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Approximate solutions for integrodifferential equations of the neutral type

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Abstract. The main objective of the present paper is to study the approximate solutions for integrodifferential equations of the neutral type with given initial condition. A variant of a certain fundamental integral inequality with explicit estimate is used to establish the results. The discrete analogues of the main results are also given.

Keywords: approximate solutions, integrodifferential equation, neutral type, explicit estimate, discrete analogues, dependency of solutions, closeness of solutions

Classification: 34G20, 34A60

1. Introduction

Consider the initial value problem (IVP, for short) for the integrodifferential equation of the form

$$x'(t) = f(t, x(t), x'(t), Hx(t)), \tag{1.1}$$

for $t \in \mathbb{R}_+ = [0, \infty)$, with the given initial condition

$$x(0) = x_0, \tag{1.2}$$

where

$$Hx(t) := \int_0^t h(t, \sigma, x(\sigma), x'(\sigma))\, d\sigma, \tag{1.3}$$

f, h are given functions, x is the unknown function to be found and $'$ denotes the derivative. We assume that $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and for $\sigma \leq t$, $h \in C(\mathbb{R}_+^2 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with appropriate norm denoted by $|\cdot|$. The special version of IVP of the form

$$x'(t) = f(t, x(t), Hx(t)), x(0) = x_0,$$

(often referred to as neutral integrodifferential equations) is discussed in the recent monograph by H. Brunner [3, pp.155, 175] (see also [4]). In fact, analysis of the qualitative properties of solutions of IVP (1.1)–(1.2) is a challenging task, because of the occurrence of the extra factor $x'(t)$ on the right hand side in (1.1).

In practice, it is often difficult to obtain the solutions to the IVP (1.1)–(1.2) explicitly and, thus, a new insight to handle the qualitative properties of its solutions is needed. The method of approximate solutions provides the most powerful and widely used analytic tool in the study of various dynamic equations. It enables us to obtain valuable information about solutions without the need to know in advance the solutions explicitly. The problems of existence and some other basic properties of solutions of IVP (1.1)–(1.2) are recently dealt by the present author in [8]. In the present paper, we offer the conditions for the error evaluation of approximate solutions of IVP (1.1)–(1.2) by establishing some new bounds on solutions of approximate problems. We also study the dependency of solutions of IVP (1.1)–(1.2) on parameters. The main tool employed in the analysis is based on the application of a variant of a certain integral inequality with explicit estimate given in [8] (see also [5], [7]). Results on the discrete analogue of IVP (1.1)–(1.2) are also given.

2. Main results

Let $x_i(t) \in C(\mathbb{R}_+, \mathbb{R}^n)$ $(i = 1, 2)$ be functions such that $x_i'(t)$ exist for $t \in \mathbb{R}_+$ and satisfy the inequalities

$$|x_i'(t) - f(t, x_i(t), x_i'(t), Hx_i(t))| \leq \varepsilon_i, \tag{2.1}$$

for given constants $\varepsilon_i \geq 0$, where it is assumed that the initial conditions

$$x_i(0) = x_i, \tag{2.2}$$

are fulfilled. Then we call $x_i(t)$ the ε_i-approximate solutions with respect to IVP (1.1)–(1.2).

We require the following variant of the integral inequality established by the present author in [8, p. 98]. For similar results, see [5], [7].

Lemma 1. *Let $u, a, b \in C(\mathbb{R}_+, \mathbb{R}_+)$ and for $s \leq t$; $e(t,s), \frac{\partial}{\partial t}e(t,s), k(t,s) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ and $a(t)$ is nondecreasing for $t \in \mathbb{R}_+$. If*

$$u(t) \leq a(t) + \int_0^t \left[b(s)u(s) + e(t,s)u(s) + \int_0^s k(s,\sigma)u(\sigma)\, d\sigma \right] ds, \tag{2.3}$$

for $t \in \mathbb{R}_+$, then

$$u(t) \leq a(t) \exp\left(\int_0^t [b(s) + A(s)]\, ds \right), \tag{2.4}$$

for $t \in \mathbb{R}_+$, where

$$A(t) = e(t,t) + \int_0^t \left\{ k(t,\sigma) + \frac{\partial}{\partial t} e(t,\sigma) \right\} d\sigma. \tag{2.5}$$

In the following theorem we obtain estimates for the difference between the two approximate solutions of equation (1.1) with (2.2).

