On Jordan ideals and derivations in rings with involution

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Abstract. Let R be a 2-torsion free $*$-prime ring, d a derivation which commutes with $*$ and J a $*$-Jordan ideal and a subring of R. In this paper, it is shown that if either d acts as a homomorphism or as an anti-homomorphism on J, then $d = 0$ or $J \subseteq Z(R)$. Furthermore, an example is given to demonstrate that the $*$-primeness hypothesis is not superfluous.

Keywords: $*$-prime rings, Jordan ideals, derivations

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1. Introduction

Throughout this paper, R will denote an associative ring with center $Z(R)$. We will write for all $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. R is 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R admits an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is in general not true. Indeed, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.

An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. A Jordan ideal J which satisfies $J^* = J$ is called a $*$-Jordan ideal. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all x, y in R. A derivation d commutes with an involution $*$ if $d(r^*) = (d(r))^*$ for all $r \in R$. A derivation d acts as a homomorphism (resp. as an anti-homomorphism) on a subset S of R, if $d(xy) = d(x)d(y)$ (resp. $d(xy) = d(y)d(x)$), for all $x, y \in S$. In [2], Bell and Kappe proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal I of R, then $d = 0$. This result was extended by Asma et al. [1] to square closed Lie ideals of 2-torsion free prime rings. Indeed, they showed that if d is a derivation of a 2-torsion free prime ring R which acts as a homomorphism or an anti-homomorphism on a nonzero square closed Lie ideal U of R, then either $d = 0$ or $U \subseteq Z(R)$. In the year 2007, the author et al. [3] established the analogous result for Lie ideals of $*$-prime rings.

In this paper, our attempt is to extend the result of [2] to Jordan ideals of rings with involution.

2. The results

Throughout, $(R,*)$ will be a 2-torsion free ring with involution and $Sa_*(R) := \{r \in R/ r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R.

Lemma 1 ([5, Lemma 2.4])**.** *If R is a ring and J a nonzero Jordan ideal of R, then $2[R,R]J \subseteq J$ and $2J[R,R] \subseteq J$.*

Lemma 2. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $aJb = a^*Jb = 0$, then $a = 0$ or $b = 0$.*

Proof: Assume that $a \neq 0$. Since $2[R,R]J \subseteq J$ by Lemma 1, then $2a[r,s]jb = 0$ for all $r,s \in R$, $j \in J$. This implies that

$$a[r,s]jb = 0 \text{ for all } r,s \in R,\ j \in J. \tag{1}$$

Replacing s by sa in (1), because of $ajb = 0$, we find that $asarjb = 0$ and thus

$$aRarjb = 0 \text{ for all } r \in R,\ j \in J. \tag{2}$$

On the other hand, from $a^*Jb = 0$ it follows that $a^*[r,sa]jb = 0$, which leads to $a^*sarjb = 0$ for all $r,s \in R$ and therefore

$$a^*Rarjb = 0 \text{ for all } r \in R,\ j \in J. \tag{3}$$

From equations (2) and (3), because of $a \neq 0$, the $*$-primeness of R yields $arjb = 0$ for all $r \in R$, $j \in J$. Accordingly

$$aRjb = 0 \text{ for all } j \in J. \tag{4}$$

Writing sa^* instead of s in (1), because of $a^*Jb = 0$, we get $asa^*rjb = 0$ so that

$$aRa^*rjb = 0 \text{ for all } r \in R,\ j \in J. \tag{5}$$

In view of $a^*Jb = 0$, we find that $a^*[r,sa^*]jb = 0$ and thus $a^*sa^*rjb = 0$ for all $r,s \in R$, $j \in J$. Hence

$$a^*Ra^*rjb = 0 \text{ for all } r \in R,\ j \in J. \tag{6}$$

Using (5) and (6), because of $a \neq 0$, the $*$-primeness of R yields $a^*rjb = 0$ and therefore

$$a^*Rjb = 0 \text{ for all } j \in J. \tag{7}$$

Again, because of equations (4) and (7), $*$-primeness of R assures that $jb = 0$ for all $j \in J$. Whence it follows that

$$Jb = 0. \tag{8}$$

From $(j \circ r)b = 0$, by view of (8), we get $jrb = 0$ for all $r \in R$, $j \in J$ and thus

(9) $$jRb = 0 \quad \text{for all} \quad j \in J.$$

Since J is invariant under $*$, from (9) it follows that

(10) $$j^*Rb = 0 \quad \text{for all} \quad j \in J.$$

Using the $*$-primeness of R, because of $J \neq 0$, equations (9) and (10) assure that $b = 0$. $\square$

Lemma 3. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $[J, J] = 0$, then $J \subseteq Z(R)$.*

Proof: From $[2x[r,s], y] = 0$ it follows that $[x[r,s], y] = 0$ and thus $x[[r,s], y] = 0$ for all $r, s \in R$, $x, y \in J$. Hence

(11) $$J[[r,s], y] = 0 \quad \text{for all} \quad r, s \in R, \ y \in J.$$

Since equation (11) is analogous to equation (8), arguing as in the proof of Lemma 2, we arrive at

