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On Kantorovich's result on the symmetry of Dini derivatives

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Abstract. For $f : (a, b) \to \mathbb{R}$, let A_f be the set of points at which f is Lipschitz from the left but not from the right. L.V. Kantorovich (1932) proved that, if f is continuous, then A_f is a "(k_d)-reducible set". The proofs of L. Zajíček (1981) and B.S. Thomson (1985) give that A_f is a σ-strongly right porous set for an arbitrary f. We discuss connections between these two results. The main motivation for the present note was the observation that Kantorovich's result implies the existence of a σ-strongly right porous set $A \subset (a, b)$ for which no continuous f with $A \subset A_f$ exists. Using Thomson's proof, we prove that such continuous f (resp. an arbitrary f) exists if and only if there exist strongly right porous sets A_n such that $A_n \nearrow A$. This characterization improves both results mentioned above.

Keywords: Dini derivative, one-sided Lipschitzness, σ-porous set, strong right porosity, abstract porosity

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1. Introduction

For $f : (a, b) \to \mathbb{R}$, we set

$$\begin{aligned} A_f := \{x \in (a, b) :\ & \max(|D^+ f(x)|, |D_+ f(x)|) = \infty, \\ & \max(|D^- f(x)|, |D_- f(x)|) < \infty\}, \\ B_f := \{x \in (a, b) :\ & \max(|D^+ f(x)|, |D_+ f(x)|) < \infty, \\ & \max(|D^- f(x)|, |D_- f(x)|) = \infty\}, \end{aligned}$$

where $D^+ f$, $D_+ f$, $D^- f$, $D_- f$ are Dini derivatives of f.

So, A_f is the set of all points at which f is Lipschitz from the left (i.e., $\limsup_{t\to x_-} \left|\frac{f(t)-f(x)}{t-x}\right| < \infty$), but is not Lipschitz from the right.

Note that the classical Denjoy-Young-Saks theorem (see, e.g., [4, §70]) gives that A_f and B_f are always Lebesgue null.

The set A_f, in the case when f is continuous, was considered by L.V. Kantorovich in his early work [2]. His result [2, Theorem II, p. 161] reads as follows.

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Theorem K. *If f is a continuous function on (a, b), then A_f is a (k_d)-reducible set.*

The notion of a (k_d)-reducible set ("(k_d)-privodimaja sovokupnost") is defined using the notion of strong right porosity (see Definition 1.4 below) as follows. (Of course, in [2] the term "porosity" was not used, since it was first used by E.P. Dolzhenko in 1967.)

Definition 1.1. A set $E \subset \mathbb{R}$ is called (k_d)-*reducible*, if for each perfect set $F \subset E$, the set of all points $x \in F$ at which F is not strongly right porous is not dense in F.

In other words, E is (k_d)-reducible, if and only if each perfect set $F \subset E$ has a portion F^* (i.e., a nonempty set of the form $F^* = F \cap (c, d)$) which is a strongly right porous set.

Independently on [2], the sets A_f, B_f were studied in [5] (our attention was drawn to [2] by an unpublished manuscript by D.L. Renfro, where the connection between [2] and [5] was noted). The result of [5, Theorem 2] states that $A_f \cup B_f$ is σ-strongly porous even for an arbitrary function f on (a, b) and the proof clearly gives that

$$A_f \text{ is a } \sigma\text{-strongly right porous set.} \tag{1.1}$$

For an alternative proof, where (1.1) can be seen more easily, see the proof of [4, Theorem 73.2]. If f is a continuous function, Theorem K implies (1.1), but the argument needs a recent rather deep result on the structure of non-σ-porous sets (see Section 4).

The second interesting fact, which motivated the present article, is that (for a continuous function f) Theorem K is stronger than (1.1), since it easily implies (observe that, by Baire's Theorem, the set in Example 3.1 is not (k_d)-reducible) that

$(*)$ *there exists a σ-strongly right porous set $A \subset (a, b)$ such that $A \subset A_f$ for no continuous function f on (a, b).*

In the present article, we observe that, slightly changing the proof of (1.1) from [4], we obtain Theorem K, and so also $(*)$, even for an arbitrary function f. Moreover, we fully characterize the smallness of sets A_f, proving the following result.