Theorem 1. *Suppose that the functions f, h in equation* (1.1) *satisfy the conditions*

$$|f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z})| \le M\left[|x-\bar{x}| + |y-\bar{y}|\right] + |z-\bar{z}|\,, \tag{2.6}$$

$$|h(t,s,x,y) - h(t,s,\bar{x},\bar{y})| \le q(t,s)\left[|x-\bar{x}| + |y-\bar{y}|\right], \tag{2.7}$$

where $M \ge 0$ is a constant such that $M < 1$ and for $s \le t$; $q(t,s), \frac{\partial}{\partial t}q(t,s) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$. Let $x_i(t)$ $(i = 1,2)$ be respectively ε_i-approximate solutions of equation (1.1) *with* (2.2) *on $\mathbb{R}_+$ such that*

$$|x_1 - x_2| \le \delta, \tag{2.8}$$

where $\delta \ge 0$ is a constant. Then

$$|x_1(t) - x_2(t)| + |x_1'(t) - x_2'(t)| \le \alpha(t)\exp\left(\int_0^t \left[\frac{M}{1-M} + A_0(s)\right] ds\right), \tag{2.9}$$

for $t \in \mathbb{R}_+$, where

$$\alpha(t) = \frac{(\varepsilon_1 + \varepsilon_2)(t+1) + \delta}{1-M}\,, \tag{2.10}$$

$$A_0(t) = \frac{1}{1-M}\left[q(t,t) + \int_0^t \left\{q(t,\sigma) + \frac{\partial}{\partial t}q(t,\sigma)\right\} d\sigma\right]. \tag{2.11}$$

Proof: Since $x_i(t)$ $(i = 1,2)$ for $t \in \mathbb{R}_+$ are respectively ε_i-approximate solutions of equation (1.1) with (2.2), we have (2.1). By taking $t = s$ and integrating both sides with respect to s from 0 to t, we have

$$\begin{aligned}
\varepsilon_i t &\ge \int_0^t |x_i'(s) - f(s, x_i(s), x_i'(s), Hx_i(s))|\, ds \\
&\ge \left|\int_0^t \{x_i'(s) - f(s, x_i(s), x_i'(s), Hx_i(s))\}\, ds\right| \\
&= \left|\left\{x_i(t) - x_i(0) - \int_0^t f(s, x_i(s), x_i'(s), Hx_i(s))\, ds\right\}\right|,
\end{aligned} \tag{2.12}$$

for $i = 1,2$. From (2.12) and using the elementary inequalities

$$|v - z| \le |v| + |z|,\ \ |v| - |z| \le |v - z|, \tag{2.13}$$

we observe that

$$
\begin{aligned}
(\varepsilon_1+\varepsilon_2)t &\geq \left|\left\{x_1(t)-x_1(0)-\int_0^t f(s,x_1(s),x_1'(s),Hx_1(s))\,ds\right\}\right| \\
&\quad+\left|\left\{x_2(t)-x_2(0)-\int_0^t f(s,x_2(s),x_2'(s),Hx_2(s))\,ds\right\}\right| \\
&\geq \left|\left\{x_1(t)-x_1(0)-\int_0^t f(s,x_1(s),x_1'(s),Hx_1(s))\,ds\right\}\right. \\
&\quad\left.-\left\{x_2(t)-x_2(0)-\int_0^t f(s,x_2(s),x_2'(s),Hx_2(s))\,ds\right\}\right| \\
&\geq |x_1(t)-x_2(t)|-|x_1(0)-x_2(0)| \\
&\quad-\left|\int_0^t f(s,x_1(s),x_1'(s),Hx_1(s))\,ds\right. \\
&\quad\left.-\int_0^t f(s,x_2(s),x_2'(s),Hx_2(s))\,ds\right|.
\end{aligned}
\tag{2.14}
$$

Moreover, from (2.1) and using the elementary inequalities in (2.13), we observe that

$$
\begin{aligned}
\varepsilon_1+\varepsilon_2 &\geq |x_1'(t)-f(t,x_1(t),x_1'(t),Hx_1(t))| \\
&\quad+|x_2'(t)-f(t,x_2(t),x_2'(t),Hx_2(t))| \\
&\geq |\{x_1'(t)-f(t,x_1(t),x_1'(t),Hx_1(t))\} \\
&\quad-\{x_2'(t)-f(t,x_2(t),x_2'(t),Hx_2(t))\}| \\
&\geq |x_1'(t)-x_2'(t)|-|f(t,x_1(t),x_1'(t),Hx_1(t)) \\
&\quad-f(t,x_2(t),x_2'(t),Hx_2(t))|.
\end{aligned}
\tag{2.15}
$$

