A note on formal power series

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Abstract. In this note we investigate a relationship between the boundary behavior of power series and the composition of formal power series. In particular, we prove that the composition domain of a formal power series g is convex and balanced which implies that the subset $\overline{\mathbb{X}}_g$ consisting of formal power series which can be composed by a formal power series g possesses such properties. We also provide a necessary and sufficient condition for the superposition operator T_g to map $\overline{\mathbb{X}}_g$ into itself or to map $\mathbb{X}_g$ into itself, respectively.

Keywords: composition, end behavior of convergence of power series, convex and balanced set, formal power series

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1. Introduction

The behavior of power series on boundaries of convergence domains has been an interesting topic since power series were introduced. The composition of formal power series has been an important part of the formal power series theory (below we recall some interesting results about such a composition). In this paper we introduce some relationship between these two subjects and provide a condition for convergence of a power series at every point in its interval of convergence, including endpoints or boundary points.

Let S be a ring and let $l \in \mathbb{N}$ be given. A *formal power series* on S is defined to be a mapping from $\mathbb{N}^l$ to S, where $\mathbb{N}$ represents the set of all positive integers. We denote the set of all such mappings by $\mathbb{X}(S)$, or briefly by $\mathbb{X}$. If $S = \mathbb{C}$, then $\mathbb{X}$ is a commutative $\mathbb{C}$-algebra with 1. We denote by $\overline{\mathbb{X}}(\mathbb{R})$ or $\overline{\mathbb{X}}(\mathbb{C})$ the set of all *convergent power series* centered at 0, where a convergent power series means a power series with a nonzero convergent radius. We simply call such a formal power series a power series (it was called *informal* sometimes [1]). It is known that both the algebra $\mathbb{X}(\mathbb{C})$ and the subalgebra $\overline{\mathbb{X}}(\mathbb{C})$ are integral domains. The structure of the algebras $\mathbb{X}$ and $\overline{\mathbb{X}}$ has been studied and can be found in many books on complex analysis such as in [2].

The composition of formal power series or functional composition has attracted many mathematicians. Around fifty years ago, Raney [3] investigated the functional composition patterns and provided a proof of the Lagrange inversion formula, that is, if S is a commutative ring with a unit e, then for each formal power series $g(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{X} = S^{\mathbb{N}}$ there is exactly one $f \in \mathbb{X}$ such that $f(0) = 0$

and $f(x) = x \cdot g \circ f(x)$. Later, in 1997 Cheng, McKay, Towber, Wang and Wright [4] extended the Raney coefficients. Recall also that a year earlier Constantine and Savits [5] dealing with the multivariate version of the classical Faa Di Bruno formula provided a general formula for calculating the derivative of a series composition in any number of variables. Li [6] provided p-adic power series which commute under composition. Chaumat and Chollet [7] contributed a lot in 2001 when they investigated Eakin and Harris result [8], that is, if F is a holomorphic mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$ with $F(0) = 0$, then any formal power series A such that $A \circ F$ is analytic is itself analytic. They also presented a control of the radius of convergence of A by the convergence radius of $A \circ F$. Gan and Knox [9] in 2002 provided a couple of very useful necessary and sufficient conditions for the existence of a composition of formal power series that took away the restriction of nonunitness for the composed formal power series in some formal power series ring.

Applications of formal power series to equations can be found e.g. in Neelon's papers [10] and [11]. In the first one a sufficient condition for a formal power series solution of systems of real analytic equations to be necessarily real analytic is provided. In the second note the author considers properties of systems of equations which are more general than convergence. Some latest development about the composition and its applications can be found e.g. in [12], [13], and [14].

For convenience of the reader and for consistency of the notation, we introduce some definitions below.