Theorem 1.2. *Let $a, b \in \mathbb{R}^*$ and $A \subset (a, b)$. The following statements are equivalent.*

(i) *There exists a continuous function $f : (a, b) \to \mathbb{R}$ such that $A \subset A_f$.*
(ii) *There exists a function $f : (a, b) \to \mathbb{R}$ such that $A \subset A_f$.*
(iii) *There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of strongly right porous sets such that $A_1 \subset A_2 \subset \dots$ and $A = \bigcup_{n=1}^{\infty} A_n$.*

The condition (iii) gives a full simple characterization of the hereditary class generated by the sets of the form A_f, i.e. the class

$$\mathcal{H} := \{A \subset (a,b) : \ A \subset A_f \text{ for some } f : (a,b) \to \mathbb{R}\}.$$

This characterization shows (see Proposition 3.2) that $\mathcal{H}$ is strictly smaller than the class $\mathcal{S}$ of all σ-strongly right porous sets. On the other hand, Theorem 1.2 immediately implies that the σ-ideal generated by the sets of the form A_f coincides with $\mathcal{S}$.

Remark 1.3. For two equivalent reformulations of condition (iii) of Theorem 1.2 (in an abstract context) see Proposition 5.1. In particular, the fourth equivalent condition can be added in Theorem 1.2.

(iv) *There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of strongly right porous sets such that $A = \bigcup_{n=1}^{\infty} A_n$ and every A_n is relatively closed in A.*

We conclude this section with a notation and with a definition of porosity notions we need.

Notation. The symbol $A_n \nearrow A$ means that $A_1 \subset A_2 \subset \dots$ and $A = \bigcup_{n=1}^{\infty} A_n$. The length of an interval $I \subset \mathbb{R}$ is denoted by $|I|$.

Definition 1.4. Let $E \subset \mathbb{R}$ and $x \in \mathbb{R}$.

(i) We say that E is *strongly right porous* (resp. *strongly left porous*) *at* x if there exists a sequence of open intervals $\{I_k\}_{k=1}^{\infty}$ on the right (resp. left) from x such that $|I_k| \to 0$, $\operatorname{dist}(x, I_k) \to 0$, $\frac{|I_k|}{\operatorname{dist}(x,I_k)} \to \infty$ and $I_k \cap E = \emptyset$ for every $k \in \mathbb{N}$.

(ii) We say that E is *strongly right porous* (resp. *strongly left porous*) if E is strongly right porous (resp. strongly left porous) at each point of E.

(iii) We say that E is *σ-strongly right porous* (resp. *σ-strongly left porous*) if it can be expressed as a countable union of strongly right porous (resp. strongly left porous) sets.

For an alternative definition of unilateral strong porosity as well as for other definitions of similar porosity notions see e.g. [4] or [6].

The following easy lemma provides a criterion for strong right porosity.

Lemma 1.5. *Let $E \subset \mathbb{R}$, $x \in \mathbb{R}$, and let there exist, for every $\varepsilon > 0$, an open interval $J_\varepsilon = (u,v)$ such that $x \le u$, $|J_\varepsilon| \le 1$, $J_\varepsilon \cap E = \emptyset$ and $|u - x| < \varepsilon |J_\varepsilon|$. Then E is strongly right porous at x.*

Proof: For every $k \in \mathbb{N}$, we set $\varepsilon_k := \frac{1}{k^2}$, find an open interval $J_{\varepsilon_k} = (u_k, v_k)$ such that $x \le u_k$, $v_k - u_k \le 1$, $(u_k, v_k) \cap E = \emptyset$, $u_k - x < \frac{v_k - u_k}{k^2}$ and put $I_k := (u_k + \frac{v_k - u_k}{2k^2}, u_k + \frac{v_k - u_k}{k})$. It is easy and straightforward to check that $\{I_k\}_{k=1}^{\infty}$ is a sequence of open intervals as in Definition 1.4(i). $\square$

2. Proof of Theorem

First, we formulate the following lemma. We omit its obvious proof.

Lemma 2.1. *Let $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ be such that $0 < \beta - \alpha < 1$ and $I := [\alpha, \beta]$. There exists a function $g = g_{I,n} : I \to \mathbb{R}$ with the following properties:*

(i) *g is locally Lipschitz on (α, β),*
(ii) *g is linear on $[\alpha, \beta - (\beta-\alpha)^2]$ with slope $\frac{1}{n^2}$,*
(iii) *g is decreasing on $[\beta - (\beta-\alpha)^2, \beta]$,*
(iv) *$g(\alpha) = 0$, $g(\beta) = 0$ and $g'_-(\beta) = -\infty$.*

Now we are prepared to prove the following symmetric version of our main result, which is clearly equivalent to Theorem 1.2 (to see this, it is sufficient to consider the function $f^*(x) := f(-x)$ on $(-b, -a)$).

