Growth orders of Cesàro and Abel means of uniformly continuous operator semi-groups and cosine functions

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This paper is dedicated to the memory of Sen-Yen Shaw.

Abstract. It will be proved that if N is a bounded nilpotent operator on a Banach space X of order $k+1$, where $k \geq 1$ is an integer, then the γ-th order Cesàro mean $C_t^\gamma := \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} T(s)\, ds$ and Abel mean $A_\lambda := \lambda \int_0^\infty e^{-\lambda s} T(s)\, ds$ of the uniformly continuous semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X generated by $iaI+N$, where $0 \neq a \in \mathbb{R}$, satisfy that (a) $\|C_t^\gamma\| \sim t^{k-\gamma}$ $(t \to \infty)$ for all $0 < \gamma \leq k+1$; (b) $\|C_t^\gamma\| \sim t^{-1}$ $(t \to \infty)$ for all $\gamma \geq k+1$; (c) $\|A_\lambda\| \sim \lambda$ $(\lambda \downarrow 0)$. A similar result will be also proved for the uniformly continuous cosine function $(C(t))_{t\geq 0}$ of bounded linear operators on X generated by $(iaI+N)^2$.

Keywords: Cesàro mean, Abel mean, growth order, uniformly continuous operator semi-group and cosine function

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1. Introduction and results

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on a complex Banach space X. As for the γ-th order Cesàro mean C_t^γ of the semigroup $(T(t))_{t\geq 0}$, Chen-Sato-Shaw [1] studied the following question. Does there exist for any real $\delta > 0$ an example of $(T(t))_{t\geq 0}$ such that $\sup_{t>0} \|C_t^\gamma\| = \infty$ for all $0 < \gamma < \delta$, and $\sup_{t>0} \|C_t^\gamma\| < \infty$ for all $\gamma > \delta$? They proved in [1] the following result. Let $k \geq 1$ be an integer and N be a bounded nilpotent operator on X of order $k+1$ (i.e., $N^k \neq 0$ and $N^{k+1} = 0$). Let $(T(t))_{t\geq 0}$ be the uniformly continuous semigroup of bounded linear operators on X generated by $A := iaI + N$, where $0 \neq a \in \mathbb{R}$, so that $T(t)$ has the form

$$(1) \qquad T(t) := e^{tA} = e^{iat} e^{tN} = e^{iat} \sum_{n=0}^{k} \frac{t^n N^n}{n!}\,.$$

Then the γ-th order Cesàro mean C_t^γ and Abel mean A_λ of $(T(t))_{t\geq 0}$ satisfy that $\|C_t^\gamma\| \sim t^{k-\gamma}$ $(t \to \infty)$ for all integers $\gamma = 1, 2, \ldots, k+1$; $\|C_t^\gamma\| \sim t^{-1}$ $(t \to \infty)$ for all $\gamma \geq k+1$; $\|A_\lambda\| \sim \lambda$ $(\lambda \downarrow 0)$; $\sup_{t>0} \|C_t^\gamma\| = \infty$ for all $0 < \gamma < k$, and $\sup_{t>0} \|C_t^\gamma\| < \infty$ for all $\gamma \geq k$. Here $a(t) \sim b(t)$ $(t \to \infty)$ [resp. $(t \downarrow 0)$] means that both the ratios $a(t)/b(t)$ and $b(t)/a(t)$ are bounded in some open interval (ϵ, ∞) [resp. $(0, \epsilon)$]. Thus they gave a partial solution to the question. It remains still open for $\delta > 1$ which is not an integer. (If $0 < \delta < 1$, then the question has a positive answer. See Theorem 4.2 in [1].)

The aim of this article is to prove that the relation $\|C_t^\gamma\| \sim t^{k-\gamma}$ $(t \to \infty)$ holds not only for all integers $\gamma = 1, 2, \ldots, k+1$ but also for all real numbers γ with $0 < \gamma \le k+1$. That is,

Theorem 1. *Let $(T(t))_{t\ge 0}$ be the above semigroup of operators. Let C_t^γ and A_λ denote the γ-th order Cesàro and Abel means of $(T(t))_{t\ge 0}$, respectively. Then*

(a) $\|C_t^\gamma\| \sim t^{k-\gamma}$ $(t \to \infty)$ *for all* $0 < \gamma \le k+1$;
(b) $\|C_t^\gamma\| \sim t^{-1}$ $(t \to \infty)$ *for all* $\gamma \ge k+1$;
(c) $\|A_\lambda\| \sim \lambda$ $(\lambda \downarrow 0)$.

