Closure-preserving covers in function spaces

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Abstract. It is shown that if $C_p(X)$ admits a closure-preserving cover by closed σ-compact sets then X is finite. If X is compact and $C_p(X)$ has a closure-preserving cover by separable subspaces then X is metrizable. We also prove that if $C_p(X, [0,1])$ has a closure-preserving cover by compact sets, then X is discrete.

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Introduction

To see whether a space Z has a "nice" topological property it is often useful to split Z into subspaces which could possibly have this property. V. Tkachuk proved in [8] that if $C_p(X)$ is a countable union of its subspaces with a property $\mathcal{P} \in$ {hereditary π-character $\leq \kappa$, pseudo-character $\leq \kappa$, Čech-completeness, tightness $\leq \kappa$, Fréchet-Urysohn property}, then $C_p(X)$ has $\mathcal{P}$. It is easy to see that closure-preserving covers are a generalization of countable closed covers, so the following question arises naturally: given a topological property $\mathcal{P}$ assume that $C_p(X)$ is the union of a closure-preserving family $\mathcal{F}$ of closed subspaces and each element of $\mathcal{F}$ has $\mathcal{P}$. Does this imply that $C_p(X)$ has $\mathcal{P}$ or some related topological property?

We show that if the elements of a closure-preserving cover of $C_p(X)$ are compact then X is finite. We also establish that a compact space X is metrizable if and only if $C_p(X)$ admits a closure-preserving cover by separable subspaces. We prove that if $C_p(X, [0,1])$ is a closure-preserving union of compact subspaces, then X is discrete.

1. Notation and terminology

Every topological space in this article is assumed to be Tychonoff. Our notation and terminology is standard. The set of real numbers with the natural topology is denoted by $\mathbb{R}$ and the interval $[0,1] \subset \mathbb{R}$ is represented by $\mathbb{I}$. For a space X the family of all subsets of X is denoted by $\exp(X)$, the family of all finite subsets of X is denoted by $[X]^{<\omega}$, the family of all open subsets of X is denoted by $\tau(X)$ and $\tau^*(X)$ is the family of nonempty open subsets of X. For $C \subset X$ the family of all open sets of X that contain C is denoted by $\tau(C, X)$; if $x \in X$ then we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. The space of all continuous functions from a space X into a space Y, endowed with the topology inherited from the product space Y^X, is denoted by $C_p(X,Y)$. The space $C_p(X,\mathbb{R})$ will be abbreviated by $C_p(X)$. Given $f \in C_p(X)$, a finite set $A \subset X$, and a positive real ε, let $O(f,A,\varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon \text{ for every } x \in A\}$. On the other hand, $C_u(X)$ denotes the space of all continuous real-valued functions on a space X, with the topology of uniform convergence. If $f \in C_u(X)$ and $r > 0$ then $B(f,r) = \{g \in C_u(X) : |g(x) - f(x)| < r \text{ for all } x \in X\} \in \tau(C_u(X))$ is the open ball centered at f of radius r. For every $f \in C_p(X,Y)$, define the dual map $f^* : C_p(Y) \to C_p(X)$ by $f^*(g) = g \circ f$ for every $g \in C_p(Y)$. A σ-compact (σ-countably compact) space is the countable union of compact (countably compact) spaces. A space is cosmic if it has a countable network. A space X is called monolithic if, for every $A \subset X$, we have $d(A) = nw(A)$.

2. Closure-preserving covers by closed subspaces

Suppose that $\mathcal{F}$ is a closure-preserving cover of a space Z and that every element of $\mathcal{F}$ is closed. Then every separable subspace of Z can be covered by a countable subfamily of $\mathcal{F}$. This simple observation has strong implications for function spaces.

Proposition 2.1. *Neither $\mathbb{R}^\omega$ nor $C_p(\mathbb{I})$ admits a closure-preserving cover by closed σ-pseudocompact subspaces.*

Proof: In [7, Theorem 3] Shakmatov and Tkachuk showed that $\mathbb{R}^\omega$ and $C_p(\mathbb{I})$ fail to be σ-pseudocompact, so it suffices to observe that they are both separable, because $d(C_p([0,1])) = nw(C_p(\mathbb{I})) = nw(\mathbb{I}) = \omega = w(\mathbb{R}^\omega)$. $\square$

Remark 2.2. Note that the word "closed" cannot be omitted in the formulation of Proposition 2.1 because $\mathbb{R}^\omega$ and $C_p(\mathbb{I})$ are both separable, and therefore each one admits a closure-preserving cover by countable subspaces. Indeed, take a countable dense set $A \subset C_p(\mathbb{I})$. Then the family $\{A \cup \{f\} : f \in C_p(\mathbb{I})\}$ is a closure-preserving cover of $C_p(\mathbb{I})$. Clearly every separable space has such a cover.