Theorem 2.2. *Let $a, b \in \mathbb{R}^*$ and $S \subset (a, b)$. The following statements are equivalent.*

(i) *There exists a continuous function $f : (a,b) \to \mathbb{R}$ such that $S \subset B_f$.*
(ii) *There exists a function $f : (a,b) \to \mathbb{R}$ such that $S \subset B_f$.*
(iii) *There exists a sequence $\{S_n\}_{n=1}^{\infty}$ of strongly left porous sets such that $S_1 \subset S_2 \subset \dots$ and $S = \bigcup_{n=1}^{\infty} S_n$.*

Proof: The implication (i)$\Rightarrow$(ii) is trivial.

Next, we will prove (ii)$\Rightarrow$(iii). It easily follows from the proof of [4, Theorem 73.2] that the sets

$$A_{m,n} := \{x \in (a,b) : \left|\tfrac{f(y)-f(x)}{y-x}\right| \le n \text{ for each } y \in (x, x + \tfrac{1}{m}) \\ \text{and } f \text{ is not Lipschitz from the left at } x\}$$

are strongly left porous for arbitrary $m, n \in \mathbb{N}$. Moreover, $B_f = \bigcup_{n=1}^{\infty} A_{n,n}$ and it suffices to put $S_n := A_{n,n} \cap S$ for every $n \in \mathbb{N}$.

Finally, we will prove (iii)$\Rightarrow$(i). First assume that $a, b \in \mathbb{R}$. We will construct a continuous function $f : [a,b] \to \mathbb{R}$ such that $S \subset B_{f|_{(a,b)}}$ and moreover $f(a) = 0$ and $f(b) = 0$. Without loss of generality we can assume that $a = 0$ and $b = 1$.

If $S = \emptyset$, it suffices to put $f(x) = 0$, $x \in [0,1]$. Otherwise, we can further assume that $S_n \neq \emptyset$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We denote by $\mathcal{D}_n$ the system of all components of $(0,1) \setminus \overline{S_n}$. For every $I \in \mathcal{D}_n$ we put $f_n(x) := g_{\overline{I},n}(x)$ for $x \in \overline{I}$ (where $g_{\overline{I},n}$ is the function from Lemma 2.1), $f_n(0) = 0$, $f_n(1) = 0$ and $f_n(x) = 0$ for $x \in \overline{S_n}$. Then the function $f_n : [0,1] \to \mathbb{R}$ has the following properties:

(a) $f_n(0) = f_n(1) = 0$,
(b) $0 \le f_n \le \frac{1}{n^2}$ on $[0,1]$ and $f_n = 0$ on $\overline{S_n}$, in particular $\frac{f_n(y)-f_n(x)}{y-x} \le 0$ for $x \in \overline{S_n}$ and $0 \le y < x$,
(c) f_n is continuous on $[0,1]$ and locally Lipschitz on $(0,1) \setminus \overline{S_n}$,
(d) $\frac{f_n(y)-f_n(x)}{y-x} \le \frac{1}{n^2}$ for every $x \in \overline{S_n}$ and $y \in (x, 1]$,
(e) $\liminf_{y \to x_-} \frac{f_n(y)-f_n(x)}{y-x} = -\infty$ for every $x \in \overline{S_n}$ at which S_n is strongly left porous.

Indeed, properties (a)–(d) easily follow from the definition of f_n and from properties (i)–(iv) of Lemma 2.1. If $x \in \overline{S_n}$ and S_n is strongly left porous at x then either there exists $I \in \mathcal{D}_n$ such that x is the right endpoint of I and therefore $(f_n)'_-(x) = -\infty$ (cf. Lemma 2.1(iv)), or there exists a sequence of open intervals $\{I_k\}_{k=1}^{\infty}$ in $\mathcal{D}_n$ on the left of x such that $|I_k| \to 0$, $\operatorname{dist}(x, I_k) \to 0$ and $\frac{|I_k|}{\operatorname{dist}(x,I_k)} \to \infty$. For every $k \in \mathbb{N}$, denote by y_k the point of I_k at which f_n achieves its maximum $\frac{|I_k|}{n^2}(1-|I_k|)$ on I_k. Then $x - y_k = |I_k|^2 + \operatorname{dist}(x, I_k)$ for every $k \in \mathbb{N}$, and so

$$\frac{f_n(x) - f_n(y_k)}{x - y_k} = -\frac{1 - |I_k|}{n^2\left(|I_k| + \frac{\operatorname{dist}(x,I_k)}{|I_k|}\right)} \to -\infty \quad \text{for} \quad k \to \infty.$$

Since $x - y_k \to 0$, the property (e) is also fulfilled.

