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Functor of extension of Λ-isometric maps between central subsets of the unbounded Urysohn universal space

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Abstract. The aim of the paper is to prove that in the unbounded Urysohn universal space $\mathbb{U}$ there is a functor of extension of Λ-isometric maps (i.e. dilations) between central subsets of $\mathbb{U}$ to Λ-isometric maps acting on the whole space. Special properties of the functor are established. It is also shown that the multiplicative group $\mathbb{R} \setminus \{0\}$ acts continuously on $\mathbb{U}$ by Λ-isometries.

Keywords: Urysohn's universal space, ultrahomogeneous spaces, functor, extensions of isometries

Classification: 54C20, 54E40, 54E50

The (unbounded) Urysohn universal space was introduced in [13], [14]. In [10], we have proved that in a bounded Urysohn space, there is a functor of extension of contractions with special properties. The purpose of this paper is to show that one can extend functorially Λ-isometric maps between central subsets of the unbounded Urysohn space $\mathbb{U}$. As an application, we shall prove that the semigroup $(\mathbb{R}, \cdot)$ acts on $\mathbb{U}$ in a very specific way which makes the space $\mathbb{U}$ similar to normed linear spaces.

Urysohn universal spaces are still investigated because of their importance in the theory of Polish groups and therefore the literature dealing with them is still growing up, especially in recent years, see e.g. [1], [2], [3], [4], [7], [8], [15], [16].

The paper is organized as follows: in Section 1 we build an auxiliary functor and establish its properties. This is done by means of a modified method of Katětov [6], already used by Uspenskij [15], Gao and Kechris [2] or Melleray [8]. The results of the first part are applied in Section 2 where we prove the existence of a special functor of extension of Λ-isometric maps in the unbounded Urysohn space $\mathbb{U}$. To do that we use our method introduced in [11] and applied also in [10]. In the last part we show that the multiplicative semigroup $\mathbb{R}$ acts on $\mathbb{U}$ in a very specific way.

1. Auxiliary functor

We begin with the following

1.1 Definition. A *Katětov map* on a metric space (X, d) is any function $f\colon X \to \mathbb{R}$ such that $|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$ for each $x, y \in X$. The space of all Katětov maps on X is denoted by $E(X)$ and it is called the *Katětov hull* of the space X. The Katětov hull is equipped with the metric induced by the supremum norm, which we denote by $\|\cdot\|$. (Katětov maps may be unbounded, their difference however is always bounded.) It is always a complete metric space.

The *Kuratowski map* $\mathfrak{e}\colon X \to E(X)$ is given by the formula $\mathfrak{e}(x) = e_x$, where $e_x(y) = d(x, y)$ $(x, y \in X)$. The map $\mathfrak{e}$ is isometric and $\|f - e_x\| = f(x)$ for each $f \in E(X)$ and $x \in X$. This says that the space X may be identified with the subset $\mathfrak{e}(X)$ of $E(X)$, which is commonly done. However, we follow Katětov [6] and we shall distinguish between X and $\mathfrak{e}(X)$.

Let A be a nonempty subset of the space X and let $f \in E(A)$. The *Katětov extension* of f (in X) is a map $\widehat{f}\colon X \to \mathbb{R}$ defined by

$$\widehat{f}(x) = \inf_{a\in A} (f(a) + d(x, a)) \qquad (x \in X).$$

The map $E(A) \ni f \mapsto \widehat{f} \in E(X)$ is a well defined isometric embedding (this is the theorem of Katětov [6]).

Let $E(X, \omega) = \{\widehat{f}\colon\ f \in E(T)$ with a finite nonempty $T \subset X\} \subset E(X)$. It is well known that if X is separable, then so is $E(X, \omega)$ (see e.g. [6]). Note also that $\mathfrak{e}(X) \subset E(X, \omega)$.

For a number $\lambda \in (0, +\infty)$, by a λ-*isometric* map between metric spaces (X, d) and (Y, ϱ) we mean any function $\varphi\colon X \to Y$ such that $\varrho(\varphi(x), \varphi(y)) = \lambda d(x, y)$ for every $x, y \in X$. A Λ-*isometric* map is a λ-isometric map for some positive λ. If an isometric [respectively, a λ-isometric, a Λ-isometric] map is bijective, it is called an *isometry* [respectively a λ-*isometry*, a Λ-*isometry*].