Let $u(t)=|x_1(t)-x_2(t)|+|x_1'(t)-x_2'(t)|$ for $t\in\mathbb{R}_+$. From (2.14), (2.15) and using the hypotheses, we observe that

$$
\begin{aligned}
u(t) &\leq (\varepsilon_1+\varepsilon_2)t+|x_1(0)-x_2(0)| \\
&\quad+\int_0^t |f(s,x_1(s),x_1'(s),Hx_1(s))-f(s,x_2(s),x_2'(s),Hx_2(s))|\,ds \\
&\quad+(\varepsilon_1+\varepsilon_2)+|f(t,x_1(t),x_1'(t),Hx_1(t))-f(t,x_2(t),x_2'(t),Hx_2(t))| \\
&\leq (\varepsilon_1+\varepsilon_2)(t+1)+\delta+\int_0^t\left\{Mu(s)+\int_0^s q(s,\sigma)u(\sigma)\,d\sigma\right\}ds \\
&\quad+Mu(t)+\int_0^t q(t,\sigma)u(\sigma)\,d\sigma.
\end{aligned}
\tag{2.16}
$$

From (2.16), it is easy to observe that

$$u(t) \leq \alpha(t) + \frac{1}{1-M}\int_0^t \left\{ Mu(s) + q(t,s)u(s) + \int_0^s q(s,\sigma)u(\sigma)\,d\sigma \right\} ds, \tag{2.17}$$

where $\alpha(t)$ is given by (2.10). Clearly $\alpha(t)$ is nondecreasing for $t \in \mathbb{R}_+$. Now a suitable application of Lemma 1 to (2.17) yields (2.9). □

Remark 1. We note that the estimate obtained in (2.9) yields not only a bound for the difference between the two approximate solutions of equation (1.1) with (2.2) but also a bound on the difference between their derivatives. If $x_1(t)$ is a solution of equation (1.1) with $x_1(0) = x_1$, then we see that $x_2(t) \to x_1(t)$ as $\varepsilon_2 \to 0$ and $\delta \to 0$. Moreover, if we put (i) $\varepsilon_1 = \varepsilon_2 = 0$ and $x_1 = x_2$ in (2.9), then the uniqueness of solutions of equation (1.1) is established and (ii) $\varepsilon_1 = \varepsilon_2 = 0$ in (2.9), then we get a bound that shows the dependency of solutions of equation (1.1) on given initial values.

The equation (1.1) contains as a special case the equation

$$x'(t) = f(t, x(t), x'(t)),$$

for $t \in \mathbb{R}_+$. Usually, the terminology *neutral* is used when $x'(t)$ in f is replaced by $x'(t-\tau)$, $\tau > 0$. Here, it is to be noted that, in this case for the existence of a unique solution with suitable initial conditions, one needs that the function f is bounded and satisfies a Lipschitz condition, but the Lipschitz constant need not be less than one, see [9, p. 185] and [10, p. 459]. In [1], Bellman and Cooke have discussed the behavior of solutions of such equations when the retardation τ tends to zero. For an excellent account on the study of such equations, see the book by Bellman and Cooke [2].

Consider the IVP (1.1)–(1.2) together with the following IVP

$$y'(t) = g(t, y(t), y'(t), Hy(t)), \tag{2.18}$$

$$y(0) = y_0, \tag{2.19}$$

for $t \in \mathbb{R}_+$, where H is given by (1.3) and $g \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.

In the next theorem we provide conditions concerning the closeness of the solutions of IVP (1.1)–(1.2) and IVP (2.18)–(2.19).