Finally, we put $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every $x \in [0,1]$. By properties (a)–(c) it follows that f is continuous on $[0,1]$, $f(0) = f(1) = 0$. It only remains to show that $S \subset B_{f|_{(0,1)}}$.

Fix an arbitrary $x \in S$. Let $k \in \mathbb{N}$ be the minimal number with the property $x \in \overline{S_k}$ (since $x \in S_n$ for some $n \in \mathbb{N}$, such a minimal number k clearly exists). If $k > 1$, there exists $I_i \in \mathcal{D}_i$ such that $x \in I_i$ for every $i \in \mathbb{N}$, $i < k$, and since by (c) f_i is locally Lipschitz on I_i, we easily conclude that

$$\text{(2.1)} \qquad \sum_{i=1}^{k-1} f_i \ \text{ is Lipschitz both from the right and from the left at } x \text{ if } k > 1.$$

If $i \geq k$, then $x \in \overline{S_i}$. By property (d), $\frac{f_i(y)-f_i(x)}{y-x} \leq \frac{1}{i^2}$ for every $y \in (x, 1]$. It follows that $\sum_{i=k}^{\infty} f_i$ is Lipschitz from the right at x. Using (2.1), we get that $f = \sum_{i=1}^{\infty} f_i$ is Lipschitz from the right at x.

Since $x \in S_n$ for some $n \in \mathbb{N}$, S_n is strongly left porous and $S_k \subset S_n$, S_k is strongly left porous at x. Thus, since $x \in S_i$ for each $i \geq k$, by properties (b) and (e) it follows that

$$\text{(2.2)} \qquad \liminf_{y \to x_-} \sum_{i \geq k} \frac{f_i(y) - f_i(x)}{y - x} \leq \liminf_{y \to x_-} \frac{f_k(y) - f_k(x)}{y - x} = -\infty.$$

Using (2.1) and (2.2), we conclude that $f = \sum_{i=1}^{\infty} f_i$ is not Lipschitz from the left at x.

Let $a \in \mathbb{R}$, $b = \infty$. Put $z_0 = a$. Since S is σ-strongly left porous, its complement in (a, ∞) is dense and thus we can find a sequence $\{z_n\}_{n=1}^{\infty}$ such that $z_n \to \infty$, $0 < z_n - z_{n-1} < 1$ and $z_n \notin S$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we put $J_n = [z_{n-1}, z_n]$, construct a continuous function f_{J_n} as above and extend it by zero to the whole interval $[a, \infty)$. Finally, we put $f = \sum_{n=1}^{\infty} f_{J_n}$ and obtain a continuous function on (a, ∞) with all the desired properties.

The cases $a = -\infty$, $b \in \mathbb{R}$ and $a = -\infty$, $b = \infty$ can be handled similarly. $\square$

Remark 2.3. Let f be a continuous function on $\mathbb{R}$ and

$$F_n := \left\{x \in (a,b) : \left|\frac{f(y)-f(x)}{y-x}\right| \le n \text{ for each } y \in (x, x+\frac{1}{n})\right\} \qquad (n \in \mathbb{N}).$$

It is easy to see that all F_n are closed and $F_n \nearrow L^+$, where L^+ is the set of all points at which f is Lipschitz from the right. So L^+ is an F_σ set. Similarly, the set L^- of all points at which f is Lipschitz from the left is an F_σ set. Set $H := \mathbb{R} \setminus L^-$. Then H is a G_δ set and $F_n \cap H = A_{n,n}$, where $A_{n,n}$ are as in the preceding proof. So, this proof gives that if

(i) $B = B_f$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$,

then

(ii) there exist closed sets $F_1 \subset F_2 \subset \dots$ and a G_δ set H such that each $F_n \cap H$ is strongly left porous and $B_f = \bigcup_{n=1}^{\infty}(F_n \cap H)$.

We do not know whether the implication (ii)$\Rightarrow$(i) holds.

3. Example

In this section, if $I \subset \mathbb{R}$ is a bounded interval, then $l(I)$ denotes the left endpoint of I.

Example 3.1. There exists a closed σ-strongly right porous set $S \subset [0,1]$ such that the set $\{x \in S : S \text{ is not strongly right porous at } x\}$ is dense in S.