Lemma 2.3. *If $C_p(X) = \bigcup \mathcal{F}$ and $\mathcal{F}$ is a closure-preserving family of closed σ-pseudocompact subspaces of $C_p(X)$, then X is pseudocompact.*

Proof: If the space X is not pseudocompact, then $C_p(X)$ contains a retract Y homeomorphic to $\mathbb{R}^\omega$. Let $r : C_p(X) \to Y$ be a retraction. The space $\mathbb{R}^\omega$ is separable so we can find a countable family $\mathcal{F}' \subset \mathcal{F}$ such that $Y \subset \bigcup \mathcal{F}'$. Then the equality $Y = \bigcup\{r(F) : F \in \mathcal{F}'\}$ shows that Y, and therefore $\mathbb{R}^\omega$ is σ-pseudocompact which is a contradiction. $\square$

Corollary 2.4. *If $C_p(X)$ is the union of a closure-preserving family of pseudocompact subsets, then X is pseudocompact.*

Proof: It suffices to observe that the closure of a pseudocompact subspace of $C_p(X)$ is still pseudocompact and then apply Lemma 2.3. $\square$

Our purpose is to show that if a function space $C_p(X)$ is a closure-preserving union of closed σ-countably compact subspaces then X is finite. We will need the following facts.

Proposition 2.5. *Whenever $C_p(X) = \bigcup \mathcal{F}$ and $\mathcal{F}$ is a closure-preserving family of closed σ-countably compact subspaces of $C_p(X)$, then $f(X)$ is finite for every $f \in C_p(X)$.*

Proof: Let $f \in C_p(X)$ and $Y = f(X)$. The equality $\omega = w(Y) = d(C_p(Y))$ shows that $C_p(Y)$ is separable. The space X being pseudocompact by Lemma 2.3, the map f is $\mathbb{R}$-quotient so the dual map f^* embeds $C_p(Y)$ into $C_p(X)$ as a closed subspace. Therefore $C_p(Y)$ is covered by a closure-preserving family of its closed σ-countably compact subspaces and hence it is σ-countably compact. It follows from [7, Theorem 3.11] that Y is finite. $\square$

The following lemma is a part of the folklore.

Lemma 2.6. *For any space X, if $f(X)$ is finite for every f in $C_p(X)$, then X is finite.*

Proof: Suppose $|f(X)| < \omega$ for all $f \in C_p(X)$. If X is infinite, then it is possible to find a countable discrete subspace $D = \{x_n : n \in \omega\} \subset X$ and a countable family of open sets $\{U_n : n \in \omega\}$ such that $U_n \cap D = \{x_n\}$ for each $n \in \omega$ and $\overline{U}_i \cap \overline{U_j} = \emptyset$ if $i \neq j$. For every $n \in \omega$ there is $f_n \in C(X, \mathbb{I})$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) \subset \{0\}$; let $g_n = \frac{1}{(n+1)} f_n$. To see that the map $g = \sum g_n$ is continuous take an arbitrary $p \in X$. If $p \in U_n$ for some $n \in \omega$ and $V \in \tau(g(p), \mathbb{I})$, then $W = g^{-1}(V) \cap U_n$ is an open set containing p such that $g(W) \subset V$ and hence g is continuous at p. If $\varepsilon > 0$ and $p \notin \bigcup_{n\in\omega} U_n$ then $g(p) = 0$; let $m = \min\{n \in \omega : \frac{1}{n+1} < \varepsilon\}$. If $p \notin \bigcup_0^m \overline{U_j}$ then $p \in W = X \setminus \bigcup_0^m \overline{U_j}$ and $g(W) \subset [0, \varepsilon)$ which implies that g is continuous at p. Otherwise suppose that $p \in \overline{U_k}$ for some $k \leq m$. The set $V = X \setminus \{\overline{U_j} : j \neq k \text{ and } j \leq m\}$, is in $\tau(p, X)$ since $\overline{U}_i \cap \overline{U_j} = \emptyset$ if $i \neq j$. If $W = \left(X \setminus g_k^{-1}([\varepsilon, \frac{1}{k+1}])\right) \cap V$, then $g(W) \subset [0, \varepsilon)$ and therefore the map g is continuous at p. The map g is continuous and by construction the set $g(X)$ is infinite. This contradiction shows that the space X is finite. $\square$

Corollary 2.7. *For a space X the following conditions are equivalent:*

(a) *there exists a compact closure-preserving cover of $C_p(X)$;*
(b) *there exists a closed σ-compact closure-preserving cover of $C_p(X)$;*
(c) *there exists a closed σ-countably compact closure-preserving cover of $C_p(X)$;*
(d) *the space X is finite.*

Proof: It is evident that (a) implies (b) and (b) implies (c). Apply Proposition 2.5 and Lemma 2.6 to verify that (c) implies (d). If $n \in \omega$ then $\mathbb{R}^n$ can be covered by an increasing countable family of compact balls which shows that (d) implies (a). $\square$

If the hypothesis that the elements of a closure-preserving cover of $C_p(X)$ are closed is removed, we still have the following result.