We also consider the uniformly continuous cosine function $(C(t))_{t\ge 0}$ of bounded linear operators on X generated by $B := A^2$ (cf. [4]). Thus $C(t)$ has the form

$$
\begin{aligned}
C(t) &= \frac{1}{2}\big(e^{tA} + e^{-tA}\big) = \sum_{n=0}^{k} \left(\frac{e^{iat}t^n + e^{-iat}(-t)^n}{2}\right)\frac{N^n}{n!} \\
&= \sum_{0\le n\le k}{}' \frac{t^n \cos at}{n!} N^n + \sum_{0\le n\le k}{}'' i\,\frac{t^n \sin at}{n!} N^n,
\end{aligned}
\tag{2}
$$

where $\sum'_{0\le n\le k}$ [resp. $\sum''_{0\le n\le k}$] means that the summation is taken for all n such that $0 \le n \le k$, and n is even [resp. odd]. In this case the γ-th order Cesàro mean C_t^γ and Abel mean A_λ of $(C(t))_{t\ge 0}$ are defined as $C_t^\gamma := \gamma t^{-\gamma}\int_0^t (t-s)^{\gamma-1}C(s)\,ds$ and $A_\lambda := \lambda\int_0^\infty e^{-\lambda s}C(s)\,ds$, respectively. It was proved in [1] that $\sup_{t>0}\|C_t^\gamma\| = \infty$ for all $0 < \gamma < k$, and $\sup_{t>0}\|C_t^\gamma\| < \infty$ for all $\gamma \ge k$. The next theorem improves the result considerably.

Theorem 2. *Let $(C(t))_{t\ge 0}$ be the above cosine function of operators. Let C_t^γ and A_λ denote the γ-th order Cesàro and Abel means of $(C(t))_{t\ge 0}$, respectively. Then*

(a) $\|C_t^\gamma\| = O(t^{k-\gamma})$ $(t \to \infty)$, $\|C_t^\gamma\| \ne o(t^{k-\gamma})$ $(t \to \infty)$, *and* $\|C_t^\gamma\| \not\sim t^{k-\gamma}$ $(t \to \infty)$ *for all* $0 < \gamma < k+2$;
(b) $\|C_t^\gamma\| \sim t^{-2}$ $(t \to \infty)$ *for all* $\gamma \ge k+2$;
(c) $\|A_\lambda\| \sim \lambda^2$ $(\lambda \downarrow 0)$.

For related topics the author would like to refer the reader to [2] and [3] (see also [5]).

2. Lemmas

For a complex-valued continuous function u on $[0, \infty)$, we let

$$\mathfrak{c}_t^\gamma(u) := \gamma t^{-\gamma}\int_0^t (t-s)^{\gamma-1}u(s)\,ds \qquad (\gamma,\ t > 0).$$

Let u_n be the function defined by $u_n(t) := t^n e^{it}$ for $t \geq 0$, where $n \geq 0$ is an integer. Then we have

$$\mathfrak{c}_t^\gamma(u_n) = \frac{\gamma}{t^\gamma}\int_0^t (t-s)^{\gamma-1} s^n e^{is}\, ds = \frac{\gamma}{t^\gamma} e^{it}\int_0^t s^{\gamma-1}(t-s)^n e^{-is}\, ds,$$

so that, by letting

$$U_n(\gamma,t) := \int_0^t s^{\gamma-1}(t-s)^n e^{-is}\, ds \qquad (\gamma,\ t>0), \tag{3}$$

we have

$$\mathfrak{c}_t^\gamma(u_n) = \frac{\gamma}{t^\gamma} e^{it}\, U_n(\gamma,t) \qquad (\gamma,\ t>0). \tag{4}$$

Integration by parts gives

$$U_0(\gamma,t) = it^{\gamma-1}e^{-it} - i(\gamma-1)\, U_0(\gamma-1,t) \qquad (\gamma>1,\ t>0). \tag{5}$$

