A note on propagation of singularities of semiconcave functions of two variables

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Abstract. P. Albano and P. Cannarsa proved in 1999 that, under some applicable conditions, singularities of semiconcave functions in $\mathbb{R}^n$ propagate along Lipschitz arcs. Further regularity properties of these arcs were proved by P. Cannarsa and Y. Yu in 2009. We prove that, for $n = 2$, these arcs are very regular: they can be found in the form (in a suitable Cartesian coordinate system) $\psi(x) = (x, y_1(x) - y_2(x))$, $x \in [0, \alpha]$, where y_1, y_2 are convex and Lipschitz on $[0, \alpha]$. In other words: singularities propagate along arcs with finite turn.

Keywords: semiconcave functions, singularities

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1. Introduction

Let u be a function defined on an open set $\Omega \subset \mathbb{R}^n$ which is locally (linearly) semiconcave; i.e., u is locally representable in the form $u(x) = g(x) + K\|x\|^2$, where g is concave (cf. [3]).

Let $\Sigma(u)$ be the singular set of u, i.e.

$$\Sigma(u) = \{x \in \Omega : \ u \ \text{ is not differentiable at } \ x\}.$$

It is clear that in many questions concerning $\Sigma(u)$ we can suppose that u is concave (or convex), since the results for semiconcave functions then easily follow. But it is reasonable to formulate theorems for semiconcave functions, since these functions are important in a number of applications (see [3]).

It is well-known that $\Sigma(u)$ is a rather small set: it can be covered by countably many Lipschitz DC hypersurfaces ([12]). (Note that for $A \subset \mathbb{R}^n$ there exists a convex (resp. semiconcave) function u on $\mathbb{R}^n$ such that $A = \Sigma(u)$, if and only if A is an F_σ set which can be covered by countably many Lipschitz DC hypersurfaces, see [8].)

The set $\Sigma(u)$ can have isolated points, but P. Albano and P. Cannarsa [1] found applicable conditions ensuring that $\Sigma(u)$ is in a sense big in each neighbourhood of a given $x_0 \in \Sigma(u)$. (The results of [1] can be found also in the book [3].) In particular, they proved that if $\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset$ (see Preliminaries for the

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definitions), then a Lipschitz arc $\xi : [0,\tau] \to \Omega$ emanating from x_0 is a subset of the singular set $\Sigma(u)$. The results of [1] were refined in [5]; in particular it is proved in [5, Corollary 4.3] that ξ has nonzero (right continuous) right derivative at all points.

The purpose of the present note is to show that in $\mathbb{R}^2$ the results of [5] and methods from [12] and [10] easily imply that the restriction of ξ to an interval $[0,\tau']$ has an equivalent parametrization of the form (in a suitable Cartesian coordinate system) $\psi(x) = (x, y_1(x) - y_2(x))$, $x \in [0,\alpha]$, where y_1, y_2 are convex and Lipschitz on $[0,\alpha]$. (This result is equivalent to the assertion that the restriction of ξ to an interval $[0,\tau^*]$ has finite turn, cf. Remark 3.3). In particular, ξ has (left continuous) left halftangents at all points.

The question whether the results can be generalized to the case $n > 2$ remains open.

2. Preliminaries

By $B(x,r)$ we denote the open ball with center x and radius r. The scalar product of $v, w \in \mathbb{R}^n$ is denoted by $\langle v, w\rangle$. If $A \subset \mathbb{R}^n$, $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then we define the sets $A + v$ and cA by the usual way and similarly set $\langle v, A\rangle := \{\langle v, a\rangle : a \in A\}$. The boundary and the convex hull of a set $A \subset \mathbb{R}^n$ are denoted by ∂A and $\operatorname{conv} A$, respectively. The (Fréchet) derivative $Df(a)$ of a function f on $\mathbb{R}^n$ at $a \in \mathbb{R}^n$ is considered as an element of $\mathbb{R}^n$. The one-sided derivatives of a real or vector function ξ of one variable at $x \in \mathbb{R}$ are denoted by $\xi'_+(x)$ and $\xi'_-(x)$.