Corollary 2.8. *If X is a space, and $C_p(X)$ admits a closure-preserving cover by countably compact subspaces then X is finite.*

Proof: Let $\mathcal{C}$ be a closure-preserving cover of $C_p(X)$ by countably compact subspaces. Apply Corollary 2.4 to check that X is pseudocompact. So every $C \in \mathcal{C}$ is compact by [1, Theorem III.4.23]. Using the equivalence (a)$\Longleftrightarrow$(d) of Corollary 2.7 we conclude that X is finite. $\square$

In the rest of this section we deal with closure-preserving covers of $C_p(X)$ for a compact space X.

Lemma 2.9. *If X is a space and A is a countable subset of X, then there exists $g \in C_p(X)$ such that $g|A$ is injective.*

Proof: For any two distinct points x and y of X, the set $P(x,y) = \{f \in C_p(X) : f(x) \neq f(y)\}$ is open in $C_p(X)$. For every finite $F \subset A$ let $G_F = \{g \in C_p(X) : g|F$ is injective$\}$. The equality $G_F = \bigcap\{P(x,y) : x, y \in F \text{ and } x \neq y\}$ shows that $G_F \in \tau(C_p(X))$. To check that each G_F is dense in $C_u(X)$ fix any open ball $B(f,\varepsilon) \in \tau(C_u(X))$ and take a finite $F \subset A$; then $F = \{a_0, \dots, a_n\}$ for some $a_0, \dots, a_n, n \in \omega$. Let $r_i = f(a_i)$ for each $i \leq n$. Choose a point $s_i \in (r_i, r_i + \frac{\varepsilon}{2})$ for every $i \leq n$ in such a way that the points $s_0, \dots, s_n$ are distinct. It is easy to find a function $g \in C_p(X, [0, \frac{\varepsilon}{2}])$ such that $g(a_i) = s_i - r_i$ for every $i \leq n$. If $h = f + g$, then $h|F$ is injective and $h \in B(f, \varepsilon)$.

Thus $\mathcal{G} = \{G_F : F \in [A]^{<\omega}\}$ is a countable family of open dense subsets of the complete metric space $C_u(X)$ and therefore it has non-empty intersection. Let $g \in \bigcap \mathcal{G}$. To check that $f|A$ is one-to-one take two distinct points x and y in A and let $F = \{x, y\}$. Since $g \in G_F$, the function $g|F$ is an injection, therefore $g(x) \neq g(y)$. $\square$

Lemma 2.10. *If X is a compact space such that $w(X) > \kappa$, for an infinite cardinal κ, then $C_p(X)$ does not admit a closure-preserving cover by subspaces of density less than or equal to κ.*

Proof: Suppose that $C_p(X)$ admits a closure-preserving cover $\mathcal{F}$ by closed subspaces of density at most κ. The space X maps continuously onto a compact space Z such that $w(Z) = \kappa^+$. The space $C_p(Z)$ embeds as a closed subspace in $C_p(X)$ so we can consider that $C_p(Z) \subset C_p(X)$. Since $d(C_p(Z)) = \kappa^+$, there is $\mathcal{F}_0 \subset \mathcal{F}$ such that $|\mathcal{F}_0| \leq \kappa^+$ and $C_p(Z) \subset \bigcup \mathcal{F}_0$. The space $C_p(X)$ is monolithic because X is compact [1, Corollary II.6.19], so $d(F) = nw(F) \geq nw(F \cap C_p(Z)) = d(F \cap C_p(Z))$ for each $F \in \mathcal{F}_0$. Therefore the family $\{F \cap C_p(Z) : F \in \mathcal{F}_0\}$ is a closure-preserving cover of $C_p(Z)$ by closed subspaces of density not greater than κ. We can rewrite the family $\{F \cap C_p(Z) : F \in \mathcal{F}_0\}$ as $\mathcal{F}' = \{F'_\alpha : \alpha < \kappa^+\}$. If for each $\alpha < \kappa^+$ we let $G_\alpha = \bigcup\{F'_\beta : \beta < \alpha\}$ then the family $\mathcal{G} = \{G_\alpha : \alpha < \kappa^+\}$ is an increasing closure-preserving cover of $C_p(X)$. Moreover each $G_\alpha \in \mathcal{G}$ has density less than or equal to κ for it is the union of at most κ-many spaces of density at most κ.