Lemma 1. *Let $n \geq 0$ be an integer. Then $U_n(\gamma,t) \sim t^n$ $(t\to\infty)$ for all $0<\gamma<1$.*

Proof: (i) First we consider the case $n=0$. Suppose $0<\gamma<1$. Then, since the function $s \mapsto s^{\gamma-1}$ is decreasing on $(0,\infty)$ and since

$$U_0(\gamma,t) = \int_0^t s^{\gamma-1}e^{-is}\, ds = \int_0^t s^{\gamma-1}\cos s\, ds - i\int_0^t s^{\gamma-1}\sin s\, ds,$$

it follows easily that

$$\begin{aligned} 0 &< \int_0^{2\pi} s^{\gamma-1}\sin s\, ds \leq \inf_{t>2\pi}\int_0^t s^{\gamma-1}\sin s\, ds \leq \sup_{t>2\pi} |U_0(\gamma,t)| \\ &\leq \int_0^{\pi/2} s^{\gamma-1}\cos s\, ds + \int_0^{\pi} s^{\gamma-1}\sin s\, ds \leq 2\int_0^\pi s^{\gamma-1}\, ds = \frac{2\pi^\gamma}{\gamma}. \end{aligned}$$

Hence $U_0(\gamma,t)\sim t^0$ $(t\to\infty)$. To use an induction argument we need to consider the case $\gamma \geq 1$. First we have $U_0(1,t) = i(e^{-it}-1)$. Suppose $1<\gamma\leq 2$. Then, since $|U_0(\gamma-1,t)| \leq 2\pi^{\gamma-1}/(\gamma-1)$ for all $t>\pi$, it follows from (5) that $U_0(\gamma,t) = it^{\gamma-1}e^{-it}(1+o(1))$ $(t\to\infty)$. We then apply an induction argument to $\gamma>2$. Suppose $U_0(\gamma,t) = it^{\gamma-1}e^{-it}(1+o(1))$ $(t\to\infty)$ for all γ with $n<\gamma\leq n+1$, where $n\geq 1$ is an integer, and suppose $n+1<\beta\leq n+2$. Then, since $U_0(\beta,t) = it^{\beta-1}e^{-it} - i(\beta-1)U_0(\beta-1,t)$ by (5), it follows that $U_0(\beta,t) = it^{\beta-1}e^{-it}(1+o(1))$ $(t\to\infty)$. Consequently, $U_0(\gamma,t) = it^{\gamma-1}e^{-it}(1+o(1))$ $(t\to\infty)$ for all $\gamma>1$, which will be used below.

(ii) We consider the case $n \geq 1$. Suppose the lemma holds for $n-1$. We note that

$$
\begin{aligned}
(6)\qquad U_n(\gamma,t) &= \int_0^t s^{\gamma-1}(t-s)^n e^{-is}\,ds \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\,t^k\int_0^t s^{\gamma+n-k-1}e^{-is}\,ds = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}t^k U_0(\gamma+n-k,t) \\
&= \sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}\,t^k\Big(it^{\gamma+n-k-1}e^{-it} - i(\gamma+n-k-1)U_0(\gamma+n-k-1,t)\Big) \\
&\quad + \binom{n}{n}(-1)^0\,t^n U_0(\gamma,t) \qquad \text{(by (5))} \\
&= -\,it^{\gamma+n-1}e^{-it} - i\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}t^k(\gamma-1+n-k)U_0(\gamma-1+n-k,t) \\
&\quad + \binom{n}{n}(-1)^0 t^n U_0(\gamma,t),
\end{aligned}
$$

where the last equality comes from the fact that $0=(1-1)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k$.