Now we are ready to introduce a certain category and an auxiliary functor acting on it. Let $\mathcal{S}$ be the class of all complete separable nonempty metric spaces. For two members X and Y of $\mathcal{S}$, let $\Lambda(X, Y)$ be the family of all pairs (φ, λ), where λ is a positive constant and $\varphi\colon X \to Y$ is λ-isometric. Note that the entry 'λ' matters only if $\operatorname{card} X = 1$ (in other cases λ is uniquely determined by φ).

For $X, Y, Z \in \mathcal{S}$ and $(\varphi, \lambda) \in \Lambda(X, Y)$ and $(\psi, \mu) \in \Lambda(Y, Z)$, put $(\psi, \mu)\circ(\varphi, \lambda) = (\psi \circ \varphi, \lambda\mu) \in \Lambda(X, Z)$. Additionally, let $\mathrm{Id}_X = (\mathrm{id}_X, 1)$, where id_X is the identity map on X. There is no difficulty in checking that the above rules well define a category. Now we shall build a functor on it.

For a nonempty metric space (X, d), let $j_X\colon E(X) \to X \sqcup (E(X) \setminus \mathfrak{e}(X))$ ('$\sqcup$' stands for the disjoint union) be a map defined by $j_X(e_x) = x$ for $x \in X$ and $j_X(f) = f$ for $f \notin \mathfrak{e}(X)$. Clearly, j_X is a bijection.

Now let $(X, d) \in \mathcal{S}$. Put $\mathcal{E}(X, d) = (\mathcal{E}(X), \mathcal{E}(d))$ where $\mathcal{E}(X) = X \sqcup (\overline{E(X, \omega)} \setminus \mathfrak{e}(X))$ and $\mathcal{E}(d)\colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathbb{R}$ is a metric defined by

$$\mathcal{E}(d)(u, v) = \|j_X^{-1}(u) - j_X^{-1}(v)\| \qquad (u, v \in \mathcal{E}(X)).$$

It is easy to check that

$$(X, d) \subset (\mathcal{E}(X), \mathcal{E}(d)) \tag{1.1}$$

(the above inclusion means, in particular, that the metric d coincides with the restriction of $\mathcal{E}(d)$ to the set $X \times X$) and that $(\mathcal{E}(X), \mathcal{E}(d)) \in \mathcal{S}$.

Now let $\varphi\colon X \to Y$ be λ-isometric, where X is nonempty. We define $j[\varphi, \lambda] : E(X) \to E(Y)$ by the formula $j[\varphi, \lambda](f) = \widehat{g}$, where $g = \lambda f \circ \varphi^{-1} \in E(\varphi(X))$. It is easily seen that $j[\varphi, \lambda]$ is a well defined λ-isometric map (because of the fact that $E(\varphi(X)) \ni g \mapsto \widehat{g} \in E(Y)$ is isometric).

Let $X, Y \in \mathcal{S}$ and $(\varphi, \lambda) \in \Lambda(X, Y)$. First of all, observe that if $f \in E(X, \omega)$, then $j[\varphi, \lambda](f) \in E(Y, \omega)$. Indeed, if $f = \widehat{u}$ with $u \in E(T)$, then $j[\varphi, \lambda](f) = \widehat{v}$ with $v = \lambda u \circ (\varphi\big|_T)^{-1} \in E(\varphi(T))$ and thus $j[\varphi, \lambda](f) \in E(Y, \omega)$. This provides us to define $\mathcal{E}(\varphi, \lambda)\colon \mathcal{E}(X) \to \mathcal{E}(Y)$ as follows: $\mathcal{E}(\varphi, \lambda)(u) = j_Y(j[\varphi, \lambda](j_X^{-1}(u)))$. It is not difficult to verify that $\mathcal{E}(\varphi, \lambda)$ is λ-isometric (and therefore $(\mathcal{E}(\varphi), \lambda) \in \Lambda(\mathcal{E}(X), \mathcal{E}(Y)))$ and that

$$\mathcal{E}(\varphi, \lambda)\big|_X = \varphi. \tag{1.2}$$

The reader will check that $\mathcal{E}(\mathrm{Id}_X) = \mathrm{id}_{\mathcal{E}(X)}$ and $\mathcal{E}(\psi, \mu) \circ \mathcal{E}(\varphi, \lambda) = \mathcal{E}(\psi \circ \varphi, \lambda\mu)$ for every $X, Y, Z \in \mathcal{S}$ and $(\varphi, \lambda) \in \Lambda(X, Y)$ and $(\psi, \mu) \in \Lambda(Y, Z)$. Now we shall establish further properties of the operator $\mathcal{E}$. Repeating the proof of Uspenskij [15] (or of Katětov [6]), one may show that

