Ridgelet transform on tempered distributions

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Abstract. We prove that ridgelet transform $R : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{Y})$ and adjoint ridgelet transform $R^* : \mathscr{S}(\mathbb{Y}) \to \mathscr{S}(\mathbb{R}^2)$ are continuous, where $\mathbb{Y} = \mathbb{R}^+ \times \mathbb{R} \times [0, 2\pi]$. We also define the ridgelet transform $\mathcal{R}$ on the space $\mathscr{S}'(\mathbb{R}^2)$ of tempered distributions on $\mathbb{R}^2$, adjoint ridgelet transform $\mathcal{R}^*$ on $\mathscr{S}'(\mathbb{Y})$ and establish that they are linear, continuous with respect to the weak*-topology, consistent with R, R^* respectively, and they satisfy the identity $(\mathcal{R}^* \circ \mathcal{R})(u) = u$, $u \in \mathscr{S}'(\mathbb{R}^2)$.

Keywords: ridgelet transform, tempered distributions, wavelets

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1. Introduction

The ridgelet transform was introduced by Candès [1] in 1999 as a refinement of the wavelet transform in image processing. It is known that $R : \mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{Y})$ and it satisfies the Parsevel's identity. For more details we refer the reader to [1], [8].

On the other hand, after the invention of the Dirac's distribution, various generalized function spaces have been constructed and various integral transforms have been extended to them, like Fourier transform, Laplace transform, Hilbert transform, Radon transform, Mellin transform, Lambert transform, Poisson transform, etc. From this point of view, the wavelet transform has also been extended to some suitable distributional spaces (cf. [4], [5]).

The ridgelet transform is extended to the space of square integrable Boehmian space and studied in [6]. It is well known that the space of square integrable Boehmians properly contains the space of square integrable functions and the space of compactly supported distributions but neither it contains the space of tempered distributions nor it is contained in the space of tempered distributions.

In this paper, we extend the ridgelet transform to the space of tempered distributions as a continuous linear bijection with respect to the weak*-convergence. It is also interesting to note that the space of tempered distributions contains the compactly supported distributions, all $\mathscr{L}^p$-spaces and all $\mathscr{D}'_{\mathscr{L}^p}$-spaces.

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2. Preliminaries

Let $\psi \in \mathscr{S}(\mathbb{R})$ be a real valued function satisfying the admissibility condition

$$\int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2/|\xi|^2 \, d\xi = 1.$$

For each $(a,b,\theta) \in \mathbb{Y} = \mathbb{R}^+ \times \mathbb{R} \times [0,2\pi]$, the bi-variate ridgelet is defined by

$$\psi_{a,b,\theta}(\mathbf{x}) = \psi_{a,b,\theta}(x_1,x_2) = \psi\left(\frac{x_1\cos\theta + x_2\sin\theta - b}{a}\right), \quad (x_1,x_2) \in \mathbb{R}^2.$$

The ridgelet transform [1], [8] of a square integrable function f on $\mathbb{R}^2$ is defined by

$$(Rf)(a,b,\theta) = \int_{\mathbb{R}^2} f(\mathbf{x})\psi_{a,b,\theta}(\mathbf{x})\, d\mathbf{x}, \quad (a,b,\theta) \in \mathbb{Y}. \tag{1}$$

Recall that the Fourier and the Radon transforms are defined, respectively by

$$\mathcal{F}_{\mathbf{x}}[f(\mathbf{x}),\mathbf{w}] = \hat{f}(\mathbf{w}) = \int_{\mathbb{R}^2} f(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{w}}\, d\mathbf{x}, \ \mathbf{w} = (w_1,w_2) \in \mathbb{R}^2, \tag{2}$$

$$(\mathrm{Rad}\, f)(\theta,t) = \int_{\mathbb{R}^2} \delta(\mathbf{x}\cdot e^{i\theta} - t) f(\mathbf{x})\, d\mathbf{x}, \ \theta \in [0,2\pi] \ \text{ and } \ t \in \mathbb{R}, \tag{3}$$

where δ is the Dirac distribution. The ridgelet transform and the Radon transform are related by

$$(Rf)(a,b,\theta) = \int_{-\infty}^{\infty} (\mathrm{Rad}\, f)(\theta,t)\psi\left((t-b)/a\right)\, dt. \tag{4}$$