On the other hand, the space Z is embeddable in $\mathbb{I}^{\kappa^+}$ so we can consider that $Z \subset \mathbb{I}^{\kappa^+}$ and for each $\alpha < \kappa^+$ we can define $Z_\alpha = p_\alpha(Z)$, where $p_\alpha : Z \to \mathbb{I}^\alpha$ is the natural projection from Z to $\mathbb{I}^\alpha$. Let $M_\alpha = p_\alpha^*(C_p(Z_\alpha))$. Since every $f \in C_p(Z)$ is factorizable by a projection to a countable face of $\mathbb{I}^\kappa$ according to [1, Lemma 0.2.3], the family $\mathcal{M} = \{M_\alpha : \alpha < \kappa^+\}$ is another increasing cover of $C_p(Z)$, and for each $\alpha < \kappa^+$ we have $d(M_\alpha) = w(Z_\alpha) \leq \kappa$. Hence, for each $M_\alpha \in \mathcal{M}$ there exists $G_\beta \in \mathcal{G}$ such that $M_\alpha \subset G_\beta$. We now verify that for each $G_\beta \in \mathcal{G}$ there is $M_\alpha \in \mathcal{M}$ such that $G_\beta \subset M_\alpha$. There is $D \subset G_\beta$ such that $\overline{D} = G_\beta$ and $|D| \leq \kappa$. For each $z \in D$ we can find M_{ξ_z} containing z. If $\alpha \geq \sup\{\xi_z : z \in D\}$ then $G_\beta \subset \overline{\bigcup_{z\in D} M_{\xi_z}} \subset M_\alpha$.

Let us now note that for every $\alpha_0 < \kappa^+$, there is $\alpha_1 > \alpha_0$ such that $G_{\alpha_0} \subset \overline{G_{\alpha_0}} \subset M_{\alpha_1}$ and the natural projection $p_{\alpha_0}^{\alpha_1} : Z_{\alpha_1} \to Z_{\alpha_0}$ is not injective, because if $p_{\alpha_0}^{\beta}$ is an injection for each $\beta > \alpha_0$ then $p_{\alpha_0} : Z \to \mathbb{I}^{\alpha_0}$ is an embedding and hence $w(X) \leq |\alpha_0| < \kappa^+$. Arguing in the same way, we obtain $\alpha_2 > \alpha_1$ such that $G_{\alpha_0} \subset M_{\alpha_1} \subset G_{\alpha_2}$ and the corresponding projection $p_{\alpha_1}^{\alpha_2} : Z_{\alpha_2} \to Z_{\alpha_1}$ is not an injection and so on. Let $\gamma = \sup\{\alpha_n : n \in \omega\}$; since the family $\mathcal{G}$ is closure-preserving and both families $\mathcal{G}$ and $\mathcal{M}$ are ascending, we have the following equalities $G_\gamma = \bigcup_{n\in\omega} G_{\alpha_{2n}} = \bigcup_{n\in\omega} M_{\alpha_{2n+1}} = \bigcup_{n\in\omega} \overline{G_{\alpha_{2n}}} = \overline{\bigcup_{n\in\omega} G_{\alpha_{2n}}} = \overline{G_\gamma}$. Thus we also have $\overline{\bigcup_{n\in\omega} M_{\alpha_{2n+1}}} = \overline{G_\gamma} = \bigcup_{n\in\omega} M_{\alpha_{2n+1}}$. This means that if $f \in \overline{\bigcup_{n\in\omega} M_{\alpha_{2n+1}}}$, then there must be $m \in \omega$ such that f is in $M_{\alpha_{2m+1}}$.

By the construction above for every $n \in \omega$ the projection $p_{\alpha_n}^{\alpha_{n+1}} : Z_{\alpha_{n+1}} \to Z_{\alpha_n}$ is not injective. So for each $n \in \omega$ there is a point $y_n \in Z_{\alpha_n}$ for which it is possible to find two distinct points x_n and z_n both in Z_γ such that $p_{\alpha_n}^\gamma(x_n) = p_{\alpha_n}^\gamma(z_n) = y_n$. Apply Lemma 2.9 to find $g \in C_p(Z_\gamma)$ such that $g(x_n) \neq g(z_n)$ for every $n \in \omega$. This makes $f = p_\gamma^*(g)$ fail to be in $M_{\alpha_{2n+1}}$ for each $n \in \omega$, in other words $f \notin G_\gamma$. Now, for an arbitrary finite $A \subset Z$, we can find $l \in \omega$ such that $p_{\alpha_l}^\gamma|A$ is one-to-one. Let $U = O(f, A, \varepsilon) \cap M_\gamma \in \tau(f, M_\gamma)$. There exists $h' : Z_{\alpha_l} \to \mathbb{R}$ such that for every $a \in A$ the equality $h'(p_{\alpha_l}^\gamma(a)) = f(a)$ holds and hence $h = (p_{\alpha_l}^\gamma)^*(h') \in U \cap M_{\alpha_l} \subset M_{\alpha_l} \subset C_p(Z)$ This implies that $f \in \overline{\bigcup_{n\in\omega} M_{\alpha_{2n+1}}} = \overline{G_\gamma} = G_\gamma$, but as noticed before, $f \notin G_\gamma$. This contradicts the fact that $\mathcal{G}$ is a closure-preserving cover of $C_p(Z)$. $\square$

Corollary 2.11. *If X is a compact space such that $w(X) > \mathfrak{c}$ then $C_p(X)$ does not admit a closure-preserving cover by subspaces of cardinality $\mathfrak{c}$.*