1.2 Proposition. *Let (X, d) and (Y, ϱ) be two arbitrary members of $\mathcal{S}$. If $(\varphi_n, \lambda_n)_n$ is a sequence of elements of $\Lambda(X, Y)$ such that $(\varphi_n)_n$ is pointwisely convergent to some $\varphi_0\colon X \to Y$ and $(\lambda_n)_n$ converges to some positive λ_0, then $(\varphi_0, \lambda_0) \in \Lambda(X, Y)$ and the sequence $(\mathcal{E}(\varphi_n, \lambda_n))_n$ is pointwisely convergent to $\mathcal{E}(\varphi_0, \lambda_0)$.*

The above result says that the operator $\mathcal{E}$ is continuous in the topologies of pointwise convergence. Indeed, since X and $\mathcal{E}(X)$ are separable, therefore the collections $\Lambda(X, Y)$ and $\Lambda(\mathcal{E}(X), \mathcal{E}(Y))$ are metrizable in these topologies.

It is clear that if φ is λ-isometric and surjective, then $\mathcal{E}(\varphi, \lambda)$ is surjective as well. It turns out that the converse implication is also true, which shows the following

1.3 Proposition. *Let $(X, d), (Y, \varrho) \in \mathcal{S}$ and $(\varphi, \lambda) \in \Lambda(X, Y)$. Then for every $y \in Y$,*

$$\operatorname{dist}_\varrho(y, \operatorname{im}\varphi) = \operatorname{dist}_{\mathcal{E}(\varrho)}(y, \operatorname{im}\mathcal{E}(\varphi, \lambda)), \tag{1.3}$$

where ‘ im’ stands for the image of a function. In particular, $Y \cap \operatorname{im}\mathcal{E}(\varphi, \lambda) = \operatorname{im}\varphi$ and $\mathcal{E}(\varphi, \lambda)$ is surjective if and only if so is φ.

Proof: To prove (1.3), it is enough to check that

$$\|j[\varphi, \lambda](f) - e_y\| \geq \operatorname{dist}_\varrho(y, \operatorname{im}\varphi)$$

for each $f \in E(X)$. Put $g = \lambda f \circ \varphi^{-1}$. Since g is nonnegative, we have

$$\begin{aligned}\|j[\varphi, \lambda](f) - e_y\| = \|\widehat{g} - e_y\| = \widehat{g}(y) &= \inf\{g(z) + \varrho(z, y)\colon\ z \in \varphi(X)\}\\ &\geq \inf\{\varrho(z, y)\colon\ z \in \varphi(X)\} = \operatorname{dist}_\varrho(y, \operatorname{im}\varphi).\end{aligned}$$

Now if $y \in Y \cap \mathcal{E}(\varphi)$, then the right-hand side expression of (1.3) is equal to 0 and thus $y \in \overline{\operatorname{im}\varphi}$. But $\operatorname{im}\varphi$ is closed, because X is complete and φ is Λ-isometric. So, $y \in \operatorname{im}\varphi$. The remainder of the claim follows from the proved part. $\square$

To simplify the notation, let us agree that whenever (Z, λ) is a metric space and $f, g\colon W \to Z$ are arbitrary functions defined on a nonempty set U, then $\lambda_{\sup}(f, g)$ denotes the number $\sup_{w \in W} \lambda(f(w), g(w)) \in [0, +\infty]$. One may think that the operator $\mathcal{E}$ preserves distances, i.e. that

$$\mathcal{E}(\varrho)_{\sup}(\mathcal{E}(\varphi, \lambda), \mathcal{E}(\psi, \mu)) = \varrho_{\sup}(\varphi, \psi) \in [0, +\infty] \tag{1.4}$$

whenever $(X, d), (Y, \varrho) \in \mathcal{S}$ and $(\varphi, \lambda), (\psi, \mu) \in \Lambda(X, Y)$. We suppose that this is not the rule in general. However, there is no difficulty in showing that