Suppose $0<\gamma<1$. If $0\leq k\leq n-2$, then, since $\gamma-1+n-k>1$, the result obtained in (i) shows that

$$t^k U_0(\gamma-1+n-k,t) \sim t^{\gamma+n-2} = o(t^{n-1})\ (t\to\infty).$$

Similarly, if $k=n-1$, then

$$t^{n-1}U_0(\gamma-1+n-(n-1),t) = t^{n-1}U_0(\gamma,t) \sim t^{n-1}\ (t\to\infty).$$

Thus, from (6) and the fact that $t^nU_0(\gamma,t)\sim t^n\,(t\to\infty)$, it follows that $U_n(\gamma,t)\sim t^n\ (t\to\infty)$. This completes the proof. □

Next, let v_n and w_n be the functions on $[0,\infty)$ defined by $v_n(t) := t^n\cos t$ and $w_n(t) := t^n \sin t$, where $n\geq 0$ is an integer. Then we have

$$
\begin{aligned}
\mathfrak{c}_t^\gamma(v_n) &= \frac{\gamma}{t^\gamma}\int_0^t (t-s)^{\gamma-1}s^n\cos s\,ds = \frac{\gamma}{t^\gamma}\int_0^t s^{\gamma-1}(t-s)^n\cos(t-s)\,ds \\
&= \frac{\gamma}{t^\gamma}\Big(\cos t\int_0^t s^{\gamma-1}(t-s)^n\cos s\,ds + \sin t\int_0^t s^{\gamma-1}(t-s)^n\sin s\,ds\Big)
\end{aligned}
$$

so that, letting

$$
\begin{aligned}
(7)\qquad F_n(\gamma,t) &:= \int_0^t s^{\gamma-1}(t-s)^n\cos s\,ds, \\
(8)\qquad G_n(\gamma,t) &:= \int_0^t s^{\gamma-1}(t-s)^n\sin s\,ds,
\end{aligned}
$$

$$V_n(\gamma,t) := F_n(\gamma,t)\cos t + G_n(\gamma,t)\sin t, \tag{9}$$
$$W_n(\gamma,t) := F_n(\gamma,t)\sin t - G_n(\gamma,t)\cos t, \tag{10}$$

we have

$$\mathfrak{c}_t^\gamma(v_n) = \frac{\gamma}{t^\gamma} V_n(\gamma,t) \qquad (\gamma,\ t>0); \tag{11}$$

and similarly

$$\mathfrak{c}_t^\gamma(w_n) = \frac{\gamma}{t^\gamma}\int_0^t (t-s)^{\gamma-1}s^n \sin s\, ds = \frac{\gamma}{t^\gamma} W_n(\gamma,t) \qquad (\gamma,\ t>0). \tag{12}$$

Lemma 2. *$F_0(\gamma,t)$ and $G_0(\gamma,t)$ satisfy that*

$$\begin{cases} \lim_{t\to\infty} F_0(\gamma,t) = \int_0^\infty s^{\gamma-1}\cos s\, ds > 0, \\ 0 < \inf_{t>\pi} G_0(\gamma,t) \le \sup_{t>\pi} G_0(\gamma,t) < \infty \end{cases} \qquad \textit{for all}\ \ 0<\gamma<1; \tag{13}$$

$$F_0(1,t) = \sin t, \quad G_0(1,t) = 1-\cos t; \tag{14}$$

$$\begin{cases} F_0(\gamma,t) = t^{\gamma-1}(\sin t + o(1))\ (t\to\infty), \\ G_0(\gamma,t) = t^{\gamma-1}(-\cos t + o(1))\ (t\to\infty) \end{cases} \qquad \textit{for all}\ \ \gamma>1. \tag{15}$$

Proof: Suppose $0<\gamma<1$. Then, since the function $s\mapsto s^{\gamma-1}$ is decreasing and convex on $(0,\infty)$, we have as in the proof of Lemma 1

$$\lim_{t\to\infty} F_0(\gamma,t) = \lim_{n\to\infty}\sum_{j=0}^n \int_{2j\pi}^{2(j+1)\pi} s^{\gamma-1}\cos s\, ds \ge \int_0^{2\pi} s^{\gamma-1}\cos s\, ds > 0.$$

Similarly

$$\int_0^\pi s^{\gamma-1}\sin s\, ds \ge G_0(\gamma,t) \ge \int_0^{2\pi} s^{\gamma-1}\sin s\, ds > 0 \qquad (t>\pi).$$

The proof of (14) is direct.