Definition 1.1. A formal power series f in x from $\mathbb{N}$ to a ring S is usually denoted by

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \quad \text{where } \ a_j \in S \ \text{ for every } \ j \in \mathbb{N} \cup \{0\}.$$

In this case, a_k is called the k-th coefficient of f, for every $k \in \mathbb{N} \cup \{0\}$. If $a_0 = 0$, then f is called a *nonunit*. We denote the set of all nonunits in $\mathbb{X}$ by $m(\mathbb{X})$. By $\operatorname{ord}(f)$ we denote the order of a formal power series f, that is,

$$\operatorname{ord}(f) = \begin{cases} \min\{n : a_n \neq 0\}, & \text{if } \ f \neq 0, \\ +\infty, & \text{if } \ f = 0, \end{cases}$$

then $m(\mathbb{X}) = \{f \in \mathbb{X} : \operatorname{ord}(f) \geq 1\}$.

Definition 1.2. Let S be a ring with a metric and let $g \in \mathbb{X}$, say $g(x) = \sum_{k=0}^{\infty} b_k x^k$. We define a subset $\mathbb{X}_g \subset \mathbb{X}$ to be

$$\mathbb{X}_g = \{f \in \mathbb{X} \mid f(x) = \sum_{k=0}^{\infty} a_k x^k, \ \sum_{n=0}^{\infty} b_n a_k^{(n)} \in S, \quad \text{for every } \ k \in \mathbb{N} \cup \{0\}\},$$

where $f^n(x) = \sum_{k=0}^{\infty} a_k^{(n)} x^k$, for all $n \in \mathbb{N} \cup \{0\}$ ($a_0^{(0)} = 1$, $a_k^{(0)} = 0$ for every $k \in \mathbb{N}$) is created by the product rule. Obviously, $\mathbb{X}_g \neq \emptyset$ because $m(\mathbb{X}) \subset \mathbb{X}_g$

(see Theorem 1.5 below). Then the mapping $T_g : \mathbb{X}_g \longrightarrow \mathbb{X}$ such that

$$T_g(f)(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where $c_k = \sum_{n=0}^{\infty} b_n a_k^{(n)}$, for $k \in \mathbb{N} \cup \{0\}$, is well-defined. We call $T_g(f)$ the *composition* of g and f. $T_g(f)$ is also denoted by $g \circ f$ (that is $(g \circ f)(x) = \sum_{n=0}^{\infty} b_n (f(x))^n$).

Definition 1.3. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series. If $a_m \neq 0$ and $a_j = 0$ for all $j > m$, then the degree of f is defined to be the number m and it is denoted by $\deg(f)$. If there is no such number m, then we say that $\deg(f) = +\infty$.

Definition 1.4. Let S be a ring, let $g \in \mathbb{X}$, say $g(x) = \sum_{n=0}^{\infty} b_n x^n$. The *derivative* of g is defined to be the formal power series g' such that

$$g'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1}.$$

A well-known result is that if a power series has a nonzero radius of convergence then all derivatives of this power series have the same radius of convergence. The existence of the derivative of the composition of a formal power series and a nonunit can be found in [1], too. A generalized chain rule of the composition of formal power series has been recently established in [15].

Let us notice that the derivative of a formal power series defined here has nothing to do with so-called sum function in analysis and has nothing to do with the limit of the *difference quotient*, although they may have some relationship if a formal power series is defined over a special ring and has some kind of convergence. For example, $g(x) = \sum_{n=0}^{\infty} n! x^n$, a formal power series over $\mathbb{R}$, converges at $x = 0$ only but the formal power series $g^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} n! x^{n-k}$ is well-defined for all $k \in \mathbb{N}$. Therefore one must be very careful when one tries to deal with the convergence and the derivatives of formal power series, especially the convergence at the boundary of the convergence. Lang provided some helpful comment in [16].

In what follows we will apply the following two results.

Theorem 1.5 ([9]). *Let S be a field with a metric and let $f, g \in \mathbb{X}(S)$ be given with the forms*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \qquad g(x) = b_0 + b_1 x + \cdots + b_n x^n \cdots,$$

and $\deg(f) > 0$. Then the composition $g \circ f$ exists if and only if

$$\sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in S \quad \text{for every } k \in \mathbb{N} \cup \{0\}, \tag{1.1}$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$.

