The distribution of the number of nodes in the relative interior of the typical I-segment in homogeneous planar anisotropic STIT Tessellations

Christoph Thäle

Abstract. A result about the distribution of the number of nodes in the relative interior of the typical I-segment in homogeneous and isotropic random tessellations stable under iteration (STIT tessellations) is extended to the anisotropic case using recent findings from Schreiber/Thäle, *Typical geometry, second-order properties and central limit theory for iteration stable tessellations*, arXiv:1001.0990 [math.PR] (2010). Moreover a new expression for the values of this probability distribution is presented in terms of the Gauss hypergeometric function ${}_2F_1$.

Keywords: hypergeometric function, iteration/nesting, random tessellation, segments, stochastic geometry, stochastic stability

Classification: Primary 60D05; Secondary 60G55, 52A22

1. Introduction and result

Since their introduction by Nagel and Weiss in [4], homogeneous (i.e. spatial stationary) random tessellations that are stable under the operation of iteration have attracted considerable interest in stochastic geometry. One of their main features is that they admit an explicit local and global construction that allows an interpretation as a random process of cell division. It can roughly be explained as follows: Let us fix a locally finite measure Λ on the space of lines in the plane $\mathbb{R}^2$ that can be written as $\Lambda = \ell \otimes \mathcal{R}$, where ℓ is the Lebesgue measure on the real axis and $\mathcal{R}$ a probability measure on the space of directions $\mathcal{L}$, i.e. the space of lines through the origin (the non-oriented Grassmannian in the sense of [5]). We use here the standard parameterization as identifying a line with its direction and its signed distance from the origin. We assume from now on that $\mathcal{R}$ is not concentrated on a single direction (line), which ensures that our tessellations will have bounded cells with probability one. Beside Λ, we fix some compact convex polygon $W \subset \mathbb{R}^2$ in which our construction is carried out. Now, assign to W an exponentially distributed random life time with parameter $\Lambda([W])$, where by $[W]$ we mean the collection of all (parameterized) lines in the plane that hit $W \subset \mathbb{R}^2$. Upon expiry of this random lifetime, the cell W dies and splits into two polygonal sub-cells W^+ and W^- separated by a line in $[W]$ that is chosen

This work was supported by the Swiss National Science Foundation, Grant PP002-114715/1.

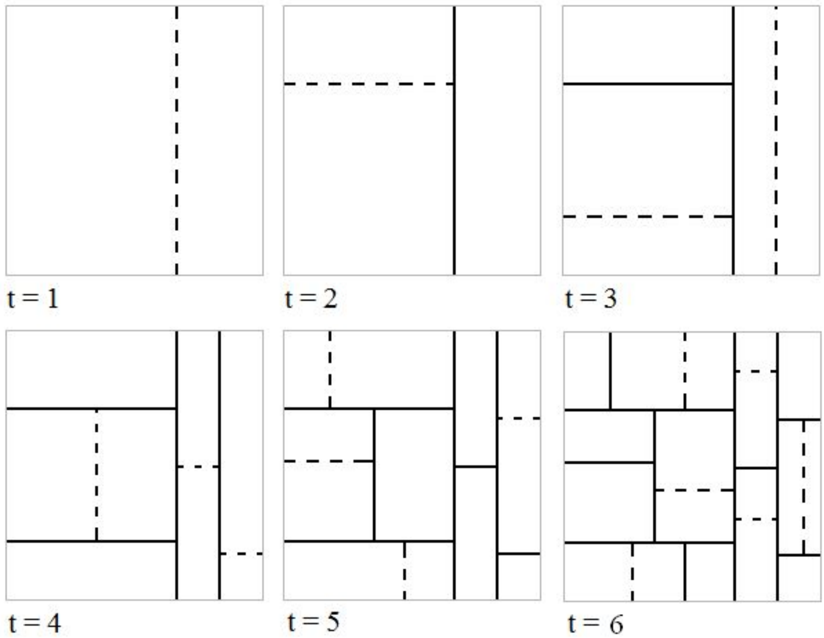

FIGURE 1. States of the random cell division process for different time instants $t \in (0, 6]$; the respective new segments are dashed