Proof: For each $I = (c,d) \subset (0,1)$ and each $k \in \mathbb{N} \cup \{0\}$, we can clearly choose a sequence $d = z_0(I,k) > z_1(I,k) > \dots$ such that $z_n(I,k) \to l(I) = c$,

$$z_1(I,k) - c < \frac{|I|}{2^{k+1}} \tag{3.1}$$

and

$$\{z_n(I,k) : n \in \mathbb{N}\} \text{ is not strongly right porous at } l(I). \tag{3.2}$$

We further put $\mathcal{M}(I,k) := \{(z_n(I,k), z_{n-1}(I,k)) : n \in \mathbb{N}\}$.

Now we inductively define $\mathcal{M}_0 := \{(0,1)\}$ and $\mathcal{M}_{k+1} := \bigcup_{I \in \mathcal{M}_k} \mathcal{M}(I,k)$ for every $k \in \mathbb{N} \cup \{0\}$. Set $G_k := \bigcup_{I \in \mathcal{M}_k} I$ and $P_k := [0,1) \setminus G_k$ for $k \in \mathbb{N} \cup \{0\}$. It is easy to see that each $\mathcal{M}_k$ is a countable disjoint system of open intervals and $P_k = \bigcup_{i=0}^{k} \bigcup_{I \in \mathcal{M}_i} \{l(I)\}$. Set $P := \bigcup_{k=0}^{\infty} P_k$ and $S := \overline{P}$.

If $x \in P$, then there exist $i \in \mathbb{N} \cup \{0\}$ and $I \in \mathcal{M}_i$ such that $x = l(I)$. Since $\{l(J) : J \in \mathcal{M}(I,i)\} = \{z_n(I,i) : n \in \mathbb{N}\} \subset S$, we obtain by (3.2) that S is not strongly right porous at x. So, since P is dense in S, it remains to prove that S is σ-strongly right porous.

Since P is countable, it is sufficient to show that

$$S \setminus (P \cup \{1\}) \text{ is a strongly right porous set.} \tag{3.3}$$

To this end, for each $k \in \mathbb{N} \cup \{0\}$ and $I = (c, d) \in \mathcal{M}_k$, set $\alpha_n(I) := \sup(P_n \cap [c, d))$, $n \geq k$. Since $\{\alpha_n(I)\}_{n=k}^{\infty}$ is increasing and bounded, there exists $\alpha(I) := \lim_{n\to\infty} \alpha_n(I)$. It is easy to see that $\alpha_k(I) = c = l(I)$ and $\alpha_{n+1}(I) = z_1((\alpha_n(I), d), n)$ for $n \geq k$. So (3.1) implies that $\alpha_{n+1}(I) - \alpha_n(I) < \frac{d-\alpha_n(I)}{2^{n+1}} \leq \frac{|I|}{2^{n+1}}$. Consequently,

$$\alpha(I) - l(I) < \sum_{n=k}^{\infty} \frac{|I|}{2^{n+1}} = \frac{|I|}{2^k}\,. \tag{3.4}$$

So it is easy to see that $\alpha(I) = \max(S \cap I) < d$.

Now fix an arbitrary $x \in S \setminus (P \cup \{1\})$. To prove (3.3), it is sufficient to show that S is strongly right porous at x. We will check that the assumptions of Lemma 1.5 hold. So let an arbitrary $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ with $2^{-k+1} < \varepsilon$ and let $I = (c, d) \in \mathcal{M}_k$ be the (unique) interval containing x. Set $J_\varepsilon = (u, v) := (\alpha(I), d)$. Then $J_\varepsilon \cap S = \emptyset$ and so $c = l(I) < x \leq u$. By (3.4) we have

$$|u - x| \leq \alpha(I) - l(I) < \frac{|I|}{2^k} \quad \text{and} \quad 1 \geq |J_\varepsilon| > |I| - \frac{|I|}{2^k} \geq \frac{|I|}{2}\,.$$

So $|u-x| < 2^{-k+1}|J_\varepsilon| < \varepsilon|J_\varepsilon|$. Thus S is strongly right porous at x by Lemma 1.5. □

The following proposition shows that Theorem 1.2 strengthens (1.1).