Let us notice that the condition (1.1) in Theorem 1.5 is equivalent to that $g^{(n)}(a_0) \in S$ for every $n \in \mathbb{N} \cup \{0\}$.

Theorem 1.6 ([9]). *Let $f, g \in \mathbb{X}(\mathbb{C})$ be given with the forms*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots \quad \text{and} \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n \cdots .$$

If the series $\sum_{n=0}^{\infty} b_n R^n$ converges for some $R > |a_0|$, then $g \circ f$ exists.

Let $g \in \overline{\mathbb{X}}(\mathbb{C})$; in particular, Theorem 1.6 implies that in this case $m(\mathbb{X}) \neq \mathbb{X}_g$.

Definition 1.7. Let $g \in \mathbb{X}(S)$ be given, where S is a metric field. We denote by $r(g)$ the radius of convergence of g and by $I(g)$ the interval or disc of convergence of g. As usual, $0 \leq r(g) \leq +\infty$. We also define the *composition domain of* g as the set

$$D(g) = \left\{ a \in S : g^{(n)}(a) \in S \ \text{ for every } \ n \in \mathbb{N} \cup \{0\} \right\}.$$

As usual, we define the Minkowski sum $A + B$ of any two sets A and B as

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

The set $\overline{\mathbb{X}}_g = D(g) + m(\mathbb{X})$ (the Minkowski sum of $D(g)$ and $m(\mathbb{X})$) is said to be the *g-composition subset* of $\mathbb{X}$, where $D(g)$ is considered as a subset of $\mathbb{X}$ consisting of all constant power series.

Let us emphasize that termwise differentiation of series is an interesting topic in analysis. There are some general results about the termwise differentiation of a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{X}}$ with $|x| < r = r(f)$. If r is a positive real number, then the existence of $f'(r) = \sum_{n=0}^{\infty} n a_n r^{n-1}$ is not a trivial question. Some results about the termwise differentiation at the endpoints of the interval of convergence could be found in [17].

Let us recall a classical example.

Example 1.8. Let us consider $\mathbb{X}(\mathbb{R})$. Let $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^n$ and let $h = 1$. Then $g(h) \in \mathbb{R}$ but $g(-h) \notin \mathbb{R}$.

Moreover, $g'(h) = \sum_{n=0}^{\infty} (-1)^{n-1} \notin \mathbb{R}$.

The rest of this paper has the following structure: in Section 2 we investigate the properties of the set $D(g)$ and we investigate the behavior of power series on boundaries of their convergence domains; in Section 3 we investigate properties of sets $\overline{\mathbb{X}}_g$ and $\mathbb{X}_g$. Finally, the results from Section 4 concern the superposition operator T_g.

2. Boundary behavior of convergence of power series

A very important character of power series is that every power series has a radius of convergence and an interval of convergence. Example 1.8 tells us how complicated is the convergence at the endpoints of the interval of convergence and how the derivative of a power series may affect the convergence at the endpoints of the interval of convergence, or at the boundary of the convergence. That is why Remmert [2, p. 119] reminded the readers to be very careful in this field. In particular, if we seek such convergence of termwise differentiation for power series not only for once, but the infinitely many times, what can we say?

Lemma 2.1. *Let $g \in \mathbb{X}(\mathbb{R})$ and $a \in \mathbb{R}$ be given. Then $a \in D(g)$ if and only if $-a \in D(g)$.*

Proof: Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$. The conclusion is true obviously if $a = 0$. We now suppose that $a \neq 0$.