according to the law $\Lambda([W])^{-1}\Lambda(\cdot \cap [W])$. The resulting two new cells W^+ and W^- are again provided independently of each other with exponential lifetimes with respective parameters $\Lambda([W^+])$ and $\Lambda([W^-])$ (this ensures that smaller cells live stochastically longer) and the entire construction continues independently and recursively, until some deterministic time threshold $t > 0$ is reached. The cell-separating 1-dimensional faces arising in subsequent cell splits are usually referred to as *I-segments*, assuming shapes similar to the letter *I*. This cell division procedure is illustrated in Figure 1 for different time instants t. The resulting random tessellation $Y(t, \Lambda, W)$ is a homogeneous random tessellation inside W, but it can be shown that its law is consistent in W, which implies the existence of random tessellation $Y(t\Lambda)$ in the whole plane with the property that for any $W \subset \mathbb{R}^d$ we have $Y(t\Lambda) \cap W \overset{D}{=} Y(t, \Lambda, W)$, where $\overset{D}{=}$ stands for equality in distribution, cf. [4]. Realizations of $Y(t\Lambda)$ inside a quadratic window W for fixed t and different choices of Λ are shown in Figure 2. We call the random tessellation $Y(t\Lambda)$ a random STIT tessellation with line measure $t\Lambda$, where the construction time t may be interpreted as the edge length intensity of $Y(t\Lambda)$, i.e. the mean length of edges per unit area. The abbreviation STIT refers to the characteristic property

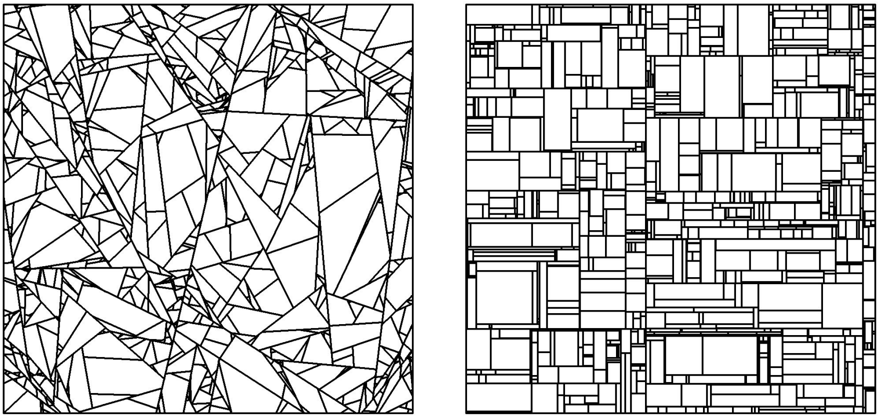

FIGURE 2. Realizations of $Y(t\Lambda)$ for fixed t and different directional distributions $\mathcal{R}$; the uniform distribution for $\mathcal{R}$ (left) and $\mathcal{R}$ concentrated with equal weight on the coordinate directions only (right)

of the tessellations of being **st**able under **it**eration, which formally means that

$$Y(t\Lambda) \boxplus Y(s\Lambda) \stackrel{D}{=} Y((s+t)\Lambda), \quad s,t>0,$$

where $\boxplus$ denotes the operation of iteration, see [4]. The line measure $t\Lambda$ is sometimes referred to as the generating or the driving measure of the homogeneous random STIT tessellation $Y(t\Lambda)$.

We are concerned in this note with the typical I-segment of a homogeneous random STIT tessellation $Y(t\Lambda)$ with driving measure $t\Lambda$, that is the almost surely uniquely determined I-segment of the tessellation that contains the origin when $Y(t\Lambda)$ is regarded under the Palm distribution with respect to the embedded point process of I-segment midpoints (see [5] for the technical background). Moreover, we can mark each of the midpoints by the direction of the I-segment through the particular point, which opens the door to the theory of Palm distributions for marked point processes, see e.g. [5, Chapter 3.5]. This technique allows us to make mathematically precise the notion of the typical I-segment of the tessellation with a particular direction l in the support of $\mathcal{R}$.