If f is a function defined on a subset of $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then we define the one-sided directional derivative as

$$f'_+(x,v) := \lim_{h\to 0+} \frac{f(x+hv) - f(x)}{h}\,.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and u a locally semiconcave function on Ω (see Introduction). Then u is locally Lipschitz and so differentiable a.e. in Ω. For $x \in \Omega$, we define (see [1] or [3, p. 54]) the set

$$D^*u(x) = \{p \in \mathbb{R}^n :\ \Omega \ni x_i \to x,\ Du(x_i) \to p\}$$

of all *reachable gradients* of u at x (note that $D^*u(x)$ is also called limiting subdifferential, cf. [1, p. 725]).

The *superdifferential* $D^+u(x)$ of u at x can be defined as the convex hull of $D^*u(x)$ (see [1, p. 723], cf. [3, Theorem 3.3.6]).

Always $D^*u(x) \subset \partial D^+u(x)$ (see [3, Proposition 3.3.4]). Note that the superdifferential $D^+u(x) = \operatorname{conv} D^*u(x)$ coincides with the Clarke's subdifferential $\partial^C u(x)$ (since $\partial^C u(x) = \operatorname{conv} D^*u(x)$, see, e.g., [4]).

Let $u(x) = g(x) + K\|x\|^2$, where g is concave, on a ball $B(x_0,\delta) \subset \Omega$. Set $f := -g$. Since $D(K\|x\|^2) = 2Kx$, we easily obtain that $D^*u(x_0) = -D^*f(x_0) + 2Kx_0$, and therefore

$$D^+u(x_0) = -\partial f(x_0) + 2Kx_0, \tag{2.1}$$

where ∂f is the classical subdifferential of the convex function f.

Recall that a function defined on an open convex subset of $\mathbb{R}^n$ is a *DC function* if it is a difference of two convex functions. We will need the following simple lemma which is a special case of the "mixing lemma" [10, Lemma 4.8].

Lemma 2.1. *Let $\varphi_1, \dots, \varphi_p$ be DC functions on $\mathbb{R}$, and let h be a continuous function on $\mathbb{R}$ such that*

$$h(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\} \quad \text{for each} \quad x \in \mathbb{R}.$$

Then h is DC on $\mathbb{R}$.

We will need also the well-known fact that convex functions are semismooth (see [7, Proposition 3], cf. also [9, Proposition 2.3]). In other words:

Lemma 2.2. *Let f be a convex function on an open convex set $C \subset \mathbb{R}^n$ and $x_0 \in C$. Let $0 \neq q \in \mathbb{R}^n$, $q_n \to q$, $t_n \searrow 0$, and $z_n \in \partial f(x_n)$, where $x_n := x_0 + t_n q_n$, be given. Then $\langle q, z_n \rangle \to f'_+(x_0, q)$. In particular,*

$$\operatorname{diam}\langle q, \partial f(x_n)\rangle \to 0. \tag{2.2}$$

3. The result and its proof

The following result is an immediate consequence of [5, Corollary 4.3].

Theorem CY. *Let u be a semiconcave function on an open set $\Omega \subset \mathbb{R}^n$, $x_0 \in \Sigma(u)$ be a singular point of u and*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset.$$

Then there exist $q \in \mathbb{R}^n$ with $\|q\| = 1$, $\tau > 0$, and a Lipschitz curve $\xi : [0, \tau] \to \Sigma(u)$ such that

(i) $\xi'_+(0) = q$,
(ii) $\lim_{s\to 0+} \xi'_+(s) = q$, *and*
(iii) $\inf_{s\in[0,\tau]} \operatorname{diam} D^+u(\xi(s)) > 0$.

Note that it is proved in [5] also that $\xi'_+(s)$ exists for each $s \in [0, \tau)$ and ξ'_+ is right continuous on $[0, \tau)$. Further note that the result without (ii) was proved already in [1].