By using the inversion theorem for 1-dimensional Fourier transforms, the convolution theorem for Fourier transforms and the projection-slice formula [3], the ridgelet transform becomes

$$\begin{aligned}(Rf)(a,b,\theta) &= \mathcal{F}_\xi\left[\mathcal{F}_b\left[\int_{-\infty}^{\infty} (\mathrm{Rad}\, f)(\theta,t)\psi\left((t-b)/a\right)\, dt, \xi\right], -b\right] \\ &= \mathcal{F}_\xi\left[\mathcal{F}_t\left[(\mathrm{Rad}\, f(\theta,t),\xi\right]\cdot\mathcal{F}_t\left[\psi(-t/a),\xi\right], -b\right] \\ &= \frac{a}{2\pi}\int_{-\infty}^{\infty} e^{i\xi b}\hat{f}(\xi e^{i\theta})\overline{\hat{\psi}(a\xi)}\, d\xi.\end{aligned} \tag{5}$$

The adjoint ridgelet transform of a suitable function on $\mathbb{Y}$ is defined by

$$(R^*F)(\mathbf{x}) = \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} F(a,b,\theta)\psi_{a,b,\theta}(\mathbf{x})\,\frac{da}{a^4}\, db\, d\theta \tag{6}$$

and it is proved in [1] that $(R^* \circ R)(f) = f$ for all $f \in \mathscr{L}^2(\mathbb{R}^2)$.

We recall that $\mathscr{S}(\mathbb{R}^2)$ is a Fréchet space, equipped with the following sequence of semi-norms [7],

$$P_N(f) = \sup_{|\mathbf{n}|\leq N} \sup_{\mathbf{x}\in\mathbb{R}^2} (1+|x|^2)^N |(D_{\mathbf{x}}^{\mathbf{n}} f)(x)|, \ N \in \mathbb{N}_0.$$

We introduced a new space consisting of smooth functions on $\mathbb{Y}$, with

$$Q_{k,\alpha;l,\beta;m}(F) = \sup_{(a,b,\theta)\in\mathbb{Y}} |a^k b^l D_a^\alpha D_b^\beta D_\theta^m F(a,b,\theta)| < \infty, \quad k,\alpha,l,\beta,m \in \mathbb{N}_0.$$

To facilitate the reader, we recall the multi-variate Faa di Bruno formula [2], which will be applied in the proof of the following theorem. Let $h(\mathbf{x}) = f[g_1(\mathbf{x}), g_2(\mathbf{x})]$, $\mathbf{x} = (x_1, x_2)$, $\mathbf{n} = (n_1, n_2)$ and we write $\mathbf{j} \prec \mathbf{k}$ if $|\mathbf{j}| < |\mathbf{k}|$ or $|\mathbf{j}| = |\mathbf{k}|$, $j_1 < k_1$ or $|\mathbf{j}| = |\mathbf{k}|$, $j_1 = k_1$, $j_2 < k_2$. Then

$$D_{\mathbf{x}}^{\mathbf{n}} h = \sum_{1\leq|\mathbf{q}|\leq|\mathbf{n}|} D_{\mathbf{x}}^{\mathbf{q}} f \sum_{p(\mathbf{n},\mathbf{q})} \prod_{j=1}^{|\mathbf{n}|} \frac{[D_{\mathbf{x}}^{\mathbf{l}_j} g_1, D_{\mathbf{x}}^{\mathbf{l}_j} g_2]^{\mathbf{k}_j}}{(\mathbf{k}_j!)(\mathbf{l}_j!)^{|\mathbf{k}_j|}},$$

where $p(\mathbf{n},\mathbf{q}) = \{(\mathbf{k}_1,\ldots,\mathbf{k}_{|\mathbf{n}|}; \mathbf{l}_1,\ldots,\mathbf{l}_{|\mathbf{n}|}) :$ for some $1 \leq s \leq |\mathbf{n}|$, $\mathbf{k}_i = \mathbf{0}$ and $\mathbf{l}_i = \mathbf{0}$ for $1 \leq i < |\mathbf{n}| - s$; $|\mathbf{k}_i| > 0$ for $|\mathbf{n}| - s + 1 \leq i \leq |\mathbf{n}|$; and $\mathbf{0} \prec \mathbf{l}_{|\mathbf{n}|-s+1} \prec \ldots \prec \mathbf{l}_{|\mathbf{n}|}$ are such that $\sum_{i=1}^{|\mathbf{n}|} \mathbf{k}_i = \mathbf{q}$, $\sum_{i=1}^{|\mathbf{n}|} |\mathbf{k}_i|\mathbf{l}_i = \mathbf{n}$.