It is our main objective here to extend a recent result from [3] concerning the probability distribution of the number of nodes in the relative interior of the typical I-segment obtained for homogeneous *and* isotropic (stochastically rotation invariant) random STIT tessellations to the anisotropic case, i.e. to general driving measures $t\Lambda$ satisfying our above formulated assumption. Our result reads as follows:

Theorem. *The probability $p(k)$ that the typical I-segment with direction $l \in \mathcal{L}$ of a homogeneous planar random STIT tessellation $Y(t\Lambda)$ has $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ nodes in its relative interior does not depend on l and t and its precise value is given by*

$$p(k) = \frac{2^{k+2}}{(k+1)(k+2)(k+3)}\ {}_2F_1(k+1, k+1; k+4; -1),$$

where ${}_2F_1$ is the Gauss hypergeometric function (*see* [1, Chapter 15] *or* (11) *below*).

Some particular values of $p(k)$ for small k are provided in the table.

	$k = 0$	$k = 1$	$k = 2$
exact	$8 \ln 2 - 5$	$32 \ln 2 - 22$	$104 \ln 2 - 72$
numerical	0.545178	0.180710	0.087307
	$k = 3$	$k = 4$	$k = 5$
exact	$304 \ln 2 - \frac{632}{3}$	$832 \ln 2 - \frac{1730}{3}$	$2176 \ln 2 - \frac{22624}{15}$
numerical	0.050076	0.031788	0.021598

From the formula for $p(k)$ it can be calculated that the mean number of points in the relative interior of the typical I-segment and of the typical I-segment with direction l equals 2. This equality was already observed in [7] and was a first indication for the distributional equality provided by the Theorem. It can further be shown that no higher moments exist.

The proof we give here will be based on recent findings from [6] obtained in the framework of martingale theory, which fills the gap from [3] of a missing Slivnyak-type result for I-segments. In fact, the martingale techniques from [6] yield a Slivnyak theorem for I-segments that holds in construction time t of the tessellation. This result plays an important rôle in the background of the proof given below.

As a byproduct, we are also able to provide an alternative formula to that from [3] for the probabilities $p(k)$ in terms of the Gauss hypergeometric function ${}_2F_1$.

Our methods provide information on anisotropic random STIT tessellations, namely that the probability distribution of the number of nodes in the relative interior of the typical I-segment in direction l is the same for all directions. This might seem implausible at first sight if one thinks for example of a directional distribution $\mathcal{R}$ that is concentrated on two different directions l_1 and l_2, l_1 with weight $\epsilon > 0$ and l_2 with weight $1 - \epsilon$. However, intuitively one could argue as follows: I-segments with direction l_1 are rare and short. But they are hit by a large number of I-segments with direction l_2. On the other hand, I-segments with direction l_2 appear more often and they are comparatively long. But they are seldom hit by I-segments with direction l_1. Surprisingly, these two effects are balancing and lead to the independence of the crucial distribution from the direction of the segment. Another way to make the result plausible is to say that the distribution of the number of nodes in the relative interior of the typical I-segment in direction l cannot depend on its length, because in contrast to the length, the number of nodes does not change under dilations of the tessellation. But the length depends on the direction, so the number of nodes and the direction could be independent, too.