Without loss of generality, we assume that $a \in D(g)$ but $-a \in D(g)$ fails. By the notice after the Theorem 1.5 in Section 1, it means that $\sum_{n=k}^{\infty} \binom{n}{k} b_n a^{n-k} \in \mathbb{R}$ for every $k \in \mathbb{N} \cup \{0\}$ but $\sum_{n=k_0}^{\infty} \binom{n}{k_0} b_n (-a)^{n-k_0} \notin \mathbb{R}$ for some $k_0 \in \mathbb{N}$. Then $\sum_{n=k_0}^{\infty} \binom{n}{k_0} b_n (-a)^{n-k_0}$ diverges. It means that there are infinitely many $n \in \mathbb{N}$ such that

$$\binom{n}{k_0} |b_n| |a|^{n-k_0} > \frac{1}{n^2} .$$

Then there exists a sequence of positive integers (n_k), $n_k \geq k_0$, such that

$$\binom{n_k}{k_0} |b_{n_k}| |a|^{n_k-k_0} > \frac{1}{n_k^2} \quad \text{for every} \quad k \in \mathbb{N}.$$

Then

$$\frac{1}{|a|^{k_0} \cdot k_0!} n_k (n_k - 1) \dots (n_k - k_0 + 1) |b_{n_k}| |a|^{n_k} > \frac{1}{n_k^2} ,$$

so

$$|b_{n_k}| |a|^{n_k} > \frac{|a|^{k_0} \cdot k_0!}{n_k^3 (n_k - 1) \dots (n_k - k_0 + 1)} > \frac{|a|^{k_0} \cdot k_0!}{n_k^{k_0+2}} .$$

Hence

$$\begin{aligned}
&\frac{1}{|a|^{k_0}} \binom{n_k}{k_0+3} |b_{n_k}| |a|^{n_k-3} \\
&\quad = \frac{n_k (n_k - 1) \dots (n_k - k_0 - 2)}{(k_0 + 3)! |a|^{k_0+3}} \cdot |b_{n_k}| |a|^{n_k} \\
&\quad > \frac{n_k (n_k - 1) \dots (n_k - k_0 - 2)}{(k_0 + 3)! |a|^{k_0+3}} \cdot \frac{|a|^{k_0} \cdot k_0!}{n_k^{k_0+2}} \\
&\quad = \frac{1}{(k_0 + 1)(k_0 + 2)(k_0 + 3) \cdot |a|^3} \cdot n_k \left(1 - \frac{1}{n_k}\right) \dots \left(1 - \frac{k_0 + 2}{n_k}\right) .
\end{aligned}$$

Then $\lim_{k \to \infty} \binom{n_k}{k_0+3} |b_{n_k}| |a|^{n_k-3} = \infty$.

Then $\lim_{n\to\infty} \binom{n}{k_0+3}|b_n||a|^{n-3} = 0$ fails which contradicts to

$$\sum_{n=k_0}^{\infty} \binom{n}{k_0+3} b_n a^{n-k_0} \in \mathbb{R}.$$

Thus, $-a \in D(g)$. We complete the proof. □

Corollary 2.2. *Let $g \in \mathbb{X}(\mathbb{C})$ and $a \in \mathbb{C}$ be given. If $a \in D(g)$, then $z \in D(g)$ for all $z \in \mathbb{C}$ with $|z| = |a|$.*

Proof: Taking $z = ae^{i\theta}$ for some real number θ and applying the similar approach in Lemma 2.1, we can obtain the conclusion. □

It is clear that $D(g) \subset I(g)$ for every $g \in \mathbb{X} = \mathbb{X}(\mathbb{C})$. By this fact and the results above, we can prove the following theorem.

Theorem 2.3. *$D(g)$, the composition domain of g, is convex and balanced for every formal power series $g \in \mathbb{X} = \mathbb{X}(\mathbb{C})$.*

Proof: Let $g \in \mathbb{X}$ be given and let $r = r(g)$. If $r = 0$ or $r = +\infty$, then $D(g) = \{0\}$ or $D(g) = \mathbb{C}$, respectively, so the conclusion is obvious.

Now, let $0 < r < +\infty$ and let $0 < t < 1$. Suppose that $a_0, c_0 \in D(g)$. If $|a_0| < r$ or $|c_0| < r$, then $|ta_0 + (1-t)c_0| < r$, and hence $ta_0 + (1-t)c_0 \in D(g)$. Further, if $|a_0| = |c_0| = r$, then $|ta_0 + (1-t)c_0| \le r$, and therefore Corollary 2.2 implies that $ta_0 + (1-t)c_0 \in D(g)$. It means that $D(g)$ is always convex.