Proposition 3.2. *There exists a closed σ-strongly right porous set $S \subset \mathbb{R}$ such that $S \neq \bigcup_{n=1}^{\infty} S_n$ whenever $S_n \subset S_{n+1}$ and S_n is strongly right porous for every $n \in \mathbb{N}$.*

Proof: Let S be the set constructed in Example 3.1. Thus S is closed, σ-strongly right porous and the set $\{x \in S : S \text{ is not strongly right porous at } x\}$ is dense in S. Suppose that $S = \bigcup_{n=1}^{\infty} S_n$ where $S_n \subset S_{n+1}$ and S_n is strongly right porous for every $n \in \mathbb{N}$. Since S is closed, by Baire's Theorem, there exist an interval (a, b) and $n_0 \in \mathbb{N}$ such that $(a, b) \cap S \neq \emptyset$ and $(a, b) \cap S_{n_0}$ is dense in $(a, b) \cap S$. Since $\{x \in S : S \text{ is not strongly right porous at } x\}$ is dense in S, there exists $x_0 \in (a, b) \cap S$ such that S is not strongly right porous at x_0. Since S_{n_0} is dense in $(a, b) \cap S$, $\overline{S_{n_0}}$ is not strongly right porous at x_0 and thus S_{n_0} is not strongly right porous at x_0. Since $x_0 \in S$ and $S_n \subset S_{n+1}$ for every $n \in \mathbb{N}$, there exists $N > n_0$ such that $x_0 \in S_N$. Since S_N is strongly right porous, it is strongly right porous at x_0. Since $S_{n_0} \subset S_N$, the set S_{n_0} is strongly right porous at x_0, which is a contradiction. □

4. Connections to Kantorovich's result

(A) First we will show that, for a continuous function f, Theorem K implies (1.1), if we know the following recent rather deep result of [1].

Theorem DZ. *Let $B \subset \mathbb{R}$ be a Borel set which is not σ-strongly right porous. Then there exists a compact set $K \subset B$ which is not σ-strongly right porous.*

To prove (1.1), suppose on the contrary that A_f is not σ-strongly right porous. Since A_f is Borel (it is even an intersection of an F_σ set and a G_δ set, cf. Remark 2.3), we obtain by Theorem DZ a compact set $K \subset A_f$ which is not σ-strongly right porous. Denote

$$K^* := \{x \in K\colon K \cap (x-\varepsilon, x+\varepsilon) \text{ is not } \sigma\text{-strongly right porous for each } \varepsilon > 0\}.$$

Then K^* is clearly closed and it is not σ-strongly right porous. Indeed, if $\mathcal{B}$ is a countable basis of open sets in $\mathbb{R}$, then

$$K \setminus K^* = \bigcup \{K \cap B : \ B \in \mathcal{B},\ K \cap B \text{ is } \sigma\text{-strongly right porous}\}.$$

Therefore $K \setminus K^*$ is σ-strongly right porous and so K^* is not σ-strongly right porous. Since A_f is a (k_d)-reducible set by Theorem K, there exists $x_0 \in K^*$ and $\varepsilon > 0$ such that $K^* \cap (x_0 - \varepsilon, x_0 + \varepsilon)$ is strongly right porous (if K^* is not perfect, the existence of x_0 and ε is obvious). But this is a contradiction, since $K \cap (x_0 - \varepsilon, x_0 + \varepsilon)$ is not σ-strongly right porous by the definition of K^* and $K \cap (x_0 - \varepsilon, x_0 + \varepsilon) \subset (K^* \cap (x_0 - \varepsilon, x_0 + \varepsilon)) \cup (K \setminus K^*)$.

(B) Further, Theorem 1.2 easily implies that Theorem K holds even for an arbitrary function f. Indeed, let $f : (a, b) \to \mathbb{R}$ be an arbitrary function and let $F \subset A_f$ be a perfect set. By Theorem 1.2, there exist strongly right porous sets A_n such that $A_n \nearrow F$. By Baire's Theorem, there exist an interval (c, d) and $n_0 \in \mathbb{N}$ such that $F^* := F \cap (c, d) \neq \emptyset$ and $A_{n_0} \cap (c, d)$ is dense in F^*. Let $x \in F^*$. Then $x \in A_n$ for some $n > n_0$, and since $A_n \cap (c, d) \supset A_{n_0} \cap (c, d)$ is dense in F^*, we obtain that F^* is strongly right porous at x. So A_f is (k_d)-reducible.

(C) We do not know the answer to the following question:

Question 4.1. Let $B \subset \mathbb{R}$ be a Borel (k_d)-reducible set. Does then exist strongly right porous sets A_n such that $A_n \nearrow B$?

If the answer to this question is positive, then Theorem K implies that, for a continuous function f, there exist strongly right porous sets A_n such that $A_n \nearrow A_f$.