Using Theorem CY and the method of the proof of the implicit function theorem for DC functions [10, Theorem 4.4], we easily prove the following result.

Theorem 3.1. *Let u be a semiconcave function on an open set $\Omega \subset \mathbb{R}^2$, $x_0 \in \Sigma(u)$ be a singular point of u and*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset.$$

Then there exist a Cartesian coordinate system in $\mathbb{R}^2$ given by an isometry $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A(x_0) = (0, 0)$, and convex Lipschitz functions y_1, y_2 on some $[0, \alpha]$ ($\alpha > 0$) such that, denoting $\psi(x) := (x, y_1(x) - y_2(x))$, $x \in [0, \alpha]$, we have $\psi(0) = (0, 0)$ and $A^{-1}(\psi([0, \alpha])) \subset \Sigma(u)$.

Proof: Let $\xi : [0,\tau] \to \Sigma(u)$ and $q \in \mathbb{R}^2$ have properties from Theorem CY. We will proceed in four steps. In Steps 1–3 we will suppose that

$$x_0 = (0,0) \ \text{ and } \ q = (1,0). \tag{3.1}$$

Step 1. Set $e_2 := (0,1)$. Let $u(x) = g(x) + K\|x\|^2$ for $x \in B(x_0,\delta) \subset \Omega$, where g is concave and Lipschitz with a constant $L > 0$ on $B(x_0,\delta)$. Set $f := -g$. Applying (2.1) to any point $x \in B(x_0,\delta)$, we obtain $D^+u(x) = -\partial f(x) + 2Kx$, $x \in B(x_0,\delta)$. So (iii) (of Theorem CY) easily implies that, for some $0 < \tau_1 < \tau$, we have that $f(\xi(s)) \in B(x_0,\delta)$ and $\partial f(\xi(s)) \subset B(0,L)$ for each $s \in [0,\tau_1]$, and

$$\inf_{s\in[0,\tau_1]} \ \operatorname{diam} \partial f(\xi(s)) \ > 0. \tag{3.2}$$

We will show that there exists $0 < \tau_2 < \tau_1$ such that

$$\delta := \inf_{s\in(0,\tau_2]} \ \operatorname{diam} \langle e_2, \partial f(\xi(s))\rangle \ > 0. \tag{3.3}$$

Suppose on the contrary that there exits a sequence (t_n) such that $t_n \searrow 0$ and

$$\lim_{n\to\infty} \operatorname{diam} \langle e_2, \partial f(\xi(t_n))\rangle = 0. \tag{3.4}$$

Set $q_n := \xi(t_n)/t_n$ and $x_n := \xi(t_n) = t_n q_n$. Since $q_n \to q$ by (i), Lemma 2.2 gives that

$$\lim_{n\to\infty} \operatorname{diam} \langle q, \partial f(\xi(t_n))\rangle = 0. \tag{3.5}$$

Since (3.4) and (3.5) clearly imply $\lim_{n\to\infty} \operatorname{diam} \partial f(\xi(t_n)) = 0$, we obtain a contradiction with (3.2).

Step 2. Let $\xi = (\xi_1, \xi_2)$. By (ii), we have $\lim_{s\to 0+} (\xi_1)'_+(s) = 1$ and therefore there exits $0 < \tau_3 < \tau_2$ such that $1/2 \le (\xi_1)'(s)$ for a.e. $s \in (0,\tau_3)$. So ξ_1 is Lipschitz strictly increasing on $[0,\tau_3]$ and $(\xi_1)^{-1}$ is Lipschitz on $[0,\alpha]$, where $\alpha := \xi_1(\tau_3)$. Set $d(x) := \xi_2 \circ (\xi_1)^{-1}(x)$, $x \in [0,\alpha]$. Then d is Lipschitz and $\psi(x) := (x, d(x))$, $x \in [0,\alpha]$, is an equivalent parametrization of $\xi|_{[0,\tau_3]}$.