2. Proof of the Theorem

Let $Y(t\Lambda)$ be a homogeneous random STIT with driving measure $t\Lambda$, $t > 0$, where Λ has the product structure $\Lambda = \ell \otimes \mathcal{R}$ with $\mathcal{R}$ as above. Denote the random segment process of I-segments of $Y(t\Lambda)$ by X_I and for each I-segment $s \in X_I$, by $\beta(s) \in (0, t]$ its birth time that it gets in the spatio-temporal construction of the tessellation. We consider now the birth time augmented process

$$\hat{X}_I := \{(s, \beta(s)) : s \in X_I\}$$

and denote by $\hat{\mathbb{Q}}^{Y(t\Lambda)}$ the distribution of the birth-time marked typical I-segment in the sense of [5, Chapter 4.1]. From the general theory presented in [5, Chapter 3] we deduce that there exists a regular family of distributions $\hat{\mathbb{Q}}_l^{Y(t\Lambda)}$, $l \in \mathcal{L}$, such that

$$\hat{\mathbb{Q}}^{Y(t\Lambda)} = \int_{\mathcal{L}} \hat{\mathbb{Q}}_l^{Y(t\Lambda)} \mathcal{R}(dl). \tag{1}$$

We may interpret a random segment with distribution $\hat{\mathbb{Q}}_l^{Y(t\Lambda)}$ as the typical birth-time marked I-segment of $Y(t\Lambda)$ with direction l.

We regard now a homogeneous Poisson line tessellation $\mathrm{PLT}(s\Lambda)$ in the plane with line measure $s\Lambda$ and $0 < s < t$. Then we can consider — similarly to what we did with the I-segments above — the distribution $\mathbb{Q}^{\mathrm{PLT}(s\Lambda)}$ of the typical edge of the Poisson line tessellation and the family $(\mathbb{Q}_l^{\mathrm{PLT}(s\Lambda)})_{l \in \mathcal{L}}$ of distributions of random segments that may be interpreted as typical edges of $\mathrm{PLT}(s\Lambda)$ with direction l.

Having these concepts in mind, we recall Equation (33) from [6], which states that the distribution $\hat{\mathbb{Q}}^{Y(t\Lambda)}$ of the typical birth-time marked I-segment of $Y(t\Lambda)$ is a mixture of distributions $\hat{\mathbb{Q}}^{\mathrm{PLT}(s\Lambda)}$, $0 < s < t$, and the mixing distribution is a beta-distribution on $[0, t]$ with parameters 2 and 1. More formally, we have

$$\hat{\mathbb{Q}}^{Y(t\Lambda)} = \int_0^t \frac{2s}{t^2} [\mathbb{Q}^{\mathrm{PLT}(s\Lambda)} \otimes \delta_s] \, ds,$$

where δ_s stands for the unit mass Dirac measure concentrated at $s \in (0, t]$. Using now (1) and its equivalent for the Poisson line tessellations, we arrive at

$$\begin{aligned} &\int_{\mathcal{L}} \int f(Z) \hat{\mathbb{Q}}_l^{Y(t\Lambda)}(dZ) \mathcal{R}(dl) \\ &= \int_{\mathcal{L}} \int_0^t \frac{2s}{t^2} \int f(Z) (\mathbb{Q}_l^{\mathrm{PLT}(s\Lambda)} \otimes \delta_s)(dZ) ds \mathcal{R}(dl), \end{aligned} \tag{2}$$

for any non-negative measurable function f on the space of birth-time marked line segments.

Above, we have constructed from the distribution $\mathbb{Q}^{\mathrm{PLT}(s\Lambda)}$ a family of distributions $\mathbb{Q}_l^{\mathrm{PLT}(s\Lambda)}$. Similarly to that, we can infer once again from the general theory that for each $\mathbb{Q}_l^{\mathrm{PLT}(s\Lambda)}$ there exists a regular family $\mathbb{Q}_{l,x}^{\mathrm{PLT}(s\Lambda)}$, such that

$$\mathbb{Q}_l^{\mathrm{PLT}(s\Lambda)} = \int_{(0,\infty)} \mathbb{Q}_{l,x}^{\mathrm{PLT}(s\Lambda)} \lambda_{l,s}(dx),$$

where by $\lambda_{l,s}$ we mean the length distribution of a random line segment with distribution $\mathbb{Q}_l^{\mathrm{PLT}(s\Lambda)}$. Consequently, (2) can be written as

$$\begin{aligned} &\int_{\mathcal{L}} \int f(Z) \hat{\mathbb{Q}}_l^{Y(t\Lambda)}(dZ) \mathcal{R}(dl) \\ &= \int_{\mathcal{L}} \int_0^t \frac{2s}{t^2} \int_{(0,\infty)} \int f(Z) (\mathbb{Q}_{l,x}^{\mathrm{PLT}(s\Lambda)} \otimes \delta_s)(dZ) \lambda_{l,s}(dx)\, ds \mathcal{R}(dl). \end{aligned} \tag{3}$$