3. Continuity of R and R^* on function spaces

Theorem 3.1. *The ridgelet transform $R : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{Y})$ is continuous.*

Proof: Let $k, l, m, \alpha, \beta \in \mathbb{N}_0$ be arbitrary. By using (5), we get

$$\begin{aligned}
&\left| a^k b^l D_a^\alpha D_b^\beta D_\theta^m (Rf)(a,b,\theta) \right| \\
&= \frac{1}{2\pi} \left| a^{k+1} b^l D_a^\alpha D_b^\beta D_\theta^m \int_{-\infty}^{\infty} e^{i\xi b} \hat{f}(\xi e^{i\theta}) \overline{\hat{\psi}(a\xi)}\, d\xi \right| \\
&= \frac{1}{2\pi} \left| b^l D_b^\beta D_\theta^m \int_{-\infty}^{\infty} e^{i\xi b} \hat{f}(\xi e^{i\theta}) a^{k+1} D_a^\alpha \overline{\hat{\psi}(a\xi)}\, d\xi \right|.
\end{aligned} \tag{7}$$

Now we have, after differentiating under integral sign with respect to a and by using integration by parts,

$$\begin{aligned}
a^{k+1} D_a^\alpha \overline{\hat{\psi}(a\xi)} &= \int_{-\infty}^{\infty} a^{k+1} (i\xi y)^\alpha e^{i\xi a y} \overline{\psi(y)}\, dy \\
&= (i\xi)^{-k-1} \int_{-\infty}^{\infty} (i\xi y)^\alpha (ia\xi)^{k+1} e^{i\xi a y} \overline{\psi(y)}\, dy \\
&= (i\xi)^{-k-1} \int_{-\infty}^{\infty} (i\xi y)^\alpha D_y^{k+1} e^{i\xi a y} \overline{\psi(y)}\, dy \\
&= (i\xi)^{-k-1} (-1)^{k+1} \int_{-\infty}^{\infty} e^{i\xi a y} D_y^{k+1} \left((i\xi y)^\alpha \overline{\psi(y)} \right) dy.
\end{aligned} \tag{8}$$

Using (8) in (7) and by using the same technique employed in (8) for the variable b, we get

$$
\begin{aligned}
(9)\quad & \left|a^k b^l D_a^\alpha D_b^\beta D_\theta^m (Rf)(a,b,\theta)\right| \\
&= \frac{1}{2\pi}\left|D_\theta^m \int_{-\infty}^{\infty} b^l D_b^\beta e^{i\xi b} \hat{f}(\xi e^{i\theta})\xi^{\alpha-k-1}\int_{-\infty}^{\infty} e^{i\xi a y} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right) dy\, d\xi\right| \\
&= \frac{1}{2\pi}\left|D_\theta^m \int_{-\infty}^{\infty} D_\xi^l e^{i\xi b} \hat{f}(\xi e^{i\theta})\xi^{\alpha+\beta-k-1}\int_{-\infty}^{\infty} e^{i\xi a y} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right) dy\, d\xi\right| \\
&= \frac{1}{2\pi}\left|D_\theta^m \int_{-\infty}^{\infty} e^{i\xi b} D_\xi^l \left[\hat{f}(\xi e^{i\theta})\xi^{\alpha+\beta-k-1}\int_{-\infty}^{\infty} e^{i\xi a y} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right)\, dy\right] d\xi\right| \\
&= \frac{1}{2\pi}\left|D_\theta^m \int_{-\infty}^{\infty} e^{i\xi b} \sum_{r=0}^{l}\binom{l}{r} D_\xi^r \hat{f}(\xi e^{i\theta}) \sum_{s=0}^{l-r}\binom{l-r}{s} A_1 \xi^{\alpha+\beta-k-1-s}\right. \\
&\qquad \left. \times D_\xi^{l-r-s}\int_{-\infty}^{\infty} e^{i\xi a y} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right) dy\, d\xi\right|
\end{aligned}
$$