Theorem 2. *Suppose that the functions f, h in equation* (1.1) *satisfy the conditions* (2.6), (2.7) *and there exist constants $\bar{\varepsilon} \geq 0$, $\bar{\delta} \geq 0$ such that*

$$|f(t,x,y,z) - g(t,x,y,z)| \leq \bar{\varepsilon}, \tag{2.20}$$

$$|x_0 - y_0| \leq \bar{\delta}, \tag{2.21}$$

where f, x_0 and g, y_0 are as in IVP (1.1)–(1.2) *and IVP* (2.18)–(2.19). *Let $x(t)$ and $y(t)$ be respectively, solutions of IVP* (1.1)–(1.2) *and IVP* (2.18)–(2.19) *on* $\mathbb{R}_+$*. Then*

$$(2.22)\qquad |x(t)-y(t)|+|x'(t)-y'(t)|\leq \beta(t)\exp\left(\int_0^t\left[\frac{M}{1-M}+A_0(s)\right]ds\right),$$

for $t\in\mathbb{R}_+$*, where*

$$(2.23)\qquad \beta(t)=\frac{\bar{\varepsilon}(t+1)+\bar{\delta}}{1-M},$$

and $A_0(t)$ *is as in* (2.11)*.*

Proof: Let $r(t)=|x(t)-y(t)|+|x'(t)-y'(t)|$ for $t\in\mathbb{R}_+$. Using the facts that $x(t)$ and $y(t)$ are solutions of IVP (1.1)–(1.2) and IVP (2.18)–(2.19) and the assumptions, we observe that

(2.24)

$$\begin{aligned}
r(t) &\leq |x_0-y_0|+\int_0^t|f(s,x(s),x'(s),Hx(s))-f(s,y(s),y'(s),Hy(s))|\,ds\\
&\quad+\int_0^t|f(s,y(s),y'(s),Hy(s))-g(s,y(s),y'(s),Hy(s))|\,ds\\
&\quad+|f(t,x(t),x'(t),Hx(t))-f(t,y(t),y'(t),Hy(t))|\\
&\quad+|f(t,y(t),y'(t),Hy(t))-g(t,y(t),y'(t),Hy(t))|\\
&\leq \bar{\delta}+\int_0^t\left\{Mr(s)+\int_0^s q(s,\sigma)r(\sigma)\,d\sigma\right\}ds+\bar{\varepsilon}t\\
&\quad+Mr(t)+\int_0^t q(t,\sigma)r(\sigma)\,d\sigma+\bar{\varepsilon}\\
&=\bar{\varepsilon}(t+1)+\bar{\delta}+Mr(t)+\int_0^t\left\{Mr(s)+q(t,s)r(s)+\int_0^s q(s,\sigma)r(\sigma)\,d\sigma\right\}ds.
\end{aligned}$$

From (2.24), we get

$$(2.25)\qquad r(t)\leq\beta(t)+\frac{1}{1-M}\int_0^t\left\{Mr(s)+q(t,s)r(s)+\int_0^s q(s,\sigma)r(\sigma)\,d\sigma\right\}ds,$$

for $t\in\mathbb{R}_+$, where $\beta(t)$ is given by (2.23). Clearly $\beta(t)$ is nondecreasing for $t\in\mathbb{R}_+$. Now an application of Lemma 1 to (2.25) yields (2.22). □

Remark 2. We note that the result given in Theorem 2 relates the solutions of IVP (1.1)–(1.2) and IVP (2.18)–(2.19) in the sense that if f is close to g and x_0 is close to y_0, then the solutions of IVP (1.1)–(1.2) and IVP (2.18)–(2.19) are also close together.

The following theorem gives conditions for an estimate of the difference between the solutions of IVP (1.1)–(1.2) and IVP (2.18)–(2.19).

Theorem 3. *Suppose that*

$$|f(t,x,y,z)-g(t,\bar{x},\bar{y},\bar{z})| \le L\,[|x-\bar{x}|+|y-\bar{y}|]+|z-\bar{z}|\,, \tag{2.26}$$

where $L \ge 0$ is a constant such that $L < 1$ and the conditions (2.7), (2.21) *hold. Let $x(t)$ and $y(t)$ be respectively, solutions of IVP* (1.1)–(1.2) *and IVP* (2.18)–(2.19) *on $\mathbb{R}_+$. Then*

$$|x(t)-y(t)|+|x'(t)-y'(t)| \le \left(\frac{\bar{\delta}}{1-L}\right)\exp\left(\int_0^t\left[\frac{L}{1-L}+A_1(s)\right]ds\right), \tag{2.27}$$

where

$$A_1(t)=\frac{1}{1-L}\left[q(t,t)+\int_0^t\left\{q(t,\sigma)+\frac{\partial}{\partial t}q(t,\sigma)\right\}d\sigma\right]. \tag{2.28}$$