PROOF: Suppose that $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ is a closure-preserving cover of $C_p(X)$ by subspaces of cardinality $\mathfrak{c}$. For each $\alpha \in I$ let $C_\alpha = \overline{F}_\alpha$; then $\mathcal{C} = \{C_\alpha : \alpha \in I\}$ is a closure-preserving closed cover of $C_p(X)$ by subspaces of density not greater than $\mathfrak{c}$. We can now apply Lemma 2.10 to obtain a contradiction. $\square$

Corollary 2.12. *For a compact space X, if $C_p(X)$ admits a closure-preserving cover by subspaces of cardinality less than or equal to $\mathfrak{c}$ then $C_p(X)$ has cardinality at most $\mathfrak{c}$.*

Proof: Let $\mathcal{C}$ be a closure-preserving cover of $C_p(X)$ by subspaces of cardinality less than or equal to $\mathfrak{c}$. The family $\mathcal{F} = \{\overline{C} : C \in \mathcal{C}\}$ is a closure-preserving cover of $C_p(X)$ by closed subspaces. Applying [1, Lemma IV.11.3] we have $|F| \leq \mathfrak{c}$ for every $F \in \mathcal{F}$. It follows from Corollary 2.11 that $w(X) \leq \mathfrak{c}$ so $d(C_p(X)) \leq \mathfrak{c}$. Therefore the cover $\mathcal{F}$ has a subcover $\mathcal{G}$ of cardinality not greater than $\mathfrak{c}$ and hence $|C_p(X)| \leq \mathfrak{c}$. □

At the beginning of this section we noted that it is useful to know whether a function space $C_p(X)$ is separable, in order to reduce any of its infinite closure-preserving closed covers to a countable one. For the case when X is compact, we will characterize separable function spaces by means of closure-preserving covers.

Corollary 2.13. *For an infinite compact space X the following conditions are equivalent:*

(a) *there exists a closure-preserving cover $\mathcal{F}$ of $C_p(X)$ such that the elements of $\mathcal{F}$ are cosmic;*
(b) *there exists a closure-preserving cover $\mathcal{F}$ of $C_p(X)$ such that the elements of $\mathcal{F}$ are separable;*
(c) *there exists a closure-preserving cover $\mathcal{F}$ of $C_p(X)$ such that the elements of $\mathcal{F}$ are countable;*
(d) *the space X is metrizable.*

Proof: It is immediate that (a) implies (b), and that (c) implies (b). Now if X is metrizable, then $C_p(X)$ is separable, and the equality $d(C_p(X)) = iw(X) = w(X) = nw(X) = nw(C_p(X))$ shows that (d) implies (a). To see that (d) implies (c), let D be a countable dense subspace of $C_p(X)$ and for every $f \in C_p(X)$ let $D_f = D \cup \{f\}$. It needs no proof that $\mathcal{D} = \{D_f : f \in C_p(X)\}$ is a closure-preserving cover of $C_p(X)$ by countable subspaces. Hence it suffices to prove that (b) implies (d). Suppose that X is not metrizable, this implies that $w(X) > \omega$ because X is compact. Apply Lemma 2.7 to conclude that (b) cannot occur. □

Corollary 2.14. *If X is an infinite compact space then $C_p(X)$ admits a closure-preserving cover by closed second countable subspaces if and only if X is countable.*

Proof: If X is countable, then $C_p(X)$ is second countable. Now suppose that $C_p(X)$ has a closure-preserving cover by closed second countable spaces then by Corollary 2.13 the space X is metrizable, which means $C_p(X)$ is separable. Hence $C_p(X)$ has a countable cover by closed second countable subspaces so we can apply [10, Corollary 1.7] to see that $|X| \leq \omega$. □

3. Topological games and their strategies

If a space Z has a compact closure-preserving cover then a topological game on Z can be defined in a natural way; in this game the first player has a winning strategy. Therefore studying analogous games in function spaces gives a possibility to strengthen some results of the previous section. The following game is a slight variation from the one presented by R. Telgársky in [8].

Definition 3.1. On a Tychonoff space Y, consider a family $\mathcal{C} \subset \exp(Y)$. We define the game $\mathcal{G}(\mathcal{C}, Y)$ of two players I and II who take turns in the following way: at the move number n, Player I chooses $C_n \in \mathcal{C}$ and Player II chooses a set $U_n \in \tau(C_n, Y)$. The game ends after the n-th move of each player has been made for every $n \in \omega$ and Player I wins if $X = \bigcup\{U_n : n \in \omega\}$; otherwise the winner is Player II.

Definition 3.2. A strategy t for the first player in the game $\mathcal{G}(\mathcal{C}, Y)$ on a space X is defined inductively in the following way. First the set $t(\emptyset) = F_0 \in \mathcal{C}$ is chosen. An open set $U_0 \in \tau(X)$ is legal if $F_0 \subset U_0$. For every legal set U_0 the set $t(U_0) = F_1 \in \mathcal{C}$ has to be defined. Let us assume that legal sequences $(U_0, \dots, U_i)$ and sets $t(U_0, \dots, U_i)$ have been defined for each $i \leq n$. The sequence $(U_0, \dots, U_{n+1})$ is legal if so is the sequence $(U_0, \dots, U_i)$ for each $i \leq n$ and $F_{n+1} = t(U_0, \dots, U_n) \subset U_{n+1}$. A strategy t for Player I is winning if it ensures victory for I in every play it is used.