(where $A_1 = \begin{cases} 0 & \text{if } \ \alpha+\beta-k-1>0 \ \text{ and } \ \alpha+\beta-k=s \\ \prod_{j=0}^{s-1}(\alpha-\beta-k-1-j) & \text{otherwise)} \end{cases}$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left|D_\theta^m \int_{-\infty}^{\infty} e^{i\xi b} \sum_{r=0}^{l}\binom{l}{r} D_\xi^r \hat{f}(\xi e^{i\theta}) \sum_{s=0}^{l-r}\binom{l-r}{s} A_1 \xi^{\alpha+\beta-k-1-s}\right. \\
&\qquad \left. \times \int_{-\infty}^{\infty} (iay)^{l-r-s} e^{i\xi a y} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right)\, dy\, d\xi\right| \\
&= \frac{1}{2\pi}\left|D_\theta^m \int_{-\infty}^{\infty} e^{i\xi b} \sum_{r=0}^{l}\binom{l}{r} D_\xi^r \hat{f}(\xi e^{i\theta}) \sum_{s=0}^{l-r}\binom{l-r}{s} A_1 \xi^{\alpha+\beta-k-1-s}\right. \\
&\qquad \left. \times \int_{-\infty}^{\infty} \xi^{-(l-r-s)} y^{l-r-s} D_y^{l-r-s} e^{i\xi a y} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right)\, dy\, d\xi\right| \\
&= \frac{1}{2\pi}\left|D_\theta^m \int_{-\infty}^{\infty} e^{i\xi b} \sum_{r=0}^{l}\binom{l}{r} D_\xi^r \hat{f}(\xi e^{i\theta}) \sum_{s=0}^{l-r}\binom{l-r}{s} A_1 \xi^{\alpha+\beta-k-1-l+r}\right. \\
&\qquad \left. \times \int_{-\infty}^{\infty} e^{i\xi a y} D_y^{l-r-s}\left[y^{l-r-s} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right)\right]\, dy\, d\xi\right| \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{r=0}^{l}\binom{l}{r} |D_\theta^m D_\xi^r(\hat{f}(\xi e^{i\theta}))| \sum_{s=0}^{l-r}\binom{l-r}{s} A_1 |\xi|^{\alpha+\beta-k-1-l+r} \\
&\qquad \times \int_{-\infty}^{\infty} \left|D_y^{l-r-s}\left[y^{l-r-s} D_y^{k+1}\left(y^\alpha \overline{\psi(y)}\right)\right]\right|\, dy\, d\xi.
\end{aligned}
$$

Now applying the multivariate Faa di Bruno formula, we get

$$
\begin{aligned}
(10)\qquad & |D_\theta^m D_\xi^r(\hat f(\xi\cos\theta,\xi\sin\theta))| \\
&= \left| \sum_{1\le|\mathbf{q}|\le|\mathbf{n}|} D_{\mathbf{x}}^{\mathbf{q}}(\hat f(\xi\cos\theta,\xi\sin\theta)) \sum_{p(\mathbf{n},\mathbf{q})} \prod_{j=1}^{|\mathbf{n}|} \frac{[D_{\mathbf{x}}^{\mathbf{l}_j} g_1, D_{\mathbf{x}}^{\mathbf{l}_j} g_2]^{\mathbf{k}_j}}{(\mathbf{k}_j!)(\mathbf{l}_j!)^{|\mathbf{k}_j|}} \right|
\end{aligned}
$$

(where $\mathbf{x} = (\xi,\theta), \mathbf{n} = (r,m)$, and $g_1(\xi,\theta) = \xi\cos\theta, g_2(\xi,\theta) = \xi\sin\theta$)

$$
\le \sum_{1\le|\mathbf{q}|\le|\mathbf{n}|} |D_{\mathbf{x}}^{\mathbf{q}}(\hat f(\xi\cos\theta,\xi\sin\theta))| \sum_{p(\mathbf{n},\mathbf{q})} \prod_{j=1}^{|\mathbf{n}|} \frac{(|\xi|+|\xi|^2)^{|\mathbf{k}_j|}}{(\mathbf{k}_j!)(\mathbf{l}_j!)^{|\mathbf{k}_j|}},
$$

since $|D_{\mathbf{x}}^{\mathbf{l}_j} g_i| = 0$ or 1 or $|\xi|$, $i = 1,2$; and $j = 1,2,\dots|n|$.