(12) $$[[r,s], y] = 0 \quad \text{for all} \quad r, s \in R, \ y \in J.$$

Replacing s by sr in (12) we get

(13) $$[r,s][r,y] = 0 \quad \text{for all} \quad r, s \in R, \ y \in J.$$

Writing xs instead of s in (13), where $x \in J$, we obtain $[r,x]s[r,y] = 0$ and thus

(14) $$[r,x]R[r,y] = 0 \quad \text{for all} \quad x, y \in J, \ r \in R.$$

Since $J^* = J$, replacing y by y^* in (14), we get

(15) $$[r,x]R[r,y^*] = 0 \quad \text{for all} \quad x, y \in J, \ r \in R.$$

Let $r \in Sa_*(R)$. From equation (15) it follows that

(16) $$[r,x]R[r,y]^* = 0 \quad \text{for all} \quad x, y \in J.$$

Using (14) together with (16), the $*$-primeness of R forces $[r,x] = 0$ for all $x \in J$. Accordingly

(17) $$[r,x] = 0 \quad \text{for all} \quad r \in Sa_*(R), \ x \in J.$$

Let $r \in R$; since $r - r^* \in Sa_*(R)$, (17) yields $[r - r^*, x] = 0$ for all $x \in J$ and therefore

(18) $$[r,x] = [r^*,x] \quad \text{for all} \quad r \in R, \ x \in J.$$

Substituting r^* for r in (15) and using (18) we obtain $[r,x]R[r^*,y^*]=0$ for all $x,y\in J$, $r\in R$, which leads to

$$[r,x]R[r,y]^*=0 \quad \text{for all} \quad x,y\in J,\ r\in R. \tag{19}$$

Using the $*$-primeness of R, equations (14) and (19) assure that $[r,x]=0$ for all $r\in R$, $x\in J$, proving that $J\subseteq Z(R)$. $\square$

Lemma 4. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If d is a derivation of R such that $d(J)=0$, then $d=0$ or $J\subseteq Z(R)$.*

Proof: From $d(j\circ r)=0$ it follows that

$$jd(r)+d(r)j=0 \quad \text{for all} \quad j\in J,\ r\in R. \tag{20}$$

Substituting rs for r in (20) and using (20) we find that

$$d(r)[s,j]+[j,r]d(s)=0 \quad \text{for all} \quad r,s\in R,\ j\in J. \tag{21}$$

Replacing s by g in (21), where $g\in J$, the fact that $d(g)=0$ yields

$$d(r)[g,j]=0 \quad \text{for all} \quad g,j\in J,\ r\in R. \tag{22}$$

Writing rt instead of r in (22), where $t\in R$, we obtain $d(r)t[g,j]=0$ and thus

$$d(r)R[g,j]=0 \quad \text{for all} \quad g,j\in J,\ r\in R. \tag{23}$$

Since $J^*=J$, from (23) it follows that

$$d(r)R[g,j]^*=0 \quad \text{for all} \quad g,j\in J,\ r\in R. \tag{24}$$

Applying the $*$-primeness of R, because of equations (23) and (24), we conclude that $d(r)=0$ for all $r\in R$ or $[g,j]=0$ for all $g,j\in J$. Hence either $d=0$ or $[J,J]=0$ and therefore $J\subseteq Z(R)$ by Lemma 3. $\square$

Theorem 1. *Let R be a 2-torsion free $*$-prime ring, d a derivation which commutes with $*$ and J a nonzero $*$-Jordan ideal and a subring of R. If d acts as a homomorphism or as an anti-homomorphism on J, then $d=0$ or $J\subseteq Z(R)$.*

Proof: Assume that $d(xy)=d(x)d(y)$ for all $x,y\in J$. Then

$$d(x)y+xd(y)=d(x)d(y) \quad \text{for all } x,y\in J. \tag{25}$$

Replacing y by yz in (25) and using (25) we obtain $(d(x)-x)yd(z)=0$ for all $x,y,z\in J$ and thus

$$(d(x)-x)Jd(z)=0 \quad \text{for all} \quad x,z\in J. \tag{26}$$

Since d commutes with $*$ and $J^*=J$, (26) yields

$$(d(x)-x)Jd(z)^*=0 \quad \text{for all} \quad x,z\in J. \tag{27}$$

Applying Lemma 2, from (26) and (27) it follows that $d(z) = 0$ for all $z \in J$ or $d(x) = x$ for all $x \in J$.

If $d(x) = x$ for all $x \in J$, then from $d(xy) = xy$ we find, because of 2-torsion freeness, that $xy = 0$ for all $x, y \in J$. Since $x(r \circ y) = 0$, we get $xry = 0$ for all $x, y \in J$, $r \in R$, whence it follows that

$$xRy = 0 = xRy^* \quad \text{for all} \quad x, y \in J. \tag{28}$$

Applying Lemma 2, equation (28) contradicts the fact that $0 \neq J$. Hence, $d(z) = 0$ for all $z \in J$ so that $d(J) = 0$ and, by Lemma 4, $d = 0$ or $J \subseteq Z(R)$.