1.4 Proposition. *Let $(X, d), (Y, \varrho) \in \mathcal{S}$ and let $\lambda > 0$. If $\varphi, \psi\colon X \to Y$ are two λ-isometric maps, then*

$$\mathcal{E}(\varrho)_{\sup}(\mathcal{E}(\varphi, \lambda), \mathcal{E}(\psi, \lambda)) = \varrho_{\sup}(\varphi, \psi).$$

2. Functor on the unbounded Urysohn space

Recall that a metric space $\mathbb{U}$ is an (unbounded) *Urysohn universal space* if $\mathbb{U}$ is complete, separable, *universal* and *ultrahomogeneous*, where universality means that every separable metric space is isometrically embeddable in $\mathbb{U}$ and ultrahomogeneity says that every isometry between finite subsets of $\mathbb{U}$ is extendable to an isometry of the whole space. The Urysohn universal space $\mathbb{U}$ is more than ultrahomogeneous — it is compact homogeneous, i.e. every isometry between two compact subsets of $\mathbb{U}$ is extendable to an isometry of the whole space. This was proved by Huhunaišvili in [5]. The existence and the uniqueness (up to isometry) of the Urysohn space were proved by Urysohn [13], [14]. He has also proved that a metric space X is Urysohn if it is the completion of an unbounded separable finitely injective space, which gives the most convenient way to construct this space. Recall that an unbounded metric space (Z, d) is *finitely injective* if for every finite nonempty subset T of Z and every Katětov map f on T there is $x \in X$ such that $f = e_x\big|_T$.

Katětov in his fundamental paper [6] described a very convenient method of constructing finitely injective spaces. Namely, starting with an arbitrary nonempty metric space X, one defines inductively an increasing sequence of spaces as follows: $X_0 = X$ and $X_n = E(X_{n-1}, \omega)$ (or $X_n = \overline{E(X_{n-1}, \omega)}$) for $n \geq 1$. Identifying X_n with $\mathfrak{e}(X_n)$, one has $X_n \subset X_{n+1}$ and thus the space $X^* = \bigcup_{n=0}^{\infty} X_n$ is well defined. Whatever X is, X^* is finitely injective and has the same density character as X. Thus, thanks to the before mentioned theorem of Urysohn, the completion of X^* is an Urysohn space, provided X is separable. We shall use these fact and idea in the sequel. But first we state the following definition (we restrict here only to unbounded spaces; for the general case see e.g. [9]).

2.1 Definition. Let X be an unbounded metric space. A subset A of X is *central*, if for each finite nonempty subset T of X and every Katětov map f on $A \cup T$ there is $x \in X$ such that $f = e_x\big|_{A \cup T}$.

Central subsets of the Urysohn space $\mathbb{U}$ are of our interest. One shows that if A is a central subset of $\mathbb{U}$, then $E(A)$ is separable (see e.g. [9]). The converse is not true.

In the sequel we shall need the following result due to Melleray [7].

2.2 Theorem. *For a nonempty metric space (X, d), the following conditions are equivalent:*

(i) *$E(X)$ is separable,*

(ii) *X is separable and $E(X, \omega)$ is dense in $E(X)$,*

(iii) *X has the collinearity property, i.e. there is no **infinite** subset A of X for which*

$$\inf\{d(x, y) + d(y, z) - d(x, z) \colon\ x, y, z \text{ are distinct points of } A\} > 0.$$

Closed balls in the completion of a metric space with the collinearity property are compact.

Melleray [7] has shown that every unbounded metric space has an isometric copy in $\mathbb{U}$ which is not central. In the opposite, we have proved in [9] that every metric space with separable Katětov hull has a central copy in $\mathbb{U}$.

Our aim is to extend Λ-isometric maps between central subsets of $\mathbb{U}$. For this, we need the following two results, the proofs of which the reader can find in [9]:

2.3 Proposition. *If A is a central subset of a metric space X, then A is central in the completion of X.*

2.4 Proposition. *Every isometry between central subsets of $\mathbb{U}$ is extendable to an isometry of the whole space.*

Let (X, d) be a metric space. For any (possibly empty) subset B of X, put $E(X, B, \omega) = \{\widehat{f} \colon\ f \in E(B \cup T) \text{ for some finite nonempty } T \subset X\} \subset E(X)$. Thus, $E(X, \omega) = E(X, \emptyset, \omega)$. Note that $E(X, B_1, \omega) \subset E(X, B_2, \omega)$, provided $B_1 \subset B_2 \subset X$.