Proof: Let $w(t)=|x(t)-y(t)|+|x'(t)-y'(t)|$ for $t\in\mathbb{R}_+$. Using the facts that $x(t)$ and $y(t)$ are respectively, solutions of IVP (1.1)–(1.2) and IVP (2.18)–(2.19) and the assumptions, we observe that

$$\begin{aligned} w(t) &\le |x_0-y_0|+\int_0^t |f(s,x(s),x'(s),Hx(s))-g(s,y(s),y'(s),Hy(s))|\,ds\\ &\quad + |f(t,x(t),x'(t),Hx(t))-g(t,y(t),y'(t),Hy(t))|\\ &\le \bar{\delta}+\int_0^t\left\{Lw(s)+\int_0^s q(s,\sigma)w(\sigma)\,d\sigma\right\}ds\\ &\quad + Lw(t)+\int_0^t q(t,\sigma)w(\sigma)\,d\sigma. \end{aligned} \tag{2.29}$$

From (2.29), we get

$$w(t)\le\frac{\bar{\delta}}{1-L}+\frac{1}{1-L}\int_0^t\left\{Lw(s)+q(t,s)w(s)+\int_0^s q(s,\sigma)w(\sigma)\,d\sigma\right\}ds. \tag{2.30}$$

Now an application of Lemma 1 to (2.30) yields (2.27). □

We next consider the following neutral type integrodifferential equations

$$z'(t)=F(t,z(t),z'(t),Hz(t),\mu), \tag{2.31}$$

$$z'(t)=F(t,z(t),z'(t),Hz(t),\mu_0), \tag{2.32}$$

with the given initial condition

$$z(0)=z_0, \tag{2.33}$$

for $t\in\mathbb{R}_+$, where H is given by (1.3), $F\in C(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n)$ and μ,μ_0 are real parameters.

The following theorem deals with the dependency of solutions of IVP (2.31)–(2.33) and IVP (2.32)–(2.33) on parameters.

Theorem 4. *Suppose that the function F in equations* (2.31), (2.32) *satisfy the conditions*

$$|F(t,x,y,z,\mu) - F(t,\bar{x},\bar{y},\bar{z},\mu)| \le N\left[|x-\bar{x}| + |y-\bar{y}|\right] + |z-\bar{z}|\,, \tag{2.34}$$

$$|F(t,x,y,z,\mu) - F(t,x,y,z,\mu_0)| \le n(t)\,|\mu-\mu_0|\,, \tag{2.35}$$

where $N \ge 0$ is a constant such that $N < 1$, and $n \in C(\mathbb{R}_+,\mathbb{R}_+)$ satisfies

$$n(t) + \int_0^t n(s)\,ds \le \bar{N}, \tag{2.36}$$

$\bar{N} \ge 0$ is a constant and the function h satisfies the condition (2.7)*. Let $z_1(t)$ and $z_2(t)$ be the solutions of IVP* (2.31)–(2.33) *and IVP* (2.32)–(2.33) *respectively. Then*

$$\begin{aligned}|z_1(t) - z_2(t)| &+ |z_1'(t) - z_2'(t)| \\ &\le \frac{\bar{N}\,|\mu-\mu_0|}{1-N}\exp\left(\int_0^t \left[\frac{N}{1-N} + A_2(s)\right] ds\right),\end{aligned} \tag{2.37}$$

where

$$A_2(t) = \frac{1}{1-N}\left[q(t,t) + \int_0^t \left\{q(t,\sigma) + \frac{\partial}{\partial t}q(t,\sigma)\right\} d\sigma\right]. \tag{2.38}$$

Proof: Let $v(t) = |z_1(t) - z_2(t)| + |z_1'(t) - z_2'(t)|$ for $t \in \mathbb{R}_+$. Using the facts that $z_1(t)$ and $z_2(t)$ are the solutions of IVP (2.31)–(2.33) and IVP (2.32)–(2.33), we observe that

$$\begin{aligned}v(t) \le &\int_0^t |F(s,z_1(s),z_1'(s),Hz_1(s),\mu) - F(s,z_2(s),z_2'(s),Hz_2(s),\mu)|\,ds \\ &+ \int_0^t |F(s,z_2(s),z_2'(s),Hz_2(s),\mu) - F(s,z_2(s),z_2'(s),Hz_2(s),\mu_0)|\,ds \\ &+ |F(t,z_1(t),z_1'(t),Hz_1(t),\mu) - F(t,z_2(t),z_2'(t),Hz_2(t),\mu)| \\ &+ |F(t,z_2(t),z_2'(t),Hz_2(t),\mu) - F(t,z_2(t),z_2'(t),Hz_2(t),\mu_0)|\,.\end{aligned} \tag{2.39}$$