Let us now assume that d acts as an anti-homomorphism on J. Then

$$d(y)d(x) = d(x)y + xd(y) \quad \text{for all} \quad x, y \in J. \tag{29}$$

Replacing x by xy in (29) we arrive at

$$d(y)xd(y) = xyd(y) \quad \text{for all} \quad x, y \in J. \tag{30}$$

Substituting zx for x in (30) and using (30) we get $[d(y), z]xd(y) = 0$ in such a way that

$$[d(y), z]Jd(y) = 0 \quad \text{for all} \quad y, z \in J. \tag{31}$$

Since d commutes with $*$, because of Lemma 2, equation (31) implies that

$$\text{for all} \quad y \in J \cap Sa_*(R) \quad \text{either} \quad d(y) = 0 \quad \text{or} \quad [d(y), z] = 0 \quad \text{for all} \quad z \in J.$$

Let $y \in J$. Since $y^* - y \in J \cap Sa_*(R)$, we have $d(y^* - y) = 0$ or $[d(y^* - y), J] = 0$.

If $d(y^* - y) = 0$, as d commutes with $*$, then $d(y) \in Sa_*(R)$ and equation (31) implies that $d(y) = 0$ or $[d(y), J] = 0$.

If $[d(y^* - y), J] = 0$, then $[d(y^*), z] = [d(y), z]$ for all $z \in J$. Substituting y^* for y in (31) we arrive at

$$[d(y), z]Jd(y^*) = 0 \quad \text{for all} \quad z \in J. \tag{32}$$

Since d commutes with $*$, (32) becomes

$$[d(y), z]J(d(y))^* = 0 \quad \text{for all} \quad z \in J. \tag{33}$$

In view of equations (31) and (33), Lemma 2 yields $d(y) = 0$ or $[d(y), J] = 0$. In conclusion, we have $d(y) = 0$ or $[d(y), J] = 0$ for all $y \in J$.

Let us consider $J_1 = \{y \in J \,/\, d(y) = 0\}$ and $J_2 = \{y \in J \,/\, [d(y), J] = 0\}$; it is clear that J_1 and J_2 are additive subgroups of J such that $J = J_1 \cup J_2$. But a group cannot be a union of two of its proper subgroups so that $J = J_1$ or $J = J_2$. If $J = J_1$, then $d(J) = 0$ and Lemma 4 forces $d = 0$ or $J \subseteq Z(R)$.

Suppose that $J = J_2$. Then

$$[d(x), y] = 0 \quad \text{for all} \quad x, y \in J. \tag{34}$$

Replacing x in (34) by xy we get

$$x[d(y), y] + [x, y]d(y) = 0 \quad \text{for all} \quad x, y \in J. \tag{35}$$

Substituting zx for x in (35) we obtain $[z, y]xd(y) = 0$ and thus

$$[z, y]Jd(y) = 0 \quad \text{for all} \quad y, z \in J. \tag{36}$$

Reasoning as above, equation (36) leads to $d(y) = 0$ or $[y, J] = 0$ for all $y \in J$. Consider $U_1 = \{y \in J \,/\, d(y) = 0\}$ and $U_2 = \{y \in J \,/\, [y, J] = 0\}$; clearly U_1 and U_2 are additive subgroups of J such that $J = U_1 \cup U_2$ and therefore $J = U_1$ or $J = U_2$. If $J = U_1$, then $d(J) = 0$ and Lemma 4 forces $d = 0$ or $J \subseteq Z(R)$. If $J = U_2$, then $[J, J] = 0$ and Lemma 3 yields $J \subseteq Z(R)$. □

The following example proves the necessity of the $*$-primeness hypothesis in Theorem 1.

Example 1. Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is nonzero. Further, suppose that $R = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) : x, y \in S\}$ and $J = \{\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right) : y \in S\}$. Consider $* : R \longrightarrow R$ defined by $\left(\begin{smallmatrix} u & v \\ 0 & u \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} -u & -v \\ 0 & -u \end{smallmatrix}\right)$; it is easy to verify that $*$ is an involution. Moreover, if we set $r = \left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)$, where $s \neq 0$, then using $sus = 0$ for all $u \in S$ we find that $aRa = 0 = aRa^*$ proving that R is a non $*$-prime ring. Furthermore, the map $d : R \longrightarrow R$ defined by $d\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$ is a derivation which commutes with $*$. Moreover, J is a $*$-Jordan ideal and a subring of R such that d acts as a homomorphism as well as an anti-homomorphism on J; but neither $d = 0$ nor J is central. Indeed, if $r = \left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)$ and $j = \left(\begin{smallmatrix} 0 & w \\ 0 & 0 \end{smallmatrix}\right)$, with $sw \neq 0$, then $[j, r] \neq 0$. Hence, the hypothesis of $*$-primeness in Theorem 1 is crucial.

Using the fact that a $*$-prime ring which admits a nonzero central $*$-ideal must be commutative (see [4], proof of Theorem 1.1), Theorem 1 yields the following result.

Theorem 2. *Let R be a 2-torsion free $*$-prime ring, d a nonzero derivation commuting with $*$ and I a nonzero $*$-ideal of R. If either d acts as a homomorphism or as an anti-homomorphism on I, then R is commutative.*

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