Step 3. Choose a partition $\{-L = y_0 < y_1 < \cdots < y_p = L\}$ of the interval $[-L,L]$ such that $\max\{y_i - y_{i-1},\ i = 1,\dots,p\} < \delta/2$. For each $x \in (0,\alpha)$, the set $\langle e_2, \partial f(\psi(x))\rangle \subset [-L,L]$ is a closed interval of length at least δ and so we can choose $i_x \in \{1,\dots,p\}$ such that

$$y_{i_x} \in \langle e_2, \partial f(\psi(x))\rangle \quad \text{and} \quad y_{i_x-1} \in \langle e_2, \partial f(\psi(x))\rangle. \tag{3.6}$$

For $i \in \{1,\dots,p\}$, set $A_i := \{x \in (0,\alpha) : i_x = i\}$. We will show that, for each $i \in \{1,\dots,p\}$ with $A_i \neq \emptyset$, the function $d|_{A_i}$ can be extended to a Lipschitz DC function φ_i on $\mathbb{R}$.

To this end, fix a such i and set

$$\omega_1(x) := f(x, d(x)) - y_i d(x) \ \text{ and } \ \omega_2(x) := f(x, d(x)) - y_{i-1} d(x) \ \text{ for } \ x \in A_i.$$

Since $\omega_1(x) - \omega_2(x) = (y_{i-1} - y_i)d(x)$, $x \in A_i$, it is sufficient to prove that ω_j $(j = 1, 2)$ can be extended to a Lipschitz convex function c_j defined on $\mathbb{R}$.

For each $x \in A_i$, choose $p_x \in \mathbb{R}$ such that $(p_x, y_i) \in \partial f(x, d(x))$ and consider the affine function

$$a_x(t) := \omega_1(x) + p_x(t - x), \quad t \in \mathbb{R}.$$

Set

$$c_1(t) := \sup\{a_x(t) : \ x \in A_i\}, \quad t \in \mathbb{R}.$$

Since ω_1 is clearly bounded on A_i and $|p_x| \leq L$ for $x \in A_i$, it is easy to see that c_1 is a Lipschitz convex function on $\mathbb{R}$.

Now consider arbitrary $x, t \in A_i$, $x \neq t$. Since $(p_x, y_i) \in \partial f(x, d(x))$, we have

$$f(t, d(t)) - f(x, d(x)) \geq p_x(t - x) + y_i(d(t) - d(x)),$$

and therefore

$$\omega_1(t) = f(t, d(t)) - y_i d(t) \geq f(x, d(x)) - y_i d(x) + p_x(t - x) = a_x(t).$$

Since $a_t(t) = \omega_1(t)$, $t \in A_i$, we obtain that c_1 extends ω_1. Quite similarly we can find a convex Lipschitz extension c_2 of ω_2.

Since $d(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\}$ for each $x \in (0, \alpha)$, and d, $\varphi_1, \dots, \varphi_p$ are continuous on $[0, \alpha]$, we can clearly find $i_0, i_\alpha \in \{1, \dots, p\}$ such that $d(0) = \varphi_{i_0}(0)$ and $d(\alpha) = \varphi_{i_\alpha}(\alpha)$.

Let h be the extension of d with $h(x) = \varphi_{i_0}(x)$, $x < 0$ and $h(x) = \varphi_{i_\alpha}(x), x > \alpha$. Then h is continuous on $\mathbb{R}$ and $h(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\}$ for each $x \in \mathbb{R}$. Thus Lemma 2.1 implies that h is DC on $\mathbb{R}$, i.e., $h = \gamma_1 - \gamma_2$, where γ_1 and γ_2 are convex on $\mathbb{R}$. Then $y_j := \gamma_j|_{[0,\alpha]}$, $j = 1, 2$, are clearly convex Lipschitz functions, and $\psi(x) = (x, y_1(x) - y_2(x))$, $x \in [0, \alpha]$.

