and the unit sphere of H be Lipschitz isomorphic. We will apply Lemma 1.1 to a suitable Lipschitz isomorphism of the ball $B(\frac{1}{2t}\mathbf{e}, \frac{1}{2t})$ onto the sphere $S(\frac{1}{2t}\mathbf{e}, \frac{1}{2t})$. Let $h\colon B(\frac{1}{2t}\mathbf{e}, \frac{1}{2t}) \to S(\frac{1}{2t}\mathbf{e}, \frac{1}{2t})$ be a Lipschitz isomorphism and denote $\mathbf{a} = h(\mathbf{0}) \in S$. We may compose h with a suitable isometry of $S(\frac{1}{2t}\mathbf{e}, \frac{1}{2t})$ for which $\mathbf{a} \mapsto \mathbf{0}$; for example, the reflection in the plane passing through the centre $\frac{1}{2t}\mathbf{e}$ of S and having $\mathbf{a}$ as normal is such an isometry. We may therefore assume that $h(\mathbf{0}) = \mathbf{0}$.

To apply Lemma 1.1 with $A = B_t$, we have to verify its assumptions. The map $h\colon A \subset H \setminus \{0\} \to H$ is bilipschitz; in particular (since h^{-1} is Lipschitz), for some

$c > 0$, $\|h(\mathbf{x}) - h(\mathbf{y})\| \geq c\|\mathbf{x} - \mathbf{y}\|$. Putting $\mathbf{y} = \mathbf{0}$ we get that $\|h(\mathbf{x})\| \geq c\|\mathbf{x}\|$ as required. Applying Lemma 1.1 with $A = B_t$ and recalling from Lemma 1.2 that $T(B_t) = H_t$ and $T(h(A)) = T(S_t) = V_t$, we get $ThT\colon H_t \to V_t$ is Lipschitz. The argument applies also to h^{-1} with $A = S_t$ in Lemma 1.1 to get that $(ThT)^{-1} = Th^{-1}T\colon V_t \to H_t$ is Lipschitz. Hence $ThT\colon H_t \to V_t$ is a Lipschitz isomorphism. Since V_t is linearly isomorphic with H, this gives us a Lipschitz isomorphism of a half space onto the whole space.

Conversely, suppose that H is Lipschitz isomorphic to its closed half space. Hence there is a Lipschitz isomorphism h of H_t onto V_t. Composing such an isomorphism with a suitable shift in V_t, we may also assume that $h(t\mathbf{e}) = t\mathbf{e}$. To apply Lemma 1.1 with $A = H_t$, we have to find $c > 0$ so that $\|h(\mathbf{x})\| \geq c\|\mathbf{x}\|$ for every $\mathbf{x} \in H_t$. Since h^{-1} is Lipschitz, there is $k > 0$ so that $\|h(\mathbf{y}) - h(\mathbf{x})\| \geq k\|\mathbf{y} - \mathbf{x}\|$. It follows that

$$\|h(\mathbf{x})\| \geq \|h(\mathbf{x}) - h(t\mathbf{e})\| - \|h(t\mathbf{e})\| \geq k\|\mathbf{x} - t\mathbf{e}\| - t \geq k\|\mathbf{x}\| - kt - t \geq \tfrac{1}{2}k\|\mathbf{x}\|$$

if $\|\mathbf{x}\| \geq 2(k+1)t/k$ and $\|h(\mathbf{x})\| \geq t \geq \frac{k}{2(k+1)}\|\mathbf{x}\|$ if $\|\mathbf{x}\| < 2(k+1)t/k$. Hence the assumptions of Lemma 1.1 hold with $c = \frac{k}{2(k+1)}$, and we infer that $ThT\colon B_t \to S_t$ is Lipschitz. An analogous argument applies also to h^{-1} with $A = S_t$ in Lemma 1.1 to get that $(ThT)^{-1} = Th^{-1}T\colon S_t \to B_t$ is Lipschitz. Hence $ThT\colon B_t \to S_t$ is a Lipschitz isomorphism, and it suffices to use Lemma 2.1 to extend it to a Lipschitz isomorphism of the ball $B(\frac{1}{2t}\mathbf{e}, \frac{1}{2t})$ onto the sphere $S(\frac{1}{2t}\mathbf{e}, \frac{1}{2t})$. $\square$

References

[1] Benyamini Y., Lindenstrauss J., *Geometric Nonlinear Functional Analysis*, Amer. Math. Soc. Colloquium Publications, 48, American Mathematical Society, Providence, RI, 2000.

[2] Bessaga C., Pelczynski A., *Selected Topics in Infinite Dimensional Topology*, PWN, Warsaw, 1975.

[3] Keller O.H., *Die Homeomorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.

[4] Klee V.L., *Convex Bodies and Periodic Homeomorphisms in Hilbert Space*, Trans. Amer. Math. Soc. **74** (1953), no. 1, 10–43.

[5] Nahum R., *On the Lipschitz equivalence of unit balls and spheres in normed spaces*, preprint.

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