Using (10) in (9), we get

$$
\begin{aligned}
&\left|a^{k+1} b^l D_a^\alpha D_b^\beta D_\theta^m (Rf)(a,b,\theta)\right| \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{r=0}^{l} \binom{l}{r} \sum_{1\le|\mathbf{q}|\le|\mathbf{n}|} |D_{\mathbf{x}}^{\mathbf{q}}(\hat f(\xi\cos\theta,\xi\sin\theta))| \sum_{p(\mathbf{n},\mathbf{q})} \prod_{j=1}^{|\mathbf{n}|} \frac{(|\xi|+|\xi|^2)^{|\mathbf{k}_j|}}{(\mathbf{k}_j!)(\mathbf{l}_j!)^{|\mathbf{k}_j|}} \\
&\quad \sum_{s=0}^{l-r} \binom{l-r}{s} A_1 |\xi|^{\alpha+\beta-k-1-l+r} \int_{-\infty}^{\infty} \left| D_y^{l-r-s}\left[y^{l-r-s} D_y^{k+1}\left(y^\alpha \overline{\psi(y)} \right)\right]\right| dy\, d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{r=0}^{l} \binom{l}{r} \sum_{1\le|\mathbf{q}|\le|\mathbf{n}|} |D_{\mathbf{x}}^{\mathbf{q}}(\hat f(\xi\cos\theta,\xi\sin\theta))| \sum_{p(\mathbf{n},\mathbf{q})} M_{\mathbf{n},\mathbf{q}} \\
&\quad \sum_{\Omega\in\mathscr{P}(S)} |\xi|^{\sum_{j=1}^{|\mathbf{n}|}|\mathbf{k}_j| + \prod_{|\mathbf{k}_j|\in\Omega}|\mathbf{k}_j|} \sum_{s=0}^{l-r} \binom{l-r}{s} A_1 C_1 |\xi|^{\alpha+\beta-k-1-l+r}\, d\xi
\end{aligned}
$$

$\big($where $C_1 = \int_{-\infty}^{\infty} |D_y^{l-r-s}[y^{l-r-s} D_y^{k+1}(y^\alpha\overline{\psi(y)})]|\,dy$, $M_{\mathbf{n},\mathbf{q}} = \prod_{j=1}^{|\mathbf{n}|} \frac{1}{(\mathbf{k}_j!)(\mathbf{l}_j!)^{|\mathbf{k}_j|}}$ and $\mathscr{P}(S)$ is the power set of $S = \{|\mathbf{k}_j| : 1\le j\le|\mathbf{n}|\}\big)$

$$
\le \frac{1}{2\pi}\sum_{r=0}^{l}\binom{l}{r} \sum_{1\le|\mathbf{q}|\le|\mathbf{n}|} \sum_{p(\mathbf{n},\mathbf{q})} M_{\mathbf{n},\mathbf{q}} \sum_{\Omega\in\mathscr{P}(S)} \sum_{s=0}^{l-r} \binom{l-r}{s} A_1 C_1 C_2 P_N(\hat f),
$$

where $C_2 = \int_{-\infty}^{\infty} \frac{d\xi}{1+|\xi|^2}$ and $N = |\mathbf{n}| + \sum_{j=1}^{|\mathbf{n}|}|\mathbf{k}_j| + \prod_{|\mathbf{k}_j|\in\Omega}|\mathbf{k}_j| + \alpha+\beta-k-l+r$. □

Definition 3.2. For $F \in \mathscr{S}(\mathbb{Y})$, the adjoint ridgelet transform R^* is defined by

$$
(11)\qquad (R^*F)(\mathbf{x}) = \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} F(a,b,\theta)\psi_{a,b,\theta}(\mathbf{x}) \frac{da}{a^4}\, db \frac{d\theta}{4\pi},\ (a,b,\theta)\in\mathbb{Y}.
$$