Definition 3.3. A strategy s for Player II in the game $\mathcal{G}(\mathcal{C}, Y)$ on a space X is simply a function that assigns to every finite sequence $(F_0, \dots, F_n)$ of elements of $\mathcal{C}$ an open set $U \in \tau(F_n, X)$. Such a strategy for Player II is winning if it ensures victory for II in every play it is used.

Theorem 3.4. *Given a non-empty space X, if $Y = C_p(X, \mathbb{I})$ and $\mathcal{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X, \mathbb{I})\}$, then Player II has a winning strategy in the game $\mathcal{G}(\mathcal{F}, Y)$.*

Proof: Recall that if $f \in C(X, \mathbb{I})$ and $\varepsilon \geq 0$ then the set $I(f, \varepsilon) = \{g \in Y : |g(x) - f(x)| \leq \varepsilon$ for all $x \in X\}$ is closed in the space Y. Define inductively a winning strategy s for Player II in the game $\mathcal{G}(\mathcal{F}, Y)$ on Y in the following way: let $F_0 \in \mathcal{F}$ be the first move of Player I. If $B_0 \in \tau^*(C_u(X, \mathbb{I}))$ is an open ball of radius 1 in Y, then $\emptyset \neq (B_0 \setminus F_0) \in \tau^*(C_u(X, \mathbb{I}))$ and therefore there is a point $f_0 \in B_0 \setminus F_0$ and a positive real number $\varepsilon_0 < 1$ such that $I(f_0, \varepsilon_0) \subset B_0 \setminus F_0$; then $F_0 \subset (Y \setminus I(f_0, \varepsilon_0)) \in \tau^*(C_p(X, \mathbb{I}))$. Consequently, we can define $U_0 = s(F_0) = Y \setminus I(f_0, \varepsilon_0)$ as the first choice of Player II.

Assume that for each $i \leq j < n$ and every legal finite sequence $(F_0, \dots, F_j)$ of elements of the family $\mathcal{F}$ selected by Player I, we have defined the set $U_i = s(F_0, \dots, F_i)$ and the open ball B_i of radius at most 2^{-i}, together with a positive real number $\varepsilon_i < 2^{-i}$ as well as a function $f_i \in Y \setminus F_i$ with the following properties. If we fix $j < n$, then for all $k < i \leq j$ we have $I(f_i, \varepsilon_i) \subset B_i \subset I(f_k, \varepsilon_k) \subset B_k$ and $U_i = Y \setminus I(f_i, \varepsilon_i)$.

Let F_n be the n-th move of Player I. As above, if $B_n \in \tau^*(C_u(X, \mathbb{I}))$ is an open ball of radius not greater than 2^{-n} contained in $I(f_{n-1}, \varepsilon_{n-1})$ then $\emptyset \neq B_n \setminus F_n \in \tau^*(C_u(X, \mathbb{I}))$, and hence, we can find a point $f_n \in B_n \setminus F_n$ and a positive real number $\varepsilon_n < 2^{-n}$ such that $I(f_n, \varepsilon_n) \subset B_n \setminus F_n$. The set F_n is contained in

$Y \setminus I(f_n, \varepsilon_n)$ so we can take $U_n = s(F_n) = Y \setminus I(f_n, \varepsilon_n)$ to be the n-th move of Player II.

The definition of the strategy s is complete, let us convince ourselves that it is a winning one. Let $P = \{(F_n, U_n)\}_{n\in\omega}$ be a play in which Player II applies the strategy s. By definition of s we have the equality $U_n = s(F_n) = Y \setminus I(f_n, \varepsilon_n)$ where $\varepsilon_n < 2^{-n}$. The family $\{I(f_n, \varepsilon_n) : n \in \omega\}$ is a decreasing sequence of closed non-empty subsets of the complete metric space $C_u(X, \mathbb{I})$, and the corresponding sequence of diameters converges to zero. This means that $\bigcap\{I(f_n, \varepsilon_n) : n \in \omega\} \neq \emptyset$ and therefore $\bigcup\{U_n : n \in \omega\} \neq Y$. This shows that Player II wins whenever she (or he) applies the strategy s. □

Remark 3.5. It is possible to reformulate Theorem 3.4 for the set $Y = C_p(X)$ and the family $\mathcal{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X)\}$, applying the same method to prove that Player II has a winning strategy in the game $\mathcal{G}(\mathcal{F}, Y)$.

Remark 3.6. Given a space X consider the set $Y = C_p(X, \mathbb{I})$ (or $Y = C_p(X)$), and let $\mathcal{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X, \mathbb{I})\}$ (or $\mathcal{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X)\}$). If $\mathcal{C}$ is a family of non-empty closed subsets of Y for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{C}, Y)$ then $\mathcal{C} \not\subseteq \mathcal{F}$.

