Integrability for solutions to quasilinear elliptic systems

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Abstract. In this paper we prove an estimate for the measure of superlevel sets of weak solutions to quasilinear elliptic systems in divergence form. In some special cases, such an estimate allows us to improve on the integrability of the solution.

Keywords: level set, integrability, solution, quasilinear, elliptic, system

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1. Introduction

We deal with regularity properties for weak solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of quasilinear systems in divergence form

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) \right) = 0, \quad x \in \Omega, \quad \alpha = 1, \dots, N. \tag{1.1}$$

The coefficients $a_{ij}^{\alpha\beta}(x,u)$ are only measurable with respect to x; they are continuous with respect to u; moreover, they are bounded and elliptic. We assume that the solution u of (1.1) is bounded on $\partial\Omega$:

$$u \in L^\infty(\partial\Omega). \tag{1.2}$$

De Giorgi's counterexample [1] shows that, in general, boundedness on $\partial\Omega$ does not imply boundedness inside Ω for weak solutions u of elliptic systems (1.1); see also [8]. In order to get boundedness inside Ω, we need additional assumptions on the coefficients. If $a_{ij}^{\alpha\beta}(x,u)$ are diagonal

$$a_{ij}^{\gamma\beta}(x,u) = 0 \qquad \text{for } \beta \neq \gamma \tag{1.3}$$

then the N equations (1.1) are decoupled and maximum principle applies to every component u^γ of $u = (u^1, \dots, u^N)$:

$$\sup_{\Omega} u^\gamma \leq \sup_{\partial\Omega} u^\gamma. \tag{1.4}$$

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In [7] the authors assume that coefficients are diagonal only for large values of u^γ:

$$\theta^\gamma \le u^\gamma \quad \Longrightarrow \quad a_{ij}^{\gamma\beta}(x,u) = 0 \qquad \text{for } \beta \ne \gamma; \tag{1.5}$$

then it results that

$$\sup_\Omega u^\gamma \le \max\left[\theta^\gamma; \sup_{\partial\Omega} u^\gamma\right], \tag{1.6}$$

see also [6] and [5]. In this paper we no longer assume that off-diagonal coefficients vanish; we only know that they are small when u^γ is large: there exists $q > 0$ such that

$$0 < \theta^\gamma \le |u^\gamma| \quad \Longrightarrow \quad |a_{ij}^{\gamma\beta}(x,u)| \le \frac{c_1}{|u^\gamma|^q} \quad \text{for } \beta \ne \gamma; \tag{1.7}$$

then we are able to estimate the measure of superlevels as follows

$$|\{|u^\gamma| > s\}| \le \frac{c_2}{s^{2^*(1+q)}} \tag{1.8}$$

for every $s > 0$; note that $2^* = 2n/(n-2)$ is the Sobolev exponent. Such an inequality is a special case of a general estimate that we prove for every system (1.1): in the special case (1.7) we get (1.8). Then u^γ turns out to be in the weak Lebesgue (or Marcinkiewicz) space with exponent $2^*(1+q)$:

$$u^\gamma \in L^{2^*(1+q)}_{\text{weak}}(\Omega). \tag{1.9}$$

Note that weak solutions u of system (1.1) are taken from the Sobolev space $W^{1,2}(\Omega;\mathbb{R}^N)$; the embedding guarantees that the integrability of u reaches 2^*: our result (1.9) improves such integrability, since $2^* < 2^*(1+q)$. In Section 2 we collect precise assumptions and results; Section 3 is devoted to the proof. We end this introduction by recalling that [4] deals with the linear case and the author proves L^∞ bounds under an assumption on the dispersion of eigenvalues. Eventually, we thank the referee for valuable remarks.

2. Assumptions and results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \ge 3$. For $N \ge 2$, let $a_{ij}^{\alpha\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be Carathéodory functions, that is, $a_{ij}^{\alpha\beta}(x,y)$ are measurable with respect to x and continuous with respect to y. We assume that coefficients are bounded: there exists $c_3 \in (0,+\infty)$ such that

$$|a_{ij}^{\alpha\beta}(x,y)| \le c_3 \tag{2.1}$$

for almost every $x \in \Omega$, for every $y \in \mathbb{R}^N$, for all $i,j \in \{1,\dots,n\}$, for any $\alpha,\beta \in \{1,\dots,N\}$. Let $\nu \in (0,+\infty)$; we assume ellipticity of diagonal coefficients