Next suppose $\gamma>1$. Then integration by parts gives

$$\begin{aligned} F_0(\gamma,t) &= \int_0^t s^{\gamma-1}\cos s\, ds = t^{\gamma-1}\sin t - (\gamma-1)\int_0^t s^{\gamma-2}\sin s\, ds \\ &= t^{\gamma-1}\sin t - (\gamma-1)G_0(\gamma-1,t); \end{aligned} \tag{16}$$

and similarly

$$G_0(\gamma,t) = \int_0^t s^{\gamma-1}\sin s\, ds = -t^{\gamma-1}\cos t + (\gamma-1)F_0(\gamma-1,t). \tag{17}$$

Thus, if $1 < \gamma \leq 2$, then, by the results for $0 < \gamma - 1 \leq 1$, we see that $F_0(\gamma, t) = t^{\gamma-1}(\sin t + o(1))$ $(t \to \infty)$, and $G_0(\gamma, t) = t^{\gamma-1}(-\cos t + o(1))$ $(t \to \infty)$. We can repeat this process to prove (15) for all $\gamma > 1$. □

Lemma 3. *Let $n \geq 1$. Then $F_n(\gamma, t)$ and $G_n(\gamma, t)$ satisfy that*

$$\begin{cases} F_n(\gamma, t) = t^n (F_0(\gamma, t) + o(1)) \ (t \to \infty), \\ G_n(\gamma, t) = t^n (G_0(\gamma, t) + o(1)) \ (t \to \infty) \end{cases} \quad \textit{for all} \ \ 0 < \gamma < 1; \tag{18}$$

$$\begin{cases} F_n(1, t) = O(t^{n-1}) \ (t \to \infty), \\ G_n(1, t) = t^n (1 + o(1)) \ (t \to \infty); \end{cases} \tag{19}$$

$$\begin{cases} F_n(\gamma, t) = -(\gamma - 1) G_n(\gamma - 1, t) + n G_{n-1}(\gamma, t), \\ G_n(\gamma, t) = (\gamma - 1) F_n(\gamma - 1, t) - n F_{n-1}(\gamma, t) \end{cases} \quad \textit{for all} \ \ \gamma > 1. \tag{20}$$

Proof: It follows from (7) that

$$\begin{aligned} F_n(\gamma, t) &= \int_0^t s^{\gamma-1}(t-s)^n \cos s \, ds \qquad (21) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} t^k \int_0^t s^{\gamma+n-k-1} \cos s \, ds \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} t^k F_0(\gamma + n - k, t) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k \Big(t^{\gamma+n-k-1} \sin t - (\gamma + n - k - 1) G_0(\gamma + n - k - 1, t) \Big) \\ &\quad + t^n F_0(\gamma, t) \qquad \text{(by (16))} \\ &= -t^{\gamma+n-1} \sin t - \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (\gamma - 1 + n - k) G_0(\gamma - 1 + n - k, t) \\ &\quad + t^n F_0(\gamma, t). \end{aligned}$$

Similarly

$$\begin{aligned} G_n(\gamma, t) &= \int_0^t s^{\gamma-1}(t-s)^n \sin s \, ds \\ &= t^{\gamma+n-1} \cos t + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (\gamma - 1 + n - k) F_0(\gamma - 1 + n - k, t) \qquad (22) \\ &\quad + t^n G_0(\gamma, t). \end{aligned}$$

First suppose $0 < \gamma < 1$. Then by Lemma 2

$$t^k F_0(\gamma - 1 + n - k, t) = \begin{cases} O(t^{\gamma+n-2}) \ (t \to \infty) & \text{for } \ 0 \le k \le n-2, \\ O(t^{n-1}) \ (t \to \infty) & \text{for } \ k = n-1; \end{cases}$$

and

$$t^k G_0(\gamma - 1 + n - k, t) = \begin{cases} O(t^{\gamma+n-2}) \ (t \to \infty) & \text{for } \ 0 \le k \le n-2, \\ O(t^{n-1}) \ (t \to \infty) & \text{for } \ k = n-1. \end{cases}$$

Thus (18) follows from (21) and (22).

Next suppose $\gamma = 1$. Then by (21) and (14)

$$\begin{aligned} F_n(1,t) &= -t^n \sin t - \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (n-k) G_0(n-k,t) + t^n F_0(1,t) \\ &= -\sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (n-k) G_0(n-k,t); \end{aligned}$$

and similarly by (22) and (14)

$$G_n(1,t) = t^n + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (n-k) F_0(n-k,t).$$

Here it follows from (14) and (15) that

$$\begin{cases} t^k F_0(n-k,t) = O(t^{n-1}) \ (t \to \infty) \\ t^k G_0(n-k,t) = O(t^{n-1}) \ (t \to \infty) \end{cases} \qquad \text{for all } \ 0 \le k \le n-1,$$

whence (19) follows.