In the sequel we shall need the following property, which is an almost immediate consequence of Theorem 2.2:

2.5 Lemma. *If A is a subset of a metric space X and $E(A)$ is separable, then the closures of the sets $E(X, A, \omega)$ and $E(X, \omega)$ coincide.*

Proof: We may assume that A is nonempty. Let $g \in E(X, A, \omega)$. This means that there is a finite subset T of X and a Katětov map f on $A \cup T$ such that

$g = \widehat{f}$. Let $\varepsilon > 0$. Since $E(A)$ is separable, so is $E(A \cup T)$ and therefore, by Theorem 2.2(ii), there are a finite nonempty subset B of $A \cup T$ and a Katětov map u on B such that $\|f - \widehat{u}\big|_{A\cup T}\| \leq \varepsilon$. Put $v = \widehat{u}\big|_{A\cup T}$. We have then $\widehat{v} = \widehat{u} \in E(X,\omega)$ and, by the Katětov theorem, $\|g - \widehat{v}\| = \|\widehat{f} - \widehat{v}\| = \|f - v\| \leq \varepsilon$. This shows that $g \in \overline{E(X,\omega)}$ and thus $\overline{E(X,A,\omega)} \subset \overline{E(X,\omega)}$. The converse inclusion follows from the one $E(X,\omega) \subset E(X,A,\omega)$. $\square$

Now, using the Katětov method, we shall iterate the operator $\mathcal{E}$ built in the first section. This idea is analogous to that of Gao and Kechris [2] and Uspenskij [15].

For any $(X,d) \in \mathcal{S}$, put $\mathcal{E}_0(X,d) = (\mathcal{E}_0(X), \mathcal{E}_0(d)) = (X,d)$ and for $n \geq 1$, let $\mathcal{E}_n(X,d) = (\mathcal{E}_n(X), \mathcal{E}_n(d)) = (\mathcal{E}(\mathcal{E}_{n-1}(X)), \mathcal{E}(\mathcal{E}_{n-1}(d)))$. By (1.1), $(\mathcal{E}_n(X), \mathcal{E}_n(d)) \subset (\mathcal{E}_{n+1}(X), \mathcal{E}_{n+1}(d))$ and therefore we may define a metric space $(\mathcal{E}_\infty(X), \mathcal{E}_\infty(d)) \in \mathcal{S}$ as the completion of the space $\bigcup_{n=0}^{\infty}(\mathcal{E}_n(X), \mathcal{E}_n(d))$. Similarly, for $(\varphi,\lambda) \in \Lambda(X,Y)$ with (X,d), $(Y,\varrho) \in \mathcal{S}$, put $\mathcal{E}_0(\varphi,\lambda) = \varphi$ and $\mathcal{E}_n(\varphi,\lambda) = \mathcal{E}(\mathcal{E}_{n-1}(\varphi,\lambda),\lambda)$ $(n \geq 1)$. One easily checks that $(\mathcal{E}_n(\varphi,\lambda),\lambda) \in \Lambda(\mathcal{E}_n(X), \mathcal{E}_n(Y))$ and that $\mathcal{E}_{n+1}(\varphi,\lambda)$ extends $\mathcal{E}_n(\varphi,\lambda)$ (thanks to (1.2)). So, $\bigcup_{n=0}^{\infty} \mathcal{E}_n(\varphi,\lambda)$ is a well defined λ-isometric map and therefore we may define $\mathcal{E}_\infty(\varphi,\lambda)$ as the unique continuous extension of it. We have $(\mathcal{E}_\infty(\varphi,\lambda),\lambda) \in \Lambda(\mathcal{E}_\infty(X), \mathcal{E}_\infty(Y))$.