The rest of the proof can be completed by closely looking at the proofs of the above theorems and hence we omit the details. □

Remark 3. We note that an important feature of our approach here is that it is elementary and can be extended to obtain similar results as given in this paper for the IVP for higher order Volterra integrodifferential equation of the form

$$x^{(n)}(t) = f(t,x(t),x'(t),\ldots,x^{(n)}(t),Gx(t)), \tag{2.40}$$

with the prescribed initial values

$$x^{(k)}(0) = x_0^k, \quad (k = 0, 1, \dots, n-1), \tag{2.41}$$

for $t \in \mathbb{R}_+$, where $n \geq 2$ is a given integer and

$$Gx(t) := \int_0^t g(t, \sigma, x(\sigma), x'(\sigma), \dots, x^{(n)}(\sigma))\, d\sigma. \tag{2.42}$$

For a brief discussion on the existence, uniqueness and estimates on the solutions of the special version of IVP

$$x^{(n)}(t) = f(t, x(t), \dots, x^{(n-1)}(t), Gx(t)),$$

with given initial values (3.41) by using different method, see [1, pp. 156, 176].

3. Discrete analogue

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and for any functions $z(n)$, $w(n,s)$; $n, s \in \mathbb{N}_0$ define the operators Δ, Δ_1 by $\Delta z(n) = z(n+1) - z(n)$, $\Delta_1 w(n,s) = w(n+1,s) - w(n,s)$. Let $D(S_1, S_2)$ denote the class of discrete functions from the set S_1 to the set S_2. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. We now explore our idea to obtain results similar to the ones given above, concerning the discrete analogue of IVP (1.1)–(1.2) which can be written as

$$\Delta x(n) = \bar{f}(n, x(n), \Delta x(n), \bar{H}x(n)), \tag{3.1}$$

for $n \in \mathbb{N}_0$, with the given initial condition

$$x(0) = \bar{x}_0, \tag{3.2}$$

where

$$\bar{H}x(n) := \sum_{\sigma=0}^{n-1} \bar{h}(n, \sigma, x(\sigma), \Delta x(\sigma)), \tag{3.3}$$

$\bar{f}$, $\bar{h}$ are given functions and x is unknown function to be found. We assume that $\bar{f} \in D(\mathbb{N}_0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and for $\sigma \leq n$; $\bar{h} \in D(\mathbb{N}_0^2 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$. In this section we formulate in brief the results analogous to Lemma 1 and Theorems 1, 2 related to the solutions of IVP (3.1)–(3.2) only. One can formulate results similar to those in Theorems 3 and 4 with suitable modifications.

Let $x_i(n) \in D(\mathbb{N}_0, \mathbb{R})$ $(i = 1, 2)$ be functions such that $\Delta x_i(n)$ exist for $n \in \mathbb{N}_0$ and satisfy

$$\left|\Delta x_i(n) - \bar{f}(n, x_i(n), \Delta x_i(n), \bar{H}x_i(n))\right| \leq \varepsilon_i, \tag{3.4}$$

for given constants $\varepsilon_i \geq 0$, where it is supposed that the initial conditions

$$x_i(0) = \bar{x}_i, \tag{3.5}$$

are fulfilled. Then we call $x_i(n)$ the ε_i-approximate solutions with respect to the IVP (3.1)–(3.2).

Discrete analogues of Lemma 1 and Theorems 1, 2 are given as follows.