Now, suppose that $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$. If $|a_0| < r$ or $|\alpha| < 1$, then $|\alpha a_0| < r$ and therefore $\alpha a_0 \in D(g)$. In the case when $|a_0| = r$ and $|\alpha| = 1$, then $|\alpha a_0| = r$ and it is enough to apply Corollary 2.2 again to obtain the conclusion. Hence $D(g)$ is balanced and the proof is completed. □

Lemma 2.4. *Let $g \in \overline{\mathbb{X}}(\mathbb{C})$ be given such that $r = r(g) < \infty$. If $g^{(k)}(a) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ for some $a \in \mathbb{C}$ such that $|a| = r$, then the power series $g^{(k)}$ converges absolutely on the closed disc $\{z \in \mathbb{C} : |z| \le r\}$.*

Proof: Since $g^{(k)}(a) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ for some $a \in \mathbb{C}$ such that $|a| = r$, it follows that $g^{(k)}(r) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ by Corollary 2.2. It suffices to show that $g^{(k)}(r)$ converges absolutely for every $k \in \mathbb{N} \cup \{0\}$.

Suppose that $\sum_{n=k_0}^{\infty} \binom{n}{k_0}|b_n|r^{n-k_0} = +\infty$ for some $k_0 \in \mathbb{N}$. Then there are infinitely many $n \in \mathbb{N}$ such that

$$\binom{n}{k_0}|b_n|r^{n-k_0} > \frac{1}{n^2}\,.$$

Further, by similar reasoning as in the proof of Lemma 2.1,

$$\lim_{n\to\infty} \binom{n}{k_0+3} b_n r^{n-k_0-3} = 0$$

fails which contradicts that $\sum_{n=k_0}^{\infty} \binom{n}{k_0+3} b_n r^{n-k_0-3} \in \mathbb{C}$. Thus, the power series $g^{(k)}$ converges absolutely on the boundary of the closed disc $\{z \in \mathbb{C} : |z| \leq r\}$ and hence it converges on this closed disc. $\square$

We know that every power series $g \in \overline{\mathbb{X}}(\mathbb{C})$ is continuous on the open disc $D = \{z \in \mathbb{C} : |z| < r(g)\}$. It may happen if $r(g) < \infty$, however, that $g(z_0)$ is defined for some $|z_0| = r(g)$ but g is not continuous on $D \cup \{z_0\}$. We provide a sufficient condition for the uniform continuity of a power series on closed disc $\overline{D}$ below.

Lemma 2.5. *Let $g \in \overline{\mathbb{X}}(\mathbb{C})$ be given such that $r = r(g) < \infty$. If $g^{(k)}(z_0) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ for some $z_0 \in \mathbb{C}$ such that $|z_0| = r$, then the power series $g^{(k)}$ is uniformly continuous on the closed disc $\overline{D} = \{z \in \mathbb{C} : |z| \leq r(g)\}$ for every $k \in \mathbb{N} \cup \{0\}$.*

Proof: Let $z \in \mathbb{C}$ be such that $|z| = r$. Since $g^{(k)}(z_0) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$, it follows that $g^{(k)}(z) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ by Corollary 2.2.

Let $h \in \mathbb{C}$ be such that $0 < |h| < r$ and $|z - h| \leq r$. Then

$$\begin{aligned}
&|g(z-h) - g(z)| = \Big| \sum_{n=0}^{\infty} b_n[(z-h)^n - z^n] \Big| \\
&\leq \sum_{n=0}^{\infty} |b_n||h[(z-h)^{n-1} + (z-h)^{n-2}z + \cdots + (z-h)z^{n-2} + z^{n-1}]| \\
&\leq \sum_{n=0}^{\infty} |b_n||h|[|z-h|^{n-1} + |z-h|^{n-2}|z| + \cdots + |z-h||z|^{n-2} + |z|^{n-1}] \\
&\leq |h| \sum_{n=0}^{\infty} |b_n| n r^{n-1}.
\end{aligned}$$

By Lemma 2.4, g' converges absolutely at r which means that

$$\sum_{n=0}^{\infty} |b_n| n r^{n-1} \in \mathbb{R}.$$

Letting $h \to 0$ we deduce that g is continuous on the closed disc $\overline{D}$. Therefore g is uniformly continuous on $\overline{D}$.