Lemma 3.7. *Let Y be a space, and define $\mathcal{C}$ to be the family of all nonvoid closed locally compact subspaces of Y; let $\mathcal{C}'$ be the family of all nonempty closed discrete unions of compact subspaces of Y. If there exists a closure-preserving compact cover $\mathcal{F}$ of the space Y, then Player I has a winning strategy in the games $\mathcal{G}(\mathcal{C}, Y)$ and $\mathcal{G}(\mathcal{C}', Y)$.*

Proof: For each $y \in Y$ let $K(y) = Y \setminus \bigcup\{F \in \mathcal{F} : y \notin F\}$. In [5] it is verified that $K(y)$ is an open set which contains y, and if $x \in K(y)$, then $K(x) \subset K(y)$. Call a point $m \in Y$ maximal if $K(m)$ is not properly contained in the set $K(y)$ for any $y \in Y \setminus \{m\}$. Potoczny showed in [4] that if $M(Y)$ is the set of all the maximal elements of Y then $M(Y)$ is a discrete union of compact subspaces of Y, and therefore $M(Y) \in \mathcal{C}' \subset \mathcal{C}$. Moreover in [5] it is established that if $\{U_n : n \in \omega\}$ is a family of open subsets of Y such that $M(Y) \subset U_0$, and for each $n \in \omega$ we have $M(Y \setminus \bigcup\{U_i : i = 0, \ldots, n\}) \subset U_{n+1}$; then $Y = \bigcup\{U_n : n \in \omega\}$. This shows that Player I has a winning strategy in the game $\mathcal{G}(\mathcal{C}, Y)$. □

Corollary 3.8. *For every space X, if the space $Y = C_p(X, \mathbb{I})$ is covered by a closure-preserving family of compact subspaces, then X is discrete.*

Proof: Let $\mathcal{C}$ be the family of all nonempty closed discrete unions of compact subspaces of Y. By Lemma 3.7, Player I has a winning strategy in the game $\mathcal{G}(\mathcal{C}, Y)$. By Remark 3.6, not all members of $\mathcal{C}$ are nowhere dense in $C_u(X, \mathbb{I})$. Therefore there exists $C \in \mathcal{C}$ that contains an open ball $B(f, 2r) \in \tau(C_u(X, \mathbb{I}))$ and hence $I(f, r) = \{g \in C_p(X, \mathbb{I}) : |g(x) - f(x)| \leq r$ for all $x \in X\} \subset C$. It is easy to see using connectedness of $I(f, r)$ that $I(f, r)$ is compact; therefore $C_p(X, \mathbb{I})$ is also compact being homeomorphic to $I(f, r)$ and hence X is discrete. □

Corollary 3.9. *For every space X, let $Y = C_p(X, \mathbb{I})$ and let $\mathcal{C}$ be the family of all σ-compact subspaces of Y. If Player I has a winning strategy on Y for the game $\mathcal{G}(\mathcal{C}, Y)$, then the space X is discrete.*

PROOF: Remark 3.6 states that not every element of $\mathcal{C}$ is nowhere dense in $C_u(X, \mathbb{I})$. Therefore, there is a σ-compact subspace F of Y such that F contains a closed subspace homeomorphic to $C_p(X, \mathbb{I})$. Thus $C_p(X, \mathbb{I})$ is σ-compact and hence X is discrete by [9, Theorem 1.5.2]. $\square$

Lemma 3.10. *If $\mathcal{C}$ is the family of all σ-compact subspaces of $Y = \omega^\omega$, then Player II has a winning strategy for the game $\mathcal{G}(\mathcal{C}, Y)$.*

PROOF: Suppose $(F_0, \dots, F_n)$ is any sequence of σ-compact subspaces of Y. For the set F_n it is possible to find a countable family $\{K_m^n : m \in \omega\}$ of compact subsets of Y such that $F_n = \bigcup_{m \in \omega} K_m^n$. For each $m \in \omega$, let $\pi_m : Y \to \omega$ be the natural projection from Y to the factor determined by m.

Define the finite set $C_n^m = \bigcup_{i=0}^n \pi_n(K_i^n)$, let $b_n^n = \sup C_n^n + 1$. For each $m > n$ let $b_m^n = \sup \pi_m(K_m^n) + 1$. For every $m \geq n$ let $V_m^n = \{0, \dots, b_m^n - 1\} \times \omega^{\omega \setminus \{m\}}$. Observe that $K_i^n \subset V_n^n$ whenever $i \leq n$ and $K_j^n \subset V_j^n$ for all $j > n$. Also if $y \in V_m^n$ then $y(m) < b_m^n$ for all $m \geq n$. Now, let $U_n = \bigcup_{m \geq n} V_m^n$. Define $t((F_0, \dots, F_n)) = U_n$.