However, it is well known that $\lambda_{l,s}$ has a density $p_{l,s}(x)$ with respect to the Lebesgue measure on the positive real half-axis (see the remarks in [5] or equation (6) below). To simplify the notations, we denote by I_l a random birth-time marked segment with distribution $\hat{\mathbb{Q}}_l^{Y(t\Lambda)}$ and by $Z_{l,s,x}$ the random marked line segment with distribution $\mathbb{Q}_{l,x}^{\mathrm{PLT}(s\Lambda)} \otimes \delta_s$. Then (3) can be rewritten as

$$\int_{\mathcal{L}} \mathbb{E} f(I_l) \mathcal{R}(dl) = \int_{\mathcal{L}} \int_0^t \frac{2s}{t^2} \int_0^\infty p_{l,s}(x) \cdot \mathbb{E} f(Z_{l,s,x})\, dx\, ds \mathcal{R}(dl). \tag{4}$$

We define now the following special function F on the space of birth-time marked line segments by

$$\begin{aligned} F(Z) := \mathbf{1}\ &[\text{in the relative interior of } Z \text{ there appear} \\ &\quad \text{after its birth time and until time } t \text{ exactly } k \text{ nodes}] \end{aligned}$$

with $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Furthermore, we introduce the abbreviations

$$p(k|l) := \mathbb{E} F(I_l) \quad \text{and} \quad p(Z_{l,s,x}, k) := \mathbb{E} F(Z_{l,s,x}).$$

Then (4) with f replaced by $F \cdot h(l)$ with an arbitrary non-negative measurable function $h(l)$ on $\mathcal{L}$ reads

$$\int_{\mathcal{L}} h(l)p(k|l)\mathcal{R}(dl) = \int_{\mathcal{L}} h(l) \int_0^t \frac{2s}{t^2} \int_0^\infty p_{s,l}(x)p(Z_{l,s,x},k)\,dx\,ds\mathcal{R}(dl),$$

whence

$$p(k|l) = \int_0^t \frac{2s}{t^2} \int_0^\infty p_{s,l}(x)p(Z_{l,s,x},k)\,dx\,ds \tag{5}$$

follows.

In order to calculate $p(k|l)$, it remains to determine the length density $p_{l,s}(x)$ with respect to the Lebesgue measure on the positive real half-axis and the counting density $p(Z_{l,s,x},k)$.

To this end, notice at first that it is well known from the theory of Poisson line tessellations that the length distribution of $Z_{l,s}$ is an exponential distribution with parameter $s\Lambda([e(l)])$, where $e(l)$ denotes a segment of unit length on l (see for example the remarks on Gamma-type results for Poisson based random tessellation in [5, Chapter 10] or [6] and the references given therein). Thus, we have

$$p_{l,s}(x) = s\Lambda([e(l)])e^{-s\Lambda([e(l)])x}. \tag{6}$$

It remains to determine the counting density $p(Z_{l,s,x},k)$. To this end, we make the following considerations: Note, that at time s, the random segment $Z_{l,s}$ has by definition the same distribution as the typical marked I-segment with direction l that has birth-time s. Furthermore, at the time of its birth $s \in (0,t]$, such an I-segment does not contain any node in its relative interior due to the construction, whence the nodes can only appear during the remaining time interval $(s,t]$. Recall now that homogeneous random STIT tessellations have the following section property (cf. [2] and the references cited therein): The intersection of $Y(t\Lambda)$ with any line g induces a homogeneous Poisson point process on g with intensity $t\Lambda([e(l)])$, where $l \in \mathcal{L}$ is the unique line parallel to g. Moreover, observe that from the iteration stability of $Y(t\Lambda)$ it follows