Step 4. If (3.1) does not hold, we can choose a Cartesian system of coordinates given by an isometry $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A(x_0) = (0, 0)$ and $A(q) = (1, 0)$. Applying steps 1-3 to $u^* := u \circ A^{-1}$ and $\xi^* := A \circ \xi$, we obtain ψ of the demanded form with $\psi([0, \alpha]) \subset \Sigma(u^*) = A(\Sigma(u))$. □

Remark 3.2. Well-known elementary properties of convex functions on $\mathbb{R}$ easily imply that the one-sided derivative ψ'_+ (ψ'_-) exists and is right (left) continuous on $[0, \alpha)$ ($(0, \alpha]$) and has finite variation on this interval. In other words, ψ has *bounded convexity* (see [11, Theorem 3.1] or [6, Lemma 5.5]). Further, since clearly $|\psi'_+| \geq 1$, $|\psi'_-| \geq 1$ we obtain that the curve ψ has *finite turn* (see [2, Theorem 5.4.2] or [6, Theorem 5.11]). So the curve $\psi^* := A^{-1} \circ \psi$, for which $\psi^*([0, \alpha]) \subset \Sigma(u)$, has also bounded convexity and finite turn.

Remark 3.3. The proof of Theorem 3.1 and Remark 3.2 show that, for the curve $\xi : [0, \tau] \to \Sigma(u)$ from Theorem CY, there exists $0 < \tau^* < \tau$ such that $\xi|_{[0,\tau^*]}$ has finite turn. In fact, this assertion "is not weaker" than Theorem 3.1, since it implies quickly Theorem 3.1 by standard methods.

Remark 3.4. We *did not show* that the curve ξ from Theorem CY has near 0 (left-continuous) left derivative ξ'_- at all points. However, the proof of Theorem 3.1 clearly implies that ξ has (left-continuous) left half-tangent on $(0, \tau^*]$ for some $0 < \tau^* < \tau$.

We will not give detailed proofs of facts from Remarks 3.2–3.4, since they would be inadequately long, and these facts are not essential for the present short note.

References

[1] Albano P., Cannarsa P., *Structural properties of singularities of semiconcave functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 719–740.

[2] Alexandrov A.D., Reshetnyak Yu.G., *General theory of irregular curves*, Mathematics and its Applications (Soviet Series), Vol. 29., Kluwer Academic Publishers, Dordrecht, 1989.

[3] Cannarsa P., Sinestrari C., *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and their Applications, 58, Birkhäuser, Boston, 2004.

[4] Clarke F.H., *Optimization and nonsmooth analysis*, 2nd edition, Classics in Applied Mathematics, 5, SIAM, Philadelphia, 1990.

[5] Cannarsa P., Yu Y., *Singular dynamics for semiconcave functions*, J. Eur. Math. Soc. **11** (2009), 999–1024.

[6] Duda J., *Curves with finite turn*, Czechoslovak Math. J. **58** (133) (2008), 23–49.

[7] Mifflin R., *Semismooth and semiconvex functions in constrained optimization*, SIAM J. Control Optimization **15** (1977), 959–972.

[8] Pavlica D., *On the points of non-differentiability of convex functions*, Comment. Math. Univ. Carolin. **45** (2004), 727–734.

[9] Spingarn J.E., *Submonotone subdifferentials of Lipschitz functions*, Trans. Amer. Math. Soc. **264** (1981), 77–89.

[10] Veselý L., Zajíček L., *Delta-convex mappings between Banach spaces and applications*, Dissertationes Math. (Rozprawy Mat.) **289** (1989).

[11] Veselý L., Zajíček L., *On vector functions of bounded convexity*, Math. Bohemica **133** (2008), 321–335.

[12] Zajíček L., *On the differentiation of convex functions in finite and infinite dimensional spaces*, Czechoslovak Math. J. **29** (1979) 340–348.

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