To see that t is a winning strategy, take a play $\mathcal{P} = \{(F_n, U_n)\}_{n \in \omega}$ in which Player II uses t. We will find a point in $Y \setminus \bigcup_{n \in \omega} U_n$. For every $n \in \omega$ there exists $x_n \in Y$ such that $x_n(m) = b_m^n$ for every $m \geq n$. Observe that $x_n \notin U_n$. Let $y_0(0) = x_0(0) + 1$. Now, once the value $y(n)$ has been determined, let $y(n+1) = y(n) + \sum_0^{n+1} x_i(n+1)$. The point y has the property that for every $n \in \omega$, if $m \geq n$ then $y(m) > x_n(m)$ which implies that $y \notin U_n$. It has been verified that $Y \neq \bigcup_{n \in \omega} U_n$ and therefore Player II wins the play $\mathcal{P}$ proving that t is a winning strategy. $\square$

Note that, since the elements of $\mathcal{C}$ in Lemma 3.10 are not necessarily closed, such result does not follow from [8].

Corollary 3.11. *If X is a non-σ-compact analytic space and $\mathcal{C}$ is the family of all σ-compact subsets of X, then Player II has a winning strategy in the game $\mathcal{G}(\mathcal{C}, X)$.*

PROOF: Apply [6, Theorem 3.5.3] to see that X has a closed subspace Y homeomorphic to ω^ω. By Lemma 3.10 Player II has a winning strategy for the game $\mathcal{G}(\mathcal{C}, Y)$. It is easy to see that this implies that Player II has a winning strategy in the game $\mathcal{G}(\mathcal{C}, X)$. $\square$

Corollary 3.12. *For a space X, let $\mathcal{C}$ be the family of all σ-compact subspaces of $C_p(X)$. If Player I has a winning strategy on $C_p(X)$ for the game $\mathcal{G}(\mathcal{C}, C_p(X))$, then the space X is finite.*

PROOF: It is easy to see that if Y is closed in $C_p(X)$ then Player I has a winning strategy for the corresponding game on Y. Therefore if $\mathcal{C}'$ is the family of all

σ-compact subspaces of $C_p(X,\mathbb{I})$ then Player I has a winning strategy for game $\mathcal{G}(\mathcal{C}', C_p(X,\mathbb{I}))$. Now apply Corollary 3.9 to verify that X is discrete. By Corollary 3.11 the space $\mathbb{R}^\omega$ does not embed into $C_p(X)$ as a closed subspace. Thus X is pseudocompact and hence finite. □

4. Open problems

Reminding ourselves that all spaces considered in this text are Tychonoff, let us observe that the study of closure-preserving covers of function spaces has turned out to be a huge task and this paper only scratches its surface. To show that the topic is far from being exhausted, we give below a list of some interesting open problems.

Problem 4.1. *Suppose that $C_p(X)$ is the union of a closure-preserving family of its pseudocompact subspaces. Must $C_p(X)$ be σ-pseudocompact?*

Problem 4.2. *Suppose that $C_p(X,\mathbb{I})$ is the union of a closure-preserving family of its pseudocompact subspaces. Must $C_p(X,\mathbb{I})$ be pseudocompact?*

Problem 4.3. *Suppose that $C_p(X,\mathbb{I})$ is the union of a closure-preserving family of its closed σ-compact subspaces. Does this imply that X is discrete?*

Problem 4.4. *Let X be a space, not necessarily compact, such that $C_p(X)$ is the union of a closure-preserving family of its separable subspaces. Must $C_p(X)$ be separable?*

Problem 4.5. *Suppose that $C_p(X)$ is the union of a closure-preserving family of closed subspaces of cardinality $\mathfrak{c}$. Must $C_p(X)$ have cardinality $\mathfrak{c}$?*

Problem 4.6. *Suppose that $C_p(X)$ is the union of a closure-preserving family $\mathcal{F}$ of its second countable subspaces. Must X be countable? What happens if all the elements of $\mathcal{F}$ are closed in $C_p(X)$?*

Problem 4.7. *Suppose that $C_p(X)$ is the union of a closure-preserving family $\mathcal{F}$ of closed subspaces of countable tightness. Is it true that $t(C_p(X)) = \omega$?*

Problem 4.8. *Suppose that $C_p(X)$ is the union of a closure-preserving family of its closed metrizable subspaces. Must X be countable?*

Problem 4.9. *Suppose that X is compact and $C_p(X)$ is the union of a closure-preserving family of its metrizable subspaces. Must X be countable?*

Problem 4.10. *Suppose that $C_p(X)$ is the union of a closure-preserving family of closed Lindelöf subspaces. Must $C_p(X)$ be Lindelöf?*

Problem 4.11. *Suppose that $C_p(X)$ is the union of a closure-preserving family of closed Lindelöf Σ-subspaces. Must $C_p(X)$ be Lindelöf Σ?*

Problem 4.12. *Can $C_p(\mathbb{I})$ be represented as the union of a closure-preserving family of its second countable subspaces?*

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