Let us notice that since the convergence of $g^{(k+1)}$ on the closed disc $\overline{D}$ is absolute, we can similarly prove that $g^{(k)}$ is uniformly continuous on $\overline{D}$ for every $k \in \mathbb{N}$. The proof is completed. $\square$

3. Properties of sets $\mathbb{X}_g$ and $\overline{\mathbb{X}}_g$

By means of the results from Section 2, we investigate some properties of g-composition subset of $\mathbb{X}(\mathbb{C})$.

Corollary 3.1. *For any formal power series $g \in \mathbb{X} = \mathbb{X}(\mathbb{C})$, the g-composition subset $\overline{\mathbb{X}}_g$ of $\mathbb{X}$ is convex and balanced.*

Proof: It is clear that $m(\mathbb{X})$ is convex and balanced. Theorem 2.3 shows that $D(g)$ is convex and balanced. Thus, $\overline{\mathbb{X}}_g$ is convex and balanced. □

Next result describes the relation between $\overline{\mathbb{X}}_g$ and $\mathbb{X}_g$, where $g \in \mathbb{X}(\mathbb{C})$.

Corollary 3.2. *Let $g \in \mathbb{X}(\mathbb{C})$ be given. Then*

(j) $\overline{\mathbb{X}}_g \subset \mathbb{X}_g$,

(jj) $\overline{\mathbb{X}}_g \neq \mathbb{X}_g$ *if and only if there exists some $a \in \mathbb{C}$ such that $|a| = r(g)$, $g(a) \in \mathbb{C}$ but $g^{(k)}(a) \notin \mathbb{C}$ for some $k \in \mathbb{N}$.*

Proof: (j) follows from Theorem 1.5 and the notice followed it.

Now we prove (jj). Suppose that there exists some $a \in \mathbb{C}$ such that $|a| = r(g)$, $g(a) \in \mathbb{C}$ but $g^{(k)}(a) \notin \mathbb{C}$ for some $k \in \mathbb{N}$. Let $f = a$. Then $f \in \mathbb{X}_g$ but $f \notin \overline{\mathbb{X}}_g$ by Theorem 1.5 and the definition of $\overline{\mathbb{X}}_g$. Then $\overline{\mathbb{X}}_g \neq \mathbb{X}_g$.

Conversely we suppose that $\overline{\mathbb{X}}_g \neq \mathbb{X}_g$. Then there exists a formal power series $f \in \mathbb{X}_g$ but $f \notin \overline{\mathbb{X}}_g$ by (j) above. Say that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $\deg(f) > 0$, then $g^{(k)}(a_0) \in \mathbb{C}$ for all $k \in \mathbb{N} \cup \{0\}$ by Theorem 1.5 and hence $f \in \overline{\mathbb{X}}_g$. Therefore $\deg(f) = 0$ or $f = a_0 \in \mathbb{C}$. If $|a_0| < r(g)$, then $g^{(k)}(a_0) \in \mathbb{C}$ for all $k \in \mathbb{N} \cup \{0\}$ by Theorem 1.6 and hence $f \in \overline{\mathbb{X}}_g$. Then $|a_0| = r(g)$ because $g(a_0) = g(f) \in \mathbb{C}$. Finally, it is clear that $g^{(k)}(a_0) \notin \mathbb{C}$ for some $k \in \mathbb{N}$ because otherwise $f \in \overline{\mathbb{X}}_g$ by Theorem 1.5.