The reader will check with no difficulty that $\mathcal{E}_\infty(\mathrm{Id}_X) = \mathrm{id}_{\mathcal{E}_\infty(X)}$ and $\mathcal{E}_\infty(\psi,\mu) \circ \mathcal{E}_\infty(\varphi,\lambda) = \mathcal{E}_\infty(\psi \circ \varphi, \lambda\mu)$ for every $(\varphi,\lambda) \in \Lambda(X,Y)$, $(\psi,\mu) \in \Lambda(Y,Z)$ and $X,Y,Z \in \mathcal{S}$. Further properties of the operator $\mathcal{E}_\infty$ are collected in the following

2.6 Proposition. *Whenever (X,d) and (Y,ϱ) are members of $\mathcal{S}$ and (φ_n,λ_n) $(n \geq 1)$, (φ,λ) and (ψ,λ) are elements of $\Lambda(X,Y)$, then:*

(i) *the space $\mathcal{E}_\infty(X)$ is isometric to $\mathbb{U}$,*
(ii) *$Y \cap \operatorname{im} \mathcal{E}_\infty(\varphi,\lambda) = \operatorname{im} \varphi$; $\mathcal{E}_\infty(\varphi,\lambda)$ is surjective if and only if so is φ,*
(iii) *if the sequence $(\varphi_n)_n$ is pointwisely convergent to φ and $\lambda = \lim_{n\to\infty} \lambda_n$, then the sequence $(\mathcal{E}_\infty(\varphi_n,\lambda_n))_n$ converges pointwisely to $\mathcal{E}_\infty(\varphi,\lambda)$,*
(iv) *$\mathcal{E}_\infty(\varrho)_{\sup}(\mathcal{E}_\infty(\varphi,\lambda), \mathcal{E}_\infty(\psi,\lambda)) = \varrho_{\sup}(\varphi,\psi)$.*

The crucial point of our further construction will be the following

2.7 Lemma. *For $X \in \mathcal{S}$, the following conditions are equivalent:*

(i) *X is a central subset of $\mathcal{E}_\infty(X)$,*
(ii) *$E(X)$ is separable.*

Proof: The implication '(ii) $\Longrightarrow$ (i)' follows from the fact that the Katětov hull of each central subset of $\mathbb{U}$ is separable.

To prove the converse implication, we shall use Lemma 2.5 and Proposition 2.3. Assume that $E(X)$ is separable and consider the space $X^* = \bigcup_{n=0}^{\infty} \mathcal{E}_n(X)$. Let T be a finite subset of X^* and let $f \in E(X \cup T)$. Since T is finite, there is $n \geq 0$ such that $T \subset \mathcal{E}_n(X)$. For simplicity, put $Z = \mathcal{E}_n(X)$ and $u = \widehat{f}\big|_Z$. We may assume that $u \notin \mathfrak{e}(Z)$. By Lemma 2.5, $u \in \overline{E(Z,\omega)}$. So, $u \in \overline{E(Z,\omega)} \setminus \mathfrak{e}(Z) \subset \mathcal{E}_{n+1}(X) \subset X^*$ and $f = e_u\big|_{X\cup T}$. This shows that X is central in X^* and hence, by Proposition 2.3, X is central in $\mathcal{E}_\infty(X)$ as well. $\square$

Now we are ready to state and prove the main result of this section.

2.8 Theorem. *Let d be the metric of $\mathbb{U}$. Let $\mathbb{L}$ be the collection of all quadruples of the form $(X,\varphi,Y;\lambda)$, where X and Y are closed nonempty central subsets of $\mathbb{U}$, λ is a positive number and φ is a λ-isometric map of X to Y. There exists an operator $\mathbb{L} \ni (X,\varphi,Y;\lambda) \mapsto \widehat{\varphi}_{X,Y;\lambda} \in \mathbb{U}^{\mathbb{U}}$ such that whenever X,Y,Z are closed nonempty central subsets of $\mathbb{U}$, (φ_n,λ_n) $(n \geq 1)$ and (φ,λ) and (ψ,μ) are members of $\Lambda(X,Y)$ and $(\theta,\nu) \in \Lambda(Y,Z)$, then:*