Lemma 2. *Let $u, a, b \in D(\mathbb{N}_0, \mathbb{R}_+)$ and for $s \leq n$; $e(n,s), \Delta_1 e(n,s), k(n,s) \in D(\mathbb{N}_0^2, \mathbb{R}_+)$. If $a(n)$ is nondecreasing in $n \in \mathbb{N}_0$ and*

$$u(n) \leq a(n) + \sum_{s=0}^{n-1} \left[b(s)u(s) + e(n,s)u(s) + \sum_{\sigma=0}^{s-1} k(s,\sigma)u(\sigma) \right], \tag{3.6}$$

for $n \in \mathbb{N}_0$, then

$$u(n) \leq a(n) \prod_{s=0}^{n-1} \left[1 + b(s) + \bar{A}(s)\right], \tag{3.7}$$

for $n \in \mathbb{N}_0$, where

$$\bar{A}(n) = e(n+1,n) + \sum_{\sigma=0}^{n-1} \{k(n,\sigma) + \Delta_1 e(n,\sigma)\}. \tag{3.8}$$

The proof follows by looking closely at the proofs of the similar results given in [7, Theorem 4.4.2] and [6, Theorems 1.3.4, 1.4.2].

Theorem 5. *Suppose that the functions $\bar{f}$, $\bar{h}$ in (3.1) satisfy*

$$\left|\bar{f}(n,x,y,z) - \bar{f}(n,\bar{x},\bar{y},\bar{z})\right| \leq \bar{M}\left[|x-\bar{x}| + |y-\bar{y}|\right] + |z-\bar{z}|, \tag{3.9}$$

$$\left|\bar{h}(n,\sigma,x,y) - \bar{h}(n,\sigma,\bar{x},\bar{y})\right| \leq \bar{q}(n,\sigma)\left[|x-\bar{x}| + |y-\bar{y}|\right], \tag{3.10}$$

where $\bar{M} \geq 0$ is a constant such that $\bar{M} < 1$ and for $s \leq n$; $\bar{q}(n,s), \Delta_1\bar{q}(n,s) \in D(\mathbb{N}_0^2, \mathbb{R}_+)$. Let $x_i(n)$ $(i = 1,2)$ be respectively ε_i-approximate solutions of equation (3.1) with (3.5) on $\mathbb{N}_0$ such that

$$|\bar{x}_1 - \bar{x}_2| \leq \delta_0, \tag{3.11}$$

where $\delta_0 \geq 0$ is a constant. Then

$$\begin{aligned} &|x_1(n) - x_2(n)| + |\Delta x_1(n) - \Delta x_2(n)| \\ &\qquad \leq \bar{\alpha}(n) \prod_{s=0}^{n-1} \left[1 + \left\{\frac{\bar{M}}{1-\bar{M}} + \bar{A}_0(s)\right\}\right], \end{aligned} \tag{3.12}$$

for $n \in \mathbb{N}_0$, where

$$\bar{\alpha}(n) = \frac{(\varepsilon_1 + \varepsilon_2)(n+1) + \delta_0}{1-\bar{M}}, \tag{3.13}$$

$$\bar{A}_0(n) = \frac{1}{1-\bar{M}}\left[\bar{q}(n+1,n) + \sum_{\sigma=0}^{n-1}\{\bar{q}(n,\sigma) + \Delta_1\bar{q}(n,\sigma)\}\right]. \tag{3.14}$$

PROOF: Since $x_i(n)$ $(i = 1,2)$ are respectively, ε_i-approximate solutions of equation (3.1) with (3.5), we have (3.4). By taking $n = s$ in (3.4) and summing up both sides over s from 0 to $n-1$, we observe that

$$\begin{aligned}
\varepsilon_i n &\geq \sum_{s=0}^{n-1}\left|\Delta x_i(s) - \bar{f}(s, x_i(s), \Delta x_i(s), \bar{H}x_i(s))\right| \\
&\geq \left|\sum_{s=0}^{n-1}\left\{\Delta x_i(s) - \bar{f}(s, x_i(s), \Delta x_i(s), \bar{H}x_i(s))\right\}\right| \\
&= \left|\left\{x_i(n) - x_i(0) - \sum_{s=0}^{n-1}\bar{f}(s, x_i(s), \Delta x_i(s), \bar{H}x_i(s))\right\}\right|,
\end{aligned} \tag{3.15}$$

for $i = 1,2$. From (3.15) and using the elementary inequalities in (2.13), we observe that