Thus, $f \in \mathbb{X}_g$ but $f \notin \overline{\mathbb{X}}_g$ if and only if

$$f = a \in \mathbb{C},\ |a| = r(g),\ g(a) \in \mathbb{C} \ \text{ but } \ g^{(k)}(a) \notin \mathbb{C} \ \text{ for some } \ k \in \mathbb{N}.$$

□

Proposition 3.3. *Let $g \in \mathbb{X} = \mathbb{X}(\mathbb{C})$ be given. Then*

(i) $\mathbb{X}_g = m(\mathbb{X})$ *if and only if* $r(g) = 0$.

(ii) $\mathbb{X}_g = \mathbb{X}$ *if and only if* $r(g) = +\infty$.

(iii) $h, g \in \mathbb{X}$ *with* $r(h) < r(g)$ *implies that* $D(h) \subset D(g)$, *and*

$$\overline{\mathbb{X}}_h \subset \mathbb{X}_h \subset \overline{\mathbb{X}}_g \subset \mathbb{X}_g.$$

(iv) *It is not always true that $h, g \in \mathbb{X}$ with $r(h) = r(g)$ implies that $\mathbb{X}_h = \mathbb{X}_g$.*

(v) $\mathbb{X} = \bigcup_{\overline{g} \in \overline{\mathbb{X}}} \mathbb{X}_{\overline{g}}$.

Proof: (i) follows from Corollary 3.2.

Properties (ii) and (v) are obvious.

(iii) follows from Theorem 1.6 and Corollary 3.2(jj).

For (iv), considering $h(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ and $g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n$, it is clear that a formal power series $1 \in \mathbb{X}_g$ but $1 \notin \mathbb{X}_h$. $\square$

Example 3.4. (a) $\mathbb{X}(\mathbb{C}) = \mathbb{X}_{e^z}$ where $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$.

(b) Let $g_a(x) = \sum_{n=0}^{\infty} \frac{1}{a^n} x^n$. This geometric series has the radius of convergence $r(g) = a$ if $a > 0$. Then, by Theorem 1.6,

$$\mathbb{X}(\mathbb{R}) = \bigcup_{a>0} \mathbb{X}_{g_a} = \bigcup_{k=1}^{\infty} \mathbb{X}_{g_k}.$$

4. Acting conditions

In this short section we provide a necessary and sufficient condition under which an operator T_g, where $g \in \overline{\mathbb{X}}(\mathbb{C})$ or $g \in \mathbb{X}(\mathbb{C})$, maps $\overline{\mathbb{X}}_g$ or $\mathbb{X}_g$ into itself.

Proposition 4.1. *Let $g = \sum_{n=0}^{\infty} b_n z^n \in \overline{\mathbb{X}}(\mathbb{C})$. Then T_g maps $\overline{\mathbb{X}}_g$ into itself if and only if $g(a) \in D(g)$ for every $a \in D(g)$.*

Proof: Let $f \in \overline{\mathbb{X}}_g = D(g) + m(\mathbb{X})$. Then $f = a + \tilde{f}$, where $a \in D(g)$ and $\tilde{f} \in m(\mathbb{X})$. By Definition 1.2, if $g \circ f$ is well-defined, the constant term of $g \circ f$ is $c_0 = \sum_{n=0}^{\infty} b_n a^n = g(a)$ and therefore $g \circ f \in \{c_0\} + m(\mathbb{X})$. Thus

$$g \circ f \in \overline{\mathbb{X}}_g \quad \text{if and only if} \quad g(a) \in D(g).$$

$\square$

Corollary 4.2. *Let $g \in \mathbb{X}(\mathbb{C})$ be given. Then T_g maps $\mathbb{X}_g$ into $\mathbb{X}_g$ if and only if the condition from Proposition 4.1 is satisfied and*

$$g(g(a)) \in \mathbb{C} \quad \textit{for every} \quad a \in \mathbb{X}_g \setminus \overline{\mathbb{X}}_g.$$

Proof: It is a consequence of Proposition 4.1 and Corollary 3.2(jj). $\square$

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