Finally suppose $\gamma > 1$. Then by (21) and (16)

$$\begin{aligned} F_n(\gamma,t) &= -\, t^{\gamma+n-1} \sin t - (\gamma-1) \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k G_0(\gamma - 1 + n - k, t) \\ &\quad + n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} G_0(\gamma - 1 + n - k, t) + t^n F_0(\gamma, t) \\ &= -(\gamma-1) \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} t^k G_0(\gamma - 1 + n - k, t) \\ &\quad + n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} t^k G_0(\gamma + n - 1 - k, t) \\ &= -(\gamma-1) G_n(\gamma - 1, t) + n G_{n-1}(\gamma, t); \end{aligned}$$

and similarly by (22) and (17)

$$G_n(\gamma,t) = (\gamma-1)F_n(\gamma-1,t) - nF_{n-1}(\gamma,t).$$

This proves (20), and hence the proof is complete. □

Lemma 4. *$V_0(\gamma,t)$ and $W_0(\gamma,t)$ satisfy that*

$$\begin{cases} V_0(\gamma,t) = O(t^0)\ (t\to\infty),\ V_0(\gamma,t) \neq o(t^0)\ (t\to\infty), \\ V_0(\gamma,t) \not\sim t^0\ (t\to\infty),\ \text{and} \\ W_0(\gamma,t) = O(t^0)\ (t\to\infty),\ W_0(\gamma,t) \neq o(t^0)\ (t\to\infty), \\ W_0(\gamma,t) \not\sim t^0\ (t\to\infty) \end{cases} \tag{23}$$

for all $0<\gamma<1$*;*

$$V_0(1,t) = \sin t, \quad W_0(1,t) = 1-\cos t\,; \tag{24}$$

$$\begin{cases} V_0(\gamma,t) = O(t^0)\ (t\to\infty),\ V_0(\gamma,t) \neq o(t^0)\ (t\to\infty), \\ V_0(\gamma,t) \not\sim t^0\ (t\to\infty),\ \text{and} \\ W_0(\gamma,t) = t^{\gamma-1}(1+o(1))\ (t\to\infty) \end{cases} \tag{25}$$

for all $1<\gamma<2$*.*

Proof: (23) and (24) follow directly from Lemma 2 together with the definitions of $V_0(\gamma,t)$ and $W_0(\gamma,t)$ (cf. (9), (10)). Suppose $\gamma>1$. Then by (16) and (17)

$$\begin{aligned} V_0(\gamma,t) &= F_0(\gamma,t)\cos t + G_0(\gamma,t)\sin t \\ &= \big(t^{\gamma-1}\sin t - (\gamma-1)G_0(\gamma-1,t)\big)\cos t \\ &\quad + \big(-t^{\gamma-1}\cos t + (\gamma-1)F_0(\gamma-1,t)\big)\sin t \\ &= -(\gamma-1)G_0(\gamma-1,t)\cos t + (\gamma-1)F_0(\gamma-1,t)\sin t \\ &= (\gamma-1)W_0(\gamma-1,t); \end{aligned} \tag{26}$$

and similarly

$$W_0(\gamma,t) = t^{\gamma-1} - (\gamma-1)V_0(\gamma-1,t). \tag{27}$$

Thus (25) follows from (23). This completes the proof. □

Lemma 5. *Let $n\ge 1$. Then $V_n(\gamma,t)$ and $W_n(\gamma,t)$ satisfy that*

$$\begin{cases} V_n(\gamma,t) = t^n(V_0(\gamma,t)+o(1))\ (t\to\infty), \\ W_n(\gamma,t) = t^n(W_0(\gamma,t)+o(1))\ (t\to\infty) \end{cases} \qquad \text{for all}\ \ 0<\gamma<1; \tag{28}$$

$$\begin{cases} V_n(1,t) = t^n(\sin t+o(1))\ (t\to\infty), \\ W_n(1,t) = -t^n(\cos t+o(1))\ (t\to\infty); \end{cases} \tag{29}$$

$$
(30) \qquad \begin{cases} V_n(\gamma,t) = (\gamma-1)W_n(\gamma-1,t) - nW_{n-1}(\gamma,t), \\ W_n(\gamma,t) = -(\gamma-1)V_n(\gamma-1,t) + nV_{n-1}(\gamma,t) \end{cases} \quad \text{for all } \gamma > 1.
$$