5. Monotone unions of P-porous sets and P-reducible sets

The main aim of this section is to present some easy but non-trivial observations on (k_d)-reducible sets and sets satisfying condition (iii) of Theorem 1.2 (i.e., monotone unions of strongly right porous sets). However, it is natural to work with an "abstract porosity P" instead of right strong porosity only, since the proofs are equally easy in the general case, which can perhaps find some other applications.

Let (X, ϱ) be a metric space. The open ball with center $x \in X$ and radius $r > 0$ will be denoted by $B(x, r)$. By an *abstract porosity* on X (called "a porosity-like point-set relation" in [7] and [8]) we will mean a relation P between points of X and subsets of X (i.e., $P \subset X \times 2^X$) fulfilling the following "axioms":

(A1) If $A \subset B \subset X$, $x \in X$ and $P(x, B)$, then $P(x, A)$.
(A2) $P(x, A)$ if and only if there exists $r > 0$ such that $P(x, A \cap B(x, r))$.
(A3) $P(x, A)$ if and only if $P(x, \overline{A})$.

We say that $A \subset X$ is

(i) *P-porous at $x \in X$*, if $P(x, A)$,
(ii) *P-porous* if $P(x, A)$ for every $x \in A$,
(iii) *σ-P-porous* if A is a countable union of P-porous sets.

We say that $A \subset X$ is *P-reducible* if each nonempty closed set $F \subset A$ contains a P-porous subset with nonempty relative interior in F.

It is easy to see that strong right porosity on $\mathbb{R}$ (as most of natural versions of porosity) is an abstract porosity which also fulfills:

(A4) If $x \in X$ is not an isolated point of X, then $P(x, \{x\})$.

Proposition 5.1. *Let (X, ϱ) be a metric space, P an abstract porosity on X and $A \subset X$. Then the following conditions are equivalent:*

(i) *$A_n \nearrow A$, where each A_n is P-porous;*
(ii) *$B_n \nearrow A$, where each B_n is P-porous and relatively closed in A;*
(iii) *$A = \bigcup_{n=1}^{\infty} C_n$, where each C_n is P-porous and relatively closed in A.*

Proof: To prove (i)$\Rightarrow$(ii), let A_n be as in (i). Set $B_n := \overline{A_n} \cap A$. Then clearly $B_n \nearrow A$ and each B_n is relatively closed in A. To show that B_n is also P-porous, consider an arbitrary $x \in B_n$. Then $x \in A_k$ for some $k \geq n$. Since $P(x, A_k)$ and $A_n \subset A_k$, we have $P(x, A_n)$ by (A1), and thus $P(x, B_n)$ by (A3) and (A1).

The implication (ii)$\Rightarrow$(iii) is trivial.

To prove (iii)$\Rightarrow$(i), let C_n be as in (iii). For $n \geq 2$ and $k \geq 1$, set

$$C_n^k := C_n \cap \{x \in X : \operatorname{dist}(x, C_1 \cup \dots \cup C_{n-1}) > 1/k\}. \tag{5.1}$$

Since $C_1 \cup \dots \cup C_{n-1}$ is closed in A, for each $n \geq 2$,

$$C_n^k \nearrow C_n \setminus (C_1 \cup \dots \cup C_{n-1}) \quad (k \to \infty). \tag{5.2}$$

Now set

$$A_1 := C_1 \quad \text{and} \quad A_k := C_1 \cup C_2^k \cup \dots \cup C_k^k \quad (k \geq 2).$$

Since $A = C_1 \cup (C_2 \setminus C_1) \cup (C_3 \setminus (C_1 \cup C_2)) \cup \dots$, it is easy to see that (5.2) implies $A_k \nearrow A$. To prove that each A_k is P-porous, consider arbitrary $k \geq 2$ and $x \in A_k$. If $x \in C_1$, then we obtain, using (5.1) for $n = 2, \dots, k$, that $A_k \cap B(x, 1/k) = C_1 \cap B(x, 1/k)$, and so $P(x, A_k)$ by (A2). If $x \in C_i^k$, $2 \leq i \leq k$, then we obtain, using (5.1) for $n = i, \dots, k$, that $A_k \cap B(x, 1/k) = C_i^k \cap B(x, 1/k) \subset C_i \cap B(x, 1/k)$, and so $P(x, A_k)$ by (A2) and (A1). $\square$

Proposition 5.2. *Let (X, ϱ) be a separable topologically complete metric space without isolated points, $A \subset X$ and let P be an abstract porosity on X satisfying* (A4). *Then the following conditions are equivalent.*

(i) *A is P-reducible.*

(ii) *Each nonempty perfect set $F \subset A$ contains a P-porous set with nonempty relative interior in F.*

(iii) *Each closed set $F \subset A$ is a countable union of closed P-porous sets.*

PROOF: The implication (i)$\Rightarrow$(ii) is trivial.