$$\begin{aligned}
(\varepsilon_1+\varepsilon_2)n &\geq \left|\left\{x_1(n) - x_1(0) - \sum_{s=0}^{n-1}\bar{f}(s, x_1(s), \Delta x_1(s), \bar{H}x_1(s))\right\}\right| \\
&\quad + \left|\left\{x_2(n) - x_2(0) - \sum_{s=0}^{n-1}\bar{f}(s, x_2(s), \Delta x_2(s), \bar{H}x_2(s))\right\}\right| \\
&\geq \left|\left\{x_1(n) - x_1(0) - \sum_{s=0}^{n-1}\bar{f}(s, x_1(s), \Delta x_1(s), \bar{H}x_1(s))\right\}\right. \\
&\quad \left. - \left\{x_2(n) - x_2(0) - \sum_{s=0}^{n-1}\bar{f}(s, x_2(s), \Delta x_2(s), \bar{H}x_2(s))\right\}\right| \\
&\geq |x_1(n) - x_2(n)| - |x_1(0) - x_2(0)| \\
&\quad - \left|\sum_{s=0}^{n-1}\bar{f}(s, x_1(s), \Delta x_1(s), \bar{H}x_1(s)) - \sum_{s=0}^{n-1}\bar{f}(s, x_2(s), \Delta x_2(s), \bar{H}x_2(s))\right|.
\end{aligned} \tag{3.16}$$

Furthermore, from (3.4) and using the elementary inequalities in (2.13), we observe that

$$\begin{aligned}
(\varepsilon_1+\varepsilon_2) &\geq \left|\Delta x_1(n) - \bar{f}(n, x_1(n), \Delta x_1(n), \bar{H}x_1(n))\right| \\
&\quad + \left|\Delta x_2(n) - \bar{f}(n, x_2(n), \Delta x_2(n), \bar{H}x_2(n))\right| \\
&\geq \left|\left\{\Delta x_1(n) - \bar{f}(n, x_1(n), \Delta x_1(n), \bar{H}x_1(n))\right\}\right. \\
&\quad \left. - \left\{\Delta x_2(n) - \bar{f}(n, x_2(n), \Delta x_2(n), \bar{H}x_2(n))\right\}\right| \\
&\geq |\Delta x_1(n) - \Delta x_2(n)| - \left|\bar{f}(n, x_1(n), \Delta x_1(n), \bar{H}x_1(n))\right.
\end{aligned} \tag{3.17}$$

$$-\bar{f}(n, x_2(n), \Delta x_2(n), \bar{H}x_2(n))\big| .$$

The rest of the proof can be completed by following the proof of Theorem 1 with suitable modifications and using Lemma 2. Here we omit the further details. □

Consider the IVP (3.1)–(3.2) together with the following IVP

$$\Delta y(n) = \bar{g}(n, y(n), \Delta y(n), \bar{H}y(n)), \tag{3.18}$$

$$y(0) = \bar{y}_0, \tag{3.19}$$

for $n \in \mathbb{N}_0$, where $\bar{H}$ is given by (3.3) and $\bar{g} \in D(\mathbb{N}_0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.

Theorem 6. *Suppose that the functions $\bar{f}$, $\bar{h}$ in equation* (3.1) *satisfy the conditions* (3.9), (3.10) *and there exist constants $\bar{\varepsilon} \geq 0$, $\bar{\delta} \geq 0$ such that*

$$\left|\bar{f}(n, x, y, z) - \bar{g}(n, x, y, z)\right| \leq \bar{\varepsilon}, \tag{3.20}$$

$$|\bar{x}_0 - \bar{y}_0| \leq \bar{\delta}, \tag{3.21}$$

where $\bar{f}$, $\bar{x}_0$ and $\bar{g}$, $\bar{y}_0$ are as in IVP (3.1)–(3.2) *and IVP* (3.18)–(3.19). *Let $x(n)$ and $y(n)$ be solutions of IVP* (3.1)–(3.2) *and IVP* (3.18)–(3.19), *respectively, on $\mathbb{N}_0$. Then*

$$|x(n) - y(n)| + |\Delta x(n) - \Delta y(n)| \leq \bar{\beta}(n) \prod_{s=0}^{n-1} \left[1 + \left\{\frac{\bar{M}}{1-\bar{M}} + \bar{A}_0(s)\right\}\right], \tag{3.22}$$

for $n \in \mathbb{N}_0$, where

$$\bar{\beta}(n) = \frac{\bar{\varepsilon}(n+1) + \bar{\delta}}{1 - \bar{M}}, \tag{3.23}$$

and $\bar{A}_0(n)$ is as in (3.14).

The proof follows by a similar argument as in the proof of Theorem 2 given above with suitable modifications. We omit the details.

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