$a_{ij}^{\gamma\gamma}$ for large values of the corresponding component of y: for every $\gamma \in \{1, \ldots, N\}$ there exists $\theta^\gamma \in (0, +\infty)$ such that

$$\theta^\gamma \le |y^\gamma| \quad \Longrightarrow \quad \nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x, y)\xi_j\xi_i \tag{2.2}$$

for almost every $x \in \Omega$, for any $\xi \in \mathbb{R}^n$. In order to deal with off-diagonal coefficients $a_{ij}^{\gamma\beta}$ we need to introduce the following supremum: for every $\gamma \in \{1, \ldots, N\}$, for every $L \in (0, +\infty)$ we define

$$g^\gamma(L) = \max_{i,j} \max_{\beta \ne \gamma} \sup_{|y^\gamma|>L} \sup_x |a_{ij}^{\gamma\beta}(x, y)|, \tag{2.3}$$

where $\max_{i,j}$ is taken over all $i, j \in \{1, \ldots, n\}$; $\max_{\beta\ne\gamma}$ is taken over all $\beta \in \{1, \ldots, N\} \setminus \{\gamma\}$; $\sup_{|y^\gamma|>L}$ is taken over all $y \in \mathbb{R}^N$ with $|y^\gamma| > L$; $\sup_x$ is the essential supremum taken over almost every $x \in \Omega$. Note that $L \to g^\gamma(L)$ is decreasing; moreover, assumption (2.1) guarantees that $0 \le g^\gamma(L) \le c_3$.

We prove the following

Theorem 2.1. *Under the previous assumptions* (2.1), (2.2), *let* $u = (u^1, \ldots, u^N)$ *be a weak solution of the system* (1.1), *that is,* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *and*

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x, u(x))D_j u^\beta(x) D_i v^\alpha(x)\, dx = 0 \qquad \forall\, v \in W_0^{1,2}(\Omega, \mathbb{R}^N). \tag{2.4}$$

Then every component u^γ *of* $u = (u^1, \ldots, u^N)$ *satisfies*

$$|\{x \in \Omega : |u^\gamma(x)| > 2L\}| \le c_4 \left(\frac{g^\gamma(L)}{L}\right)^{2^*}, \tag{2.5}$$

where

$$c_4 = 2\left(\frac{2(n-1)(N-1)n^2}{(n-2)\nu}||Du||_{L^2(\Omega)}\right)^{2^*}, \tag{2.6}$$

$|E|$ *is the Lebesgue measure of* $E \subset \mathbb{R}^n$ *and* $2^* = 2n/(n-2)$ *is the Sobolev exponent. Inequality* (2.5) *holds true for every* $L \ge \max\{\theta^\gamma; \sup_{\partial\Omega} u^\gamma; -\inf_{\partial\Omega} u^\gamma\}$.

Remark 2.1. With no extra assumption, $g^\gamma(L) \le c_3$ and decay (2.5) does not improve on Sobolev embedding $W^{1,2} \subset L^{2^*}$.

Remark 2.2. When off-diagonal coefficients $a_{ij}^{\gamma\beta}$ vanish for large values of $|y^\gamma|$

$$0 < \theta^\gamma \le |y^\gamma| \quad \Longrightarrow \quad a_{ij}^{\gamma\beta}(x, y) = 0 \qquad \text{for } \beta \ne \gamma, \tag{2.7}$$

then $g^\gamma(L) = 0$ for $L \ge \theta^\gamma$ and decay (2.5) says that some superlevel has zero measure, thus we have L^∞ estimates: this is already known since [7] and [6].

Remark 2.3. Now we assume that off-diagonal coefficients $a_{ij}^{\gamma\beta}(x,y)$ do not vanish any more, but they are small when the corresponding component y^γ is large: there exist $q, c_5 \in (0,+\infty)$ such that

$$0 < \theta^\gamma \leq |y^\gamma| \quad \Longrightarrow \quad |a_{ij}^{\gamma\beta}(x,y)| \leq \frac{c_5}{|y^\gamma|^q} \qquad \text{for} \quad \beta \neq \gamma; \tag{2.8}$$

then $g^\gamma(L) \leq c_5/L^q$ for $L \geq \theta^\gamma$ and (2.5) gives us

$$|\{|u^\gamma| > 2L\}| \leq \frac{c_6}{L^{2^*(1+q)}} \tag{2.9}$$

for every $L \geq \max\{\theta^\gamma; \sup_{\partial\Omega} u^\gamma; -\inf_{\partial\Omega} u^\gamma\}$. This allows us to improve the integrability of u as follows:

Theorem 2.2. *Under the previous assumptions* (2.1), (2.2), *let* $u = (u^1, \ldots, u^N)$ *be a weak solution of the system* (1.1), *that is,* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *and* (2.4) *holds true. In addition, we assume that off-diagonal coefficients satisfy* (2.8). *Moreover, we require that*

$$-\infty < \inf_{\partial\Omega} u^\gamma \quad \textit{and} \quad \sup_{\partial\Omega} u^\gamma < +\infty, \tag{2.10}$$

for every $\gamma = 1, \ldots, N$*; then* u *attains higher integrability:*

$$u \in L^{2^*(1+q)}_{\text{weak}}(\Omega; \mathbb{R}^N). \tag{2.11}$$

Remark 2.4. Please, note that assumption (2.10) implies $\max\{\theta^\gamma; \sup_{\partial\Omega} u^\gamma; -\inf_{\partial\Omega} u^\gamma\} < +\infty$, thus we can use (2.9): such an estimate and boundedness of Ω guarantee (2.11). Since we aim at higher integrability of u, it would be nice to have the same result only assuming enough integrability of the boundary datum, instead of requiring boundedness on the boundary of Ω as in (2.10). It would also be interesting to have a local version of the previous Theorem without any restriction on the boundary datum.

Remark 2.5. Note that reverse Hölder inequality gives us higher integrability of the gradient: $Du \in L^{2+\epsilon}$, see [2] and Chapter 6 in [3]; this improves on the integrability of u by means of Sobolev embedding: $u \in L^{(2+\epsilon)^*}$. In order to have global higher integrability, both the boundary of Ω and the boundary datum have to be regular enough, see Theorem 6.8 at page 209 in [3]. Moreover, it seems that strong ellipticity of $a_{ij}^{\alpha\beta}$ is required; in the present paper we need ellipticity only for diagonal entries $a_{ij}^{\gamma\gamma}$ and only for large values of u^γ. Please, note that we do not assume ellipticity for small values of u^γ: on such a set u^γ is bounded but it might oscillate very much and the gradient Du^γ might lose regularity.

3. Proof of Theorem 2.1

We fix $\gamma \in \{1, \ldots, N\}$ and we take $L \in \mathbb{R}$ with $L \geq \max\{\theta^\gamma; \sup_{\partial\Omega} u^\gamma\} > 0$, where u^γ is the γ-th component of $u = (u^1, \ldots, u^N)$. Since $L \geq \sup_{\partial\Omega} u^\gamma$, we have $\max\{u^\gamma - L; 0\} \in W_0^{1,2}(\Omega)$. We define $v = (v^1, \dots, v^N)$ as follows

$$\begin{cases} v^\alpha = 0 & \text{if } \alpha \neq \gamma \\ v^\gamma = \max\{u^\gamma - L; 0\} & \text{otherwise.} \end{cases} \tag{3.1}$$

Note that

$$\begin{cases} Dv^\alpha = 0 & \text{if } \alpha \neq \gamma \\ Dv^\gamma = 1_{\{u^\gamma > L\}} Du^\gamma & \text{otherwise} \end{cases} \tag{3.2}$$

where 1_E is the characteristic function of the set E, that is, $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. We insert such a test function v into (2.4):

$$\begin{aligned} 0 &= \int_\Omega \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) D_i v^\alpha(x)\, dx \\ &= \int_\Omega \sum_{i,j=1}^n \sum_{\beta=1}^N a_{ij}^{\gamma\beta}(x, u(x)) D_j u^\beta(x) 1_{\{u^\gamma > L\}}(x) D_i u^\gamma(x)\, dx \\ &= \int_{\{u^\gamma > L\}} \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x, u(x)) D_j u^\gamma(x) D_i u^\gamma(x)\, dx \\ &+ \int_{\{u^\gamma > L\}} \sum_{i,j=1}^n \sum_{\beta \neq \gamma} a_{ij}^{\gamma\beta}(x, u(x)) D_j u^\beta(x) D_i u^\gamma(x)\, dx. \end{aligned} \tag{3.3}$$

Then

$$\begin{aligned} &\int_{\{u^\gamma > L\}} \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x, u(x)) D_j u^\gamma(x) D_i u^\gamma(x)\, dx \\ &= - \int_{\{u^\gamma > L\}} \sum_{i,j=1}^n \sum_{\beta \neq \gamma} a_{ij}^{\gamma\beta}(x, u(x)) D_j u^\beta(x) D_i u^\gamma(x)\, dx. \end{aligned} \tag{3.4}$$