Theorem 3.3. *The adjoint ridgelet transform $R^* : \mathscr{S}(\mathbb{Y}) \to \mathscr{S}(\mathbb{R}\times\mathbb{R})$ is continuous.*

PROOF: Let $N, m, n \in \mathbb{N}_0$ with $N \geq 4$ and $m+n \leq N$. Now

$$\begin{aligned}
&\left(1+|\mathbf{x}|^2\right)^N \Big| D_{\mathbf{x}}^{(m,n)} \\
&\quad \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} F(a,b,\theta) D_{x_1}^m D_{x_2}^n \psi\left(\frac{x_1\cos\theta + x_2\sin\theta - b}{a}\right)\frac{da}{a^4}\,db\frac{d\theta}{4\pi}\Big| \\
&\leq \left(1+|\mathbf{x}|^2\right)^N \\
&\quad \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \left|F(a,b,\theta) D_{x_1}^m D_{x_2}^n \psi\left(\frac{x_1\cos\theta + x_2\sin\theta - b}{a}\right)\right|\frac{da}{a^4}\,db\frac{d\theta}{4\pi} \\
&= \left(1+|\mathbf{x}\cdot e^{i\theta}|^2\right)^N \\
&\quad \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} a^{m+n}|F(a,b,\theta)| \left|\psi^{(m+n)}\left(\frac{\mathbf{x}\cdot e^{i\theta}-b}{a}\right)\right|\frac{da}{a^4}\,db\frac{d\theta}{4\pi} \\
&\leq \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} a^{2N-(m+n+4)}|F(a,b,\theta)| \left(\frac{1+|\mathbf{x}\cdot e^{i\theta}-b+b|^2}{a^2}\right)^N \\
&\quad\times \left|\psi^{(m+n)}\left(\frac{\mathbf{x}\cdot e^{i\theta}-b}{a}\right)\right| da\,db\frac{d\theta}{4\pi} \\
&\leq \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} a^{2N-(m+n+4)}|F(a,b,\theta)| \left(\frac{1+|\mathbf{x}\cdot e^{i\theta}-b|^2+|b|^2}{a^2}\right)^N \\
&\quad\times \left|\psi^{(m+n)}\left(\frac{\mathbf{x}\cdot e^{i\theta}-b}{a}\right)\right| da\,db\frac{d\theta}{4\pi}\,.
\end{aligned}$$

Since the function $x \mapsto x^N$ is convex on $[0,\infty)$, we have

$$\begin{aligned}
\left(1+|\mathbf{x}\cdot e^{i\theta}-b|^2+|b|^2\right)^N &= 2^N\left(\frac{|\mathbf{x}\cdot e^{i\theta}-b|^2+(1+|b|^2)}{2}\right)^N \\
&\leq 2^{N-1}\left(|\mathbf{x}\cdot e^{i\theta}-b|^{2N}+(1+|b|^2)^N\right).
\end{aligned}$$

Hence the last expression is dominated by

$$\begin{aligned}
&2^{N-1}\Bigg(\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} a^{2N-(m+n+4)}|F(a,b,\theta)| \left(\frac{|\mathbf{x}\cdot e^{i\theta}-b|}{a}\right)^{2N} \\
&\quad\times \left|\psi^{(m+n)}\left(\frac{\mathbf{x}\cdot e^{i\theta}-b}{a}\right)\right| da\,db\frac{d\theta}{4\pi} \\
&+\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} a^{2N-(m+n+4)}|F(a,b,\theta)|(1+|b|^2)^N \\
&\quad\times \left|\psi^{(m+n)}\left(\frac{\mathbf{x}\cdot e^{i\theta}-b}{a}\right)\right| da\,db\frac{d\theta}{4\pi}\Bigg)
\end{aligned}$$