$$Y(t\Lambda) \overset{D}{=} Y(s\Lambda) \boxplus Y((t-s)\Lambda). \tag{7}$$

The last equation says that both results are the same, either to continue the random cell division process from time s until t or to stop the process at time s and to perform an iteration of $Y(s\Lambda)$ with tessellations distributed as $Y((t-s)\Lambda)$. We consider the second mentioned possibility and regard the effect of the iteration (7) on each of two the sides of the I-segment $Z_{l,s}$. On each side of $Z_{l,s}$, an iteration is performed and it induces on the line containing the segment a homogeneous Poisson point process with intensity $(t-s)\Lambda([e(l)])$, because of the section property for STIT tessellations mentioned afore. This is because iteration with $Y((t-s)\Lambda)$ is for the considered line the same as intersection with the STIT tessellation $Y((t-s)\Lambda)$. The same does independently also hold for the other side of the segment. As a result, there appears on the line containing the I-segment $Z_{l,s}$ the superposition of two independent and homogeneous Poisson point processes, which is again a homogeneous Poisson point process but with intensity $2(t-s)\Lambda([e(l)])$. Thus, in the relative interior of the random segment $Z_{l,s,x}$, which is born without an inner structure, there appears a random number of nodes that is Poisson distributed with parameter $2(t-s)\Lambda([e(l)])x$, i.e.

$$p(Z_{l,s,x},k) = \frac{(2(t-s)\Lambda([e(l)])x)^k}{k!} e^{-2(t-s)\Lambda([e(l)])x}, \quad k \in \mathbb{N}, \tag{8}$$

because we have conditioned additionally on its length $x \in (0,\infty)$. Thus, we obtain from (5) in view of (6) and (8) the following identity for the probability $p(k|l)$:

$$\begin{aligned} p(k|l) = \int_0^t \frac{2s}{t^2} \int_0^\infty & s\Lambda([e(l)]) e^{-s\Lambda([e(l)])x} \\ \times & \frac{(2(t-s)\Lambda([e(l)])x)^k}{k!} e^{-2(t-s)\Lambda([e(l)])x}\, dx\, ds. \end{aligned}$$

The last expression can further be evaluated and we obtain

$$\begin{aligned} p(k|l) &= \int_0^t \frac{2s}{t^2} \frac{(2(t-s)\Lambda([e(l)]))^k}{k!} s\Lambda([e(l)]) \int_0^\infty x^k e^{-\Lambda([e(l)])(2t-s)x}\, dx\, ds \\ &= \int_0^t \frac{2s}{t^2} \frac{(2(t-s)\Lambda([e(l)]))^k}{k!} s\Lambda([e(l)]) \cdot \frac{k!}{(\Lambda([e(l)])(2t-s))^{k+1}}\, ds \\ &= \int_0^t \frac{2s^2}{t^2} \frac{(2(t-s))^k}{(2t-s)^{k+1}} \frac{\Lambda([e(l)])\Lambda([e(l)])^k}{\Lambda([e(l)])^{k+1}}\, ds \\ &= \int_0^t \frac{2s^2}{t^2} \frac{(2(t-s))^k}{(2t-s)^{k+1}}\, ds \end{aligned} \tag{9}$$

which is *independent* of the direction l. This implies that the probability $p(k|l)$ for the typical I-segment with a particular direction l and $\int_{\mathcal{L}} p(k|l)\mathcal{R}(dl)$ for the typical I-segment itself coincide. For this reason we will write from now on $p(k)$ instead of $p(k|l)$.

To proceed, we apply the substitution $a = 1 - s/t$, which leads to

$$\begin{aligned} (9) &= \int_0^1 \frac{2((1-a)t)^2}{t^2} \frac{(2(t-(1-a)t))^k}{(2t-(1-a)t)^{k+1}} t\, da \\ &= 2^{k+1} \int_0^1 (1-a)^2 \frac{a^k}{(1+a)^{k+1}} da = p(k), \end{aligned} \tag{10}$$

which is *independent* of t.