Since $L \geq \theta^\gamma$, we can use ellipticity (2.2) and we get

$$\nu \int_{\{u^\gamma > L\}} |Du^\gamma|^2\, dx \leq \int_{\{u^\gamma > L\}} \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x, u(x)) D_j u^\gamma(x) D_i u^\gamma(x)\, dx. \tag{3.5}$$

We keep in mind the definition (2.3) for $g^\gamma(L)$ and we have

$$\begin{aligned} &- \int_{\{u^\gamma > L\}} \sum_{i,j=1}^n \sum_{\beta \neq \gamma} a_{ij}^{\gamma\beta}(x, u(x)) D_j u^\beta(x) D_i u^\gamma(x)\, dx \\ &\leq n^2 (N-1) g^\gamma(L) \int_{\{u^\gamma > L\}} |Du||Du^\gamma|\, dx. \end{aligned} \tag{3.6}$$

Equality (3.4) and estimates (3.5), (3.6) merge into

$$\nu \int_{\{u^\gamma > L\}} |Du^\gamma|^2 \, dx \le n^2(N-1) g^\gamma(L) \int_{\{u^\gamma > L\}} |Du||Du^\gamma| \, dx. \tag{3.7}$$

We use Hölder inequality on the right hand side in order to get

$$\begin{aligned} &\nu \int_{\{u^\gamma > L\}} |Du^\gamma|^2 \, dx \\ &\le n^2(N-1) g^\gamma(L) \left(\int_{\{u^\gamma > L\}} |Du|^2 \, dx \right)^{1/2} \left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 \, dx \right)^{1/2} . \end{aligned} \tag{3.8}$$

We divide both sides by $(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 \, dx)^{1/2}$ and we get

$$\left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 \, dx \right)^{1/2} \le \frac{n^2(N-1) g^\gamma(L)}{\nu} \left(\int_{\{u^\gamma > L\}} |Du|^2 \, dx \right)^{1/2} . \tag{3.9}$$

We keep in mind that $v^\gamma = \max\{u^\gamma - L; 0\} \in W_0^{1,2}(\Omega)$ and $n \ge 3$, thus Sobolev inequality and (3.9) allow us to write

$$\begin{aligned} \int_{\{u^\gamma > L\}} (u^\gamma - L)^{2^*} \, dx = ||v^\gamma||^{2^*}_{L^{2^*}(\Omega)} &\le \left(\frac{2(n-1)}{n-2} ||Dv^\gamma||_{L^2(\Omega)} \right)^{2^*} \\ &= \left(\frac{2(n-1)}{n-2} \left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 \, dx \right)^{1/2} \right)^{2^*} \\ &\le \left(\frac{2(n-1)}{n-2} \frac{n^2(N-1)}{\nu} [g^\gamma(L)] ||Du||_{L^2(\Omega)} \right)^{2^*} . \end{aligned} \tag{3.10}$$

Since $L > 0$, it turns out that $\{u^\gamma > 2L\} \subset \{u^\gamma > L\}$, thus

$$\begin{aligned} L^{2^*} |\{u^\gamma > 2L\}| &= \int_{\{u^\gamma > 2L\}} (2L - L)^{2^*} \, dx \\ &\le \int_{\{u^\gamma > 2L\}} (u^\gamma - L)^{2^*} \, dx \le \int_{\{u^\gamma > L\}} (u^\gamma - L)^{2^*} \, dx. \end{aligned} \tag{3.11}$$

Inequalities (3.10) and (3.11) merge into

$$|\{u^\gamma > 2L\}| \le \left(\frac{2(n-1)n^2(N-1)}{(n-2)\nu} ||Du||_{L^2(\Omega)} \frac{[g^\gamma(L)]}{L} \right)^{2^*} . \tag{3.12}$$

This estimate holds true for every $L \ge \max\{\theta^\gamma; \sup_{\partial\Omega} u^\gamma\} > 0$. Since $-\inf_{\partial\Omega} u^\gamma = \sup_{\partial\Omega}(-u^\gamma)$, if $L \ge \max\{\theta^\gamma; -\inf_{\partial\Omega} u^\gamma\} > 0$, then we can apply the previous inequality (3.12) to $-u$. This ends the proof of Theorem 2.1.

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