$$\leq 2^{N-1}P_{2N+m+n}(\psi)\Bigg(\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} a^{2N-(m+n+4)}|F(a,b,\theta)|\,da\,db\frac{d\theta}{4\pi}$$
$$+\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} a^{2N-(m+n+4)}(1+|b|^2)^N|F(a,b,\theta)|\,db\,da\frac{d\theta}{4\pi}\Bigg)$$
$$\leq C\big([Q_{0,0;0,0;0}(F)+Q_{2N-(m+n+2),0;0,0;0}(F)+Q_{0,0;2,0;0}(F)$$
$$+Q_{2N-(m+n+2),0;2,0;0}(F)]+[Q_{0,0;0,0;0}(F)+Q_{2N-(m+n+2),0;0,0;0}(F)$$
$$+Q_{0,0;2N+2,0;0}(F)+Q_{2N-(m+n+2),0;2N+2,0;0}(F)]\big)\,,$$

where $C=\frac{2^{N-2}}{\pi}P_{2N+m+n}(\psi)\int_0^\infty \frac{a^{2N-(m+n+4)}}{1+a^{2N-(m+n+2)}}\,da\int_{-\infty}^{\infty}\frac{1}{1+|b|^2}\,db$.
Hence $R^*:\mathscr{S}(\mathbb{Y})\to\mathscr{S}(\mathbb{R}^2)$ is continuous. □

4. Ridgelet transform on $\mathscr{S}'(\mathbb{R}^2)$

Definition 4.1. We define $\mathcal{R}$ on $\mathscr{S}'(\mathbb{R}^2)$ by $\langle \mathcal{R}u,F\rangle=\langle u,R^*F\rangle$, $F\in\mathscr{S}(\mathbb{Y})$.

By using the linearity of R^* on $\mathscr{S}(\mathbb{Y})$ and the linearity of u on $\mathscr{S}(\mathbb{R}^2)$, it follows that $\mathcal{R}u$ is linear on $\mathscr{S}(\mathbb{R}^2)$. As a consequence of Theorem 3.3, we note that $R^*F\in\mathscr{S}(\mathbb{R}^2)$ and whenever $F_n\to 0$ as $n\to\infty$ in $\mathscr{S}(\mathbb{Y})$, we have $\langle\mathcal{R}u,F_n\rangle=\langle u,R^*F_n\rangle\to\langle u,R^*0\rangle=0$ as $n\to\infty$. Thus $\mathcal{R}u\in\mathscr{S}'(\mathbb{R}^2)$.

Lemma 4.2. *The Ridgelet transform $\mathcal{R}:\mathscr{S}'(\mathbb{R}^2)\to\mathscr{S}'(\mathbb{Y})$ is consistent with the Ridgelet transform on $\mathscr{S}(\mathbb{R}^2)$.*

Proof: Let $g\in\mathscr{S}(\mathbb{R}^2)$, then this can be considered, in a natural way, as a member of $\mathscr{S}'(\mathbb{R}^2)$ by $\langle g,f\rangle=\int_{\mathbb{R}^2}f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}$, $f\in\mathscr{S}(\mathbb{R}^2)$. Hence

$$\langle\mathcal{R}f,F\rangle=\langle f,R^*F\rangle$$
$$=\int_{\mathbb{R}^2}f(\mathbf{x})\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\psi_{a,b,\theta}(\mathbf{x})F(a,b,\theta)\,\frac{da}{a^4}\,db\,\frac{d\theta}{4\pi}\,.$$

By applying the Fubini theorem, we get that

$$\int_{\mathbb{R}^2}f(\mathbf{x})\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\psi_{a,b,\theta}(\mathbf{x})F(a,b,\theta)\,\frac{da}{a^4}\,db\,\frac{d\theta}{4\pi}\,d\mathbf{x}$$
$$=\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{\mathbb{R}^2}f(\mathbf{x})\psi_{a,b,\theta}(\mathbf{x})\,d\mathbf{x}\,\frac{da}{a^4}\,db\,\frac{d\theta}{4\pi}$$
$$=\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}(Rf)(a,b,\theta)\,\frac{da}{a^4}\,db\,\frac{d\theta}{4\pi}$$

which is the identification of Rf in $\mathscr{S}'(\mathbb{Y})$. Hence the distributional ridgelet transform on $\mathscr{S}'(\mathbb{R}^2)$ is consistent with the ridgelet transform on $\mathscr{S}(\mathbb{R}^2)$. □

Definition 4.3. We define $\mathcal{R}^*:\mathscr{S}'(\mathbb{Y})\to\mathscr{S}'(\mathbb{R}^2)$ by $\langle\mathcal{R}^*\Lambda,f\rangle=\langle\Lambda,Rf\rangle$, $f\in\mathscr{S}(\mathbb{R}^2)$.