To prove (ii)$\Rightarrow$(i), it is sufficient to observe that if $F \subset A$ has an isolated point a, then $\{a\} \subset F$ is a P-porous set (by (A4)) with nonempty relative interior in F.

To prove (iii)$\Rightarrow$(i), suppose that (iii) holds and a nonempty closed set $F \subset A$ is given. By (iii), $F = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed and P-porous. Since (F, ϱ) is topologically complete, we can use Baire's Theorem in (F, ϱ) and obtain that F_n has nonempty relative interior in F for some $n \in \mathbb{N}$. So (i) holds.

To prove (i)$\Rightarrow$(iii), suppose that (i) holds and a nonempty closed set $F \subset A$ is given. Denote by $\mathcal{G}$ the system of all open sets $G \subset X$ such that $G \cap F$ can be covered by countably many closed P-porous sets. Set $G^* := \bigcup \mathcal{G}$. Since X is separable, we can write $G^* = \bigcup_{n=1}^{\infty} G_n$ with $G_n \in \mathcal{G}$, which clearly implies $G^* \in \mathcal{G}$. To prove (iii), it is clearly sufficient (by (A1)) to prove that $F \subset G^*$. So suppose, on the contrary, that $F^* := F \setminus G^* \neq \emptyset$. By (i) and (A1), there exist $c \in F^*$ and $r > 0$ such that $F^* \cap B(c,r)$ is P-porous. Since $F^* \cap B(c,r) = \bigcup_{n=1}^{\infty} (F^* \cap \overline{B(c,(n/n+1)r)})$ and $G^* \in \mathcal{G}$, we obtain that both $F^* \cap B(c,r)$ and $G^* \cap F \cap B(c,r)$ can be covered by countably many closed P-porous sets. Since $F \cap B(c,r) = (F^* \cap B(c,r)) \cup (G^* \cap F \cap B(c,r))$, we obtain that $B(c,r) \in \mathcal{G}$, which contradicts to $c \notin G^*$. □

Propositions 5.1 and 5.2 have the following corollary:

Corollary 5.3. *Let (X, ϱ) be a separable topologically complete metric space without isolated points, $A \subset X$ and let P be an abstract porosity on X satisfying* (A4).

(a) *If there exist P-porous sets P_n with $P_n \nearrow A$, then A is P-reducible.*

(b) *If A is closed and P-reducible, then there exist P-porous sets P_n with $P_n \nearrow A$.*

PROOF: It is sufficient to observe that condition (iii) from Proposition 5.1 clearly implies condition (iii) from Proposition 5.2. If A is closed, the converse implication is obvious. □

Remark 5.4. (1) If $X = \mathbb{R}$ and P is strong right porosity, then Corollary 5.3(b) does not hold for an arbitrary $A \subset X$ (and it seems that it also does not hold for all other interesting cases). Indeed, if $A \subset \mathbb{R}$ is a Bernstein set of the second category (see [3, Theorem 5.4]), then A is (k_d)-reducible (since it contains no perfect subset), but A is not a σ-strongly right porous set.

(2) Consider now the natural question, *whether Corollary* 5.3(b) *holds for Borel sets $A \subset X$.* The answer to this question is negative in some nontrivial cases and positive in others.

Let $X = \mathbb{R}$ and let $P(x, A)$ mean that $x \in \overline{\mathbb{R} \setminus \overline{A}}$. Then P-porous sets coincide with nowhere dense sets. Consequently, the set A of all irrationals is a Borel (G_δ) set which is clearly P-reducible but is not σ-P-porous. So the question has negative answer in this case.

However, the answer is positive, if $X = \mathbb{R}^n$ and P is lower porosity. It easily follows from [7] (note that in this case, for each σ-P-porous set A there exist P-porous sets P_n with $P_n \nearrow A$).

If $X = \mathbb{R}$ and P is strong right porosity, then the question coincides with Question 4.1. We also do not know the answer e.g. in the case if $X = \mathbb{R}$ (or $X = \mathbb{R}^n$) and P is the upper porosity on X.

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