Proof: Suppose $0 < \gamma < 1$. Then

$$
\begin{aligned}
V_n(\gamma,t) &= F_n(\gamma,t)\cos t + G_n(\gamma,t)\sin t \\
&= t^n(F_0(\gamma,t)+o(1))\cos t + t^n(G_0(\gamma,t)+o(1))\sin t \qquad \text{(by (18))} \\
&= t^n\Big(F_0(\gamma,t)\cos t + G_0(\gamma,t)\sin t + o(1)\Big) \\
&= t^n(V_0(\gamma,t)+o(1)) \ (t\to\infty);
\end{aligned}
$$

and similarly $W_n(\gamma,t) = t^n(W_0(\gamma,t)+o(1))$ $(t\to\infty)$. (29) follows easily from (9), (10) and (19). Finally suppose $\gamma > 1$. Then by (20)

$$
\begin{aligned}
V_n(\gamma,t) &= F_n(\gamma,t)\cos t + G_n(\gamma,t)\sin t \\
&= \Big(-(\gamma-1)G_n(\gamma-1,t) + nG_{n-1}(\gamma,t)\Big)\cos t \\
&\quad +\Big((\gamma-1)F_n(\gamma-1,t) - nF_{n-1}(\gamma,t)\Big)\sin t \\
&= (\gamma-1)\Big(F_n(\gamma-1,t)\sin t - G_n(\gamma-1,t)\cos t\Big) \\
&\quad -n\Big(F_{n-1}(\gamma,t)\sin t - G_{n-1}(\gamma,t)\cos t\Big) \\
&= (\gamma-1)W_n(\gamma-1,t) - nW_{n-1}(\gamma,t),
\end{aligned}
$$

so that the first half of (30) follows. The second half follows similarly. □

As an immediate consequence of Lemmas 4 and 5 (see especially (30)) we have the following

Lemma 6. *Let $n \ge 1$. Then $V_n(\gamma,t)$ and $W_n(\gamma,t)$ satisfy that*

$$
(31) \qquad \begin{cases} V_n(\gamma,t) = O(t^n)\ (t\to\infty),\ V_n(\gamma,t) \ne o(t^n)\ (t\to\infty), \\ V_n(\gamma,t) \not\sim t^n\ (t\to\infty),\ \text{and} \\ W_n(\gamma,t) = O(t^n)\ (t\to\infty),\ W_n(\gamma,t) \ne o(t^n)\ (t\to\infty), \\ W_n(\gamma,t) \not\sim t^n\ (t\to\infty) \end{cases}
$$

for all $0 < \gamma < 2$.

3. Proofs of the theorems

Proof of Theorem 1: We may assume without loss of generality that $a = 1$. By (1), (4) and Lemma 1, if $0 \le \gamma < 1$, then $\|C_t^\gamma\| \sim t^{k-\gamma}$ $(t\to\infty)$, where $C_t^0 := T(t)$. Further, since $0 \in \rho(A)$, we may apply [1, Theorem 3.3] to infer that $\|C_t^{\gamma+1}\| \sim \|C_t^\gamma - I\|t^{-1}$ $(t\to\infty)$. By this and an induction argument, if $0 \le \gamma < k+1$, then $\|C_t^\gamma\| \sim t^{k-\gamma}$ $(t\to\infty)$. The fact that $\|C_t^{k+1}\| \sim t^{-1}$ $(t\to\infty)$

has been proved in [1, Theorem 3.4]. Next suppose $k+1<\gamma\le k+2$. Then, since $\lim_{t\to\infty}\|C_t^{\gamma-1}\|=0$, the equation

$$AC_t^\gamma=\gamma t^{-1}(C_t^{\gamma-1}-I) \tag{32}$$

(cf. [3]) yields that

$$\lim_{t\to\infty}\frac{t}{\gamma}C_t^\gamma=\lim_{t\to\infty}A^{-1}(C_t^{\gamma-1}-I)=-A^{-1}. \tag{33}$$