- (Λ1) $\widehat{\mathrm{id}}_{X,X;1} = \mathrm{id}_{\mathbb{U}}$ *and* $\widehat{\theta\circ\varphi}_{X,Z;\lambda\nu} = \widehat{\theta}_{Y,Z;\nu} \circ \widehat{\varphi}_{X,Y;\lambda}$,
- (Λ2) $\widehat{\varphi}_{X,Y;\lambda}$ *is a λ-isometric map which extends φ,*
- (Λ3) $Y \cap \operatorname{im}\widehat{\varphi}_{X,Y;\lambda} = \operatorname{im}\varphi$; $\widehat{\varphi}_{X,Y;\lambda}$ *is surjective if and only if so is φ,*
- (Λ4) *if the sequence $(\varphi_n)_n$ is pointwisely convergent to φ and $\lambda = \lim_{n\to\infty}\lambda_n$, then the sequence $(\widehat{\varphi_n}_{X,Y;\lambda_n})_n$ converges pointwisely to $\widehat{\varphi}_{X,Y;\lambda}$,*
- (Λ5) *if $\mu = \lambda$, then $d_{\sup}(\widehat{\varphi}_{X,Y;\lambda},\widehat{\psi}_{X,Y,\mu}) = d_{\sup}(\varphi,\psi)$.*

Proof: Let X be an arbitrary closed nonempty central subset of $\mathbb{U}$. We infer from this that $E(X)$ is separable. By Proposition 2.6(i), there is an isometry $\Psi_X\colon \mathcal{E}_\infty(X) \to \mathbb{U}$. By Lemma 2.7, X is central in $\mathcal{E}_\infty(X)$ and thus $\Psi_X(X)$ is a central subset of $\mathbb{U}$, isometric to X and hence, thanks to Proposition 2.4, there is an isometry $\Theta_X\colon \mathbb{U}\to\mathbb{U}$ such that $\Theta_X\big|_X = \Psi_X\big|_X$. Put $\Phi_X = \Theta_X^{-1}\circ\Psi_X\colon \mathcal{E}_\infty(X)\to\mathbb{U}$. Then Φ_X is an isometry such that $\Phi_X\big|_X = \mathrm{id}_X$.

Now for $(X,\varphi,Y;\lambda)\in\mathbb{L}$, put

$$\widehat{\varphi}_{X,Y;\lambda} = \Phi_Y \circ \mathcal{E}_\infty(\varphi,\lambda)\circ\Phi_X^{-1}.$$

It is easy to check that all items of the statement are satisfied. □

Note that — as in the first section — the entry 'λ' of a quadruple $(X,\varphi,Y;\lambda)$ in the statement of the foregoing theorem may be omitted if $\operatorname{card} X > 1$. In that case, ($\Lambda$4) says that the operator $\Lambda(X,Y)\ni\varphi\mapsto\widehat{\varphi}_{X,Y}\in\Lambda(\mathbb{U},\mathbb{U})$ is continuous in the topologies of pointwise convergence.

3. Multiplicative action of $\mathbb{R}$ on $\mathbb{U}$

The main result of the previous section implies the following

3.1 Theorem. *Let d be the metric of $\mathbb{U}$ and let θ and Z be a fixed element of $\mathbb{U}$ and a central nonempty subset of it, respectively. There is an action $\cdot\colon \mathbb{R}\times\mathbb{U}\to\mathbb{U}$ such that for every $s,t\in\mathbb{R}$ and $x,y\in\mathbb{U}$:*

- (M1) $0\cdot x = \theta$, $1\cdot x = x$, $t\theta = \theta$ *and* $s(tx) = (st)x$,
- (M2) $d(tx,ty) = |t|d(x,y)$,
- (M3) $d(sz,tz) = |t-s|d(z,\theta)$ *for each* $z\in Z$,
- (M4) *the function* $\mathbb{R}\times\mathbb{U}\ni(t,x)\mapsto tx\in\mathbb{U}$ *is continuous.*

Proof: Let E denote the Banach space of all continuous real-valued functions on $[0,1]$ equipped with the supremum metric ϱ. Since E is universal for separable metric spaces, there is an isometric embedding $j\colon Z\cup\{\theta\}\to E$ such that $j(\theta)=0$. Now for $t\in\mathbb{R}\setminus\{0\}$, let $\varphi_t\colon E\to E$ be defined by $\varphi_t(f) = tf$. Put $m_t = \mathcal{E}_\infty(\varphi_t,t)\colon \mathcal{E}_\infty(E)\to\mathcal{E}_\infty(E)$ for $t\neq 0$ and $m_0\colon \mathcal{E}_\infty(E)\ni\xi\mapsto 0\in E\subset\mathcal{E}_\infty(E)$.