We can derive from the last formula an alternative expression for the values $p(k)$, i.e. for the probability that the number of nodes in the relative interior of the typical I-segment equals $k \in \mathbb{N}$. Recall for that purpose that the Gauss hypergeometric function ${}_2F_1(u, v; w; z)$, $u, v, w > 0$, $z \in \mathbb{R}$ is defined by

$$
\begin{aligned}
{}_2F_1(u, v; w; z) &= \sum_{n=0}^{\infty} \frac{(u)_n (v)_n}{(w)_n} \cdot \frac{z^n}{n!} \\
&= \frac{\Gamma(w)}{\Gamma(u)\Gamma(v)} \sum_{n=0}^{\infty} \frac{\Gamma(u+n)\Gamma(v+n)}{\Gamma(w+n)} \cdot \frac{z^n}{n!},
\end{aligned}
\tag{11}
$$

where $(x)_n$, $n \in \mathbb{N}$, is the Pochhammer symbol

$$
(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}.
$$

From [1, Equation (15.3.1)] we know that this function has the following integral representation (the so-called Euler integral):

$$
{}_2F_1(u, v; w; z) = \frac{\Gamma(w)}{\Gamma(v)\Gamma(w-v)} \int_0^1 \frac{da}{a^{1-v}(1-a)^{1-w+v}(1-za)^u}.
$$

Using the last formula with the parameters $u = v = k+1 > 0$, $w = k+4 > 0$, $k \in \mathbb{N}$, and $z = -1$ yields

$$
\begin{aligned}
&{}_2F_1(k+1, k+1; k+4; -1) \\
&\quad = \frac{\Gamma(k+4)}{\Gamma(k+1)\Gamma(3)} \int_0^1 \frac{da}{a^{-k}(1-a)^{-2}(1+a)^{k+1}} \\
&\quad = \frac{(k+1)(k+2)(k+3)}{2} \int_0^1 (1-a)^2 \frac{a^k}{(1+a)^{k+1}}\, da.
\end{aligned}
$$

A comparison of this expression and the expression for the probabilities $p(k)$ from (10) implies now

$$
\begin{aligned}
p(k) &= 2^{k+1} \int_0^1 (1-a)^2 \frac{a^k}{(1+a)^{1+k}}\, da \\
&= \frac{2^{k+2}}{(k+1)(k+2)(k+3)}\; {}_2F_1(k+1, k+1; k+4; -1)
\end{aligned}
$$

and completes the proof. $\square$

Acknowledgement. The author highly values the comments and remarks by Werner Nagel (Jena) who considerably helped to improve the presentation.

References

[1] Abramowitz M., Stegun I.A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965, online version under `http://www.math.ucla.edu/~cbm/aands/index.htm`.

[2] Mecke J., Nagel W., Weiss V., *Length distributions of edges in planar stationary and isotropic STIT tessellations*, J. Contemp. Math. Anal. **42** (2007), 28–43.

[3] Mecke J., Nagel W., Weiss V., *Some distributions for I-segments of planar random homogeneous STIT tessellations*, Math. Nachr. (2010), to appear.

[4] Nagel W., Weiss V., *Crack STIT tessellations: characterization of stationary random tessellations stable with respect to iteration*, Adv. in Appl. Probab. **37** (2005), 859–883.

[5] Schneider R., Weil W., *Stochastic and Integral Geometry*, Springer, Berlin, 2008.

[6] Schreiber T., Thäle C., *Typical geometry, second-order properties and central limit theory for iteration stable tessellations*, arXiv:1001.0990 [math.PR] (2010).

[7] Thäle C., *Moments of the length of line segments in homogeneous planar STIT tessellations*, Image Anal. Stereol. **28** (2009), 69–76.

University of Fribourg, Mathématiques, Chemin de Musée 23,
CH–1700 Fribourg, Switzerland

E-mail: christoph.thaele@unifr.ch

(*Received* April 1, 2009*, revised* February 24, 2010)