This argument can be repeated, and hence (33) holds for all $\gamma>k+1$. Finally

$$\lim_{\lambda\downarrow 0}\lambda^{-1}A_\lambda=\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}T(t)\,dt=\lim_{\lambda\downarrow 0}(\lambda I-A)^{-1}=-A^{-1}.$$

Hence in particular $\|A_\lambda\|\sim\lambda$ $(\lambda\downarrow 0)$, and $\|C_t^\gamma\|\sim t^{-1}$ $(t\to\infty)$ for all $\gamma>k+1$. This completes the proof. □

Proof of Theorem 2: This is similar to the above proof. We may assume that $a=1$. By (2), (11), (12), and Lemmas 4 and 6, if $0\le\gamma<2$, then (a) in Theorem 2 holds. We then use the equation

$$BC_t^\gamma=\frac{\gamma(\gamma-1)}{t^2}(C_t^{\gamma-2}-I)\qquad(\gamma\ge 2,\ t>0), \tag{34}$$

where $C_t^0:=C(t)$ (cf. [3]). Since $0\in\rho(B)=\rho(A^2)$, it follows that $\|C_t^{\gamma+2}\|\sim\|C_t^\gamma-I\|t^{-2}$ $(t\to\infty)$, and so if $0\le\gamma<k+2$, then (a) in Theorem 2 holds. Next, let $x\in X$ be such that $\|x\|=1$ and $Nx=0$. Then, since $\lim_{t\to\infty}C_t^kx=\lim_{t\to\infty}(k/t^k)\big(\int_0^t(t-s)^{k-1}\cos s\,ds\big)x=0$, it follows that $\liminf_{t\to\infty}\|C_t^k-I\|\ge\liminf_{t\to\infty}\|C_t^kx-x\|=\|x\|=1$, whence $\|C_t^{k+2}\|\sim t^{-2}$ $(t\to\infty)$. Now suppose $k+2<\gamma\le k+4$. Then $\lim_{t\to\infty}\|C_t^{\gamma-2}\|=0$, and so

$$\lim_{t\to\infty}\frac{t^2}{\gamma(\gamma-1)}C_t^\gamma=\lim_{t\to\infty}B^{-1}(C_t^{\gamma-2}-I)=-B^{-1}. \tag{35}$$

This argument can be repeated, and hence (35) holds for all $\gamma>k+2$. Finally

$$\lim_{\lambda\downarrow 0}\lambda^{-2}A_\lambda=\lim_{t\to\infty}\lambda^{-1}\int_0^\infty e^{-\lambda t}C(t)\,dt=\lim_{\lambda\downarrow 0}(\lambda^2I-B)^{-1}=-B^{-1}$$

(cf. [4]). Hence in particular $\|A_\lambda\|\sim\lambda^2$ $(\lambda\downarrow 0)$, and $\|C_t^\gamma\|\sim t^{-2}$ $(t\to\infty)$ for all $\gamma>k+2$. This completes the proof. □

Remark. $\lim_{t\to\infty}(t/(k+1))C_t^{k+1}$ does not exist in Theorem 1. To see this we write $AC_t^k=(k/t)(C_t^{k-1}-I)=:(k/t)C_t^{k-1}+D_t^1$, where $\lim_{t\to\infty}\|D_t^1\|=0$, and finally $A^kC_t^k=:(k!/t^k)T(t)+D_t^k$, where $\lim_{t\to\infty}\|D_t^k\|=0$. Since

$$\frac{k!}{t^k}T(t)=k!\,e^{iat}\sum_{n=0}^{k}\frac{t^{n-k}N^n}{n!}=:e^{iat}(N^k+E_t^k),\quad\text{where}\quad\lim_{t\to\infty}\|E_t^k\|=0,$$

it follows that if $N^k x \neq 0$, then $\lim_{t\to\infty} C_t^k x$ does not exist. Hence $\lim_{t\to\infty} (t/(k+1))C_t^{k+1}$ does not exist by (32). Similarly $\lim_{t\to\infty} (t^2/(k+2)(k+1))C_t^{k+2}$ does not exist in Theorem 2.

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