Additionally, for each $\xi \in \mathcal{E}_\infty(E)$ and $t \in \mathbb{R}$, put $t * \xi = m_t(\xi)$. If we think of the action '$*$' as a multiplication '$\cdot$' [on $\mathcal{E}_\infty(E)$] appearing in (M1)–(M4), then '$*$' satisfies (M1) for any $x \in \mathcal{E}_\infty(E)$ and with θ replaced by the zero element of E, because $m_t \circ m_s = m_{st}$. Further, since φ_t is a $|t|$-isometry for $t \neq 0$, so is m_t and, therefore, (M2) is also fulfilled, when d is replaced by $\mathcal{E}_\infty(\varrho)$ and $x, y \in \mathcal{E}_\infty(E)$. What is more, the action $*\colon \mathbb{R} \times \mathcal{E}_\infty(E) \to \mathcal{E}_\infty(E)$ is continuous. Further, by Proposition 2.6(i), there is an isometry $\psi\colon \mathcal{E}_\infty(E) \to \mathbb{U}$. By Lemma 2.7 and Proposition 2.4, there exists an isometry $\kappa\colon \mathbb{U} \to \mathbb{U}$ which extends $\psi \circ j$. Let $\Psi = \kappa^{-1} \circ \psi$. Then $\Psi\big|_{j(Z \cup \{\theta\})} = j^{-1}$. Finally, for $t \in \mathbb{R}$ and $u \in \mathbb{U}$, put $t \cdot u = \Psi(t * \Psi^{-1}(u))$. By the previously proved properties of the action '$*$' on $\mathcal{E}_\infty(E)$, the action '$\cdot$' on $\mathbb{U}$ satisfies conditions (M1), (M2) and (M4), while (M2) follows from the connection $t \cdot z = \Psi(tj(z))$ for $z \in Z$ and $t \in \mathbb{R}$. $\square$

3.2 *Remark.* One may ask whether there is an action as in the statement of Theorem 3.1 such that (M3) is fulfilled for each $z \in \mathbb{U}$. We answer that this is impossible, which can be shown as follows.

Suppose, for the contrary, that there exists such a multiplication. Take $b \in \mathbb{U}$ such that $d(b, \theta) = 1$. Put $b' = \frac{1}{2} \cdot b$. Then, by (M1)–(M3), $d(\theta, b') = d(b, b') = \frac{1}{2} d(\theta, b) = \frac{1}{2}$, which implies that the formulas $\theta, b' \mapsto 1$ and $b \mapsto \frac{1}{2}$ define a Katětov map on $\{\theta, b, b'\}$. Thus there exists $c \in \mathbb{U}$ such that $d(c, \theta) = d(c, b') = 2d(c, b) = 1$. Hence, by (M1)–(M3), $1 = d(b', c) = d(\frac{1}{2} \cdot b, 1 \cdot c) \leq d(\frac{1}{2} \cdot b, \frac{1}{2} \cdot c) + d(\frac{1}{2} \cdot c, 1 \cdot c) = \frac{1}{2} d(b, c) + \frac{1}{2} d(c, \theta) = \frac{3}{4}$, which is a contradiction.

3.3 *Remark.* Melleray [8], following Pestov [12] (who was interested in the 'bounded version' of the Urysohn universal space), thought of whether the group $\mathrm{Iso}(\mathbb{U})$ of all isometries of the unbounded Urysohn space $(\mathbb{U}, d)$ is path-connected with respect to its uniform topology, i.e. the topology induced by the metric $\min(d_{\sup}, 1)$. He pointed out this as an open problem. Below we explain that the answer to this is negative.

Theorem 3.1 implies that there is an *unbounded* isometry $\varphi \in \mathrm{Iso}(\mathbb{U})$, i.e. $d_{\sup}(\varphi, \mathrm{id}_{\mathbb{U}}) = +\infty$. Indeed, it is enough to put $\varphi(x) = (-1) \cdot x$. (The existence of an unbounded isometry was also proved by Cameron and Vershik [1], by a different method.) Now if we define an equivalence relation on $\mathrm{Iso}(\mathbb{U})$ by $\varphi \sim \psi \iff d_{\sup}(\varphi, \psi) < +\infty$, then it is easy to see that every equivalence class (with respect to '$\sim$') is open in the uniform topology. These sets are pairwise disjoint and therefore $\mathrm{Iso}(\mathbb{U})$ is disconnected with respect to the uniform topology.

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