Theorem 4.4. $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$ *is linear.*

PROOF: Let $u_1, u_2 \in \mathscr{S}'(\mathbb{R}^2)$ and $c_1, c_2 \in \mathbb{C}$ be arbitrary. Then for $F \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned}\langle \mathcal{R}(c_1u_1 + c_2u_2), F\rangle &= \langle (c_1u_1 + c_2u_2), R^*F\rangle \\ &= c_1\langle u_1, R^*F\rangle + c_2\langle u_2, R^*F\rangle \\ &= c_1\langle \mathcal{R}u_1, F\rangle + c_2\langle \mathcal{R}u_2, F\rangle \\ &= \langle (c_1\mathcal{R}u_1 + c_2\mathcal{R}u_2), F\rangle. \qquad \square\end{aligned}$$

Theorem 4.5. $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$ *is continuous with respect to weak***-topology.*

PROOF: Let $u_n \to u$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{R}^2)$. Then for each fixed $F \in \mathscr{S}(\mathbb{Y})$,

$$\begin{aligned}\langle \mathcal{R}u_n, F\rangle - \langle \mathcal{R}u, F\rangle &= \langle \mathcal{R}(u_n - u), F\rangle \\ &= \langle (u_n - u), R^*F\rangle \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

since $R^*F \in \mathscr{S}(\mathbb{R}^2)$ is fixed, and $u_n - u \to 0$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{R}^2)$. Hence $\mathcal{R}$ is continuous on $\mathscr{S}'(\mathbb{R}^2)$. $\square$

Theorem 4.6. $\mathcal{R}^* : \mathscr{S}'(\mathbb{Y}) \to \mathscr{S}'(\mathbb{R}^2)$ *is linear.*

PROOF: Let $\Lambda_1, \Lambda_2 \in \mathscr{S}'(\mathbb{Y})$ and $c_1, c_2 \in \mathbb{C}$. Then

$$\begin{aligned}\langle \mathcal{R}^*(c_1\Lambda_1 + c_2\Lambda_2), f\rangle &= c_1\langle \Lambda_1, R^*f\rangle + c_2\langle \Lambda_2, R^*f\rangle \\ &= c_1\langle \mathcal{R}\Lambda_1, f\rangle + c_2\langle \mathcal{R}\Lambda_2, f\rangle \\ &= \langle (c_1\mathcal{R}\Lambda_1 + c_2\mathcal{R}\Lambda_2), f\rangle. \qquad \square\end{aligned}$$

Theorem 4.7. $\mathcal{R}^* : \mathscr{S}'(\mathbb{Y}) \to \mathscr{S}'(\mathbb{R}^2)$ *is continuous.*

PROOF: If $\Lambda_n \to \Lambda$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{Y})$, then for each fixed $f \in \mathscr{S}(\mathbb{R}^2)$,

$$\langle \mathcal{R}^*\Lambda_n, f\rangle = \langle \Lambda_n, R^*F\rangle \to \langle \Lambda, R^*F\rangle = \langle \mathcal{R}^*\Lambda, f\rangle \quad \text{as} \quad n \to \infty,$$

since $RF \in \mathscr{S}(\mathbb{Y})$. $\square$

Theorem 4.8. $\mathcal{R}^* \circ \mathcal{R}$ *is the identity on* $\mathscr{S}'(\mathbb{R}^2)$.

PROOF: We know that $R^* \circ R$ is identity on $\mathscr{S}(\mathbb{R}^2)$. For $u \in \mathscr{S}'(\mathbb{R}^2)$ and $F \in \mathscr{S}(\mathbb{Y})$,

$$\begin{aligned}\langle (\mathcal{R}^* \circ \mathcal{R})u, f\rangle &= \langle \mathcal{R}u, Rf\rangle \\ &= \langle u, (R^* \circ R)f\rangle \\ &= \langle u, f\rangle. \qquad \square\end{aligned}$$

Thus $\mathcal{R}$ is a continuous linear bijection on $\mathscr{S}'(\mathbb{R}^2)$, with the inverse $\mathcal{R}^*$.

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