Infinite dimensional linear groups with a large family of G-invariant subspaces

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Abstract. Let F be a field, A be a vector space over F, $\mathrm{GL}(F, A)$ be the group of all automorphisms of the vector space A. A subspace B is called almost G-invariant, if $\dim_F(B/\operatorname{Core}_G(B))$ is finite. In the current article, we begin the study of those subgroups G of $\mathrm{GL}(F, A)$ for which every subspace of A is almost G-invariant. More precisely, we consider the case when G is a periodic group. We prove that in this case A includes a G-invariant subspace B of finite codimension whose subspaces are G-invariant.

Keywords: vector space, linear groups, periodic groups, soluble groups, invariant subspaces

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Introduction

Let F be a field, A be a vector space over F, $\mathrm{GL}(F, A)$ be the group of all automorphisms of the vector space A. The group $\mathrm{GL}(F, A)$ and its subgroups are called linear groups. The linear groups play very important role not only in algebra but in many other branches of mathematics. If $\dim_F A$, the dimension of A over F, is finite, say is equal to n, then G is a finite dimensional linear group. It is then well-known that $\mathrm{GL}(F, A)$ can be identified with the group of $n \times n$ matrices with entries in F. The theory of finite dimensional linear groups is one of the best developed theories in algebra. It employs not only algebraic, but also topological, geometrical, combinatorial, and many other methods.

However, in the case when A has infinite dimension over F, the situation becomes totally different. This case is much more complicated and its consideration requires some additional restrictions allowing an effective employing of some already developed techniques. The most natural and suitable restrictions here is the G-invariance. The following example justifies this statement.

Let $G \le \mathrm{GL}(F, A)$ and suppose that every subspace B is G-invariant. In particular, for each element $a \in A$, the subspace aF is G-invariant. If g, x are the arbitrary elements of G, then $ag = \alpha a, ax = \beta a$ for some elements $\alpha, \beta \in F$. We have

$$a(gx) = (ag)x = (\alpha a)x = \alpha(ax) = \alpha(\beta a) = (\alpha\beta)a,$$

and similarly,

$$a(xg) = (\beta\alpha)a = (\alpha\beta)a.$$

Hence $a(gx) = a(xg)$ and $a[g, x] = a$. Since it is valid for each element $a \in A, [g, x] \in C_G(A) = \langle 1 \rangle$. So for this case, the group G must be abelian.

This example justifies that consideration of linear groups having a quite large family of G-invariant subspaces could be fruitful. In the current article, we consider one of such types of linear groups.

Let F be a field and $G \leq \mathrm{GL}(F, A)$. If B is a subspace of A, then the sum of an arbitrary family of G-invariant subspaces of B is G-invariant. It follows that B has the largest G-invariant subspace $\mathrm{Core}_G(B)$ which is called the *G-core of B*. We observe that G-core of B can be zero. A subspace B is called *almost G-invariant*, if $\dim_F(B/\mathrm{Core}_G(B))$ is finite.

This notion has the following group-theoretical analog. In the paper [BLNSW] the following type of subgroups was introduced. A subgroup H is called normal-by-finite, if the index $|H/\mathrm{Core}_G(H)|$ is finite. It was proved there that locally finite groups with all subgroups normal-by-finite is abelian-by-finite.

Note that every G-invariant subspace is almost G-invariant, however the following simple example shows that the converse statement is not true.

Let $F = \mathbb{F}_p$ be a prime field of order p and let A be a vector space over F with a basis $\{a, b_n \mid n \in \mathbb{N}\}$. Define a linear transformation g_n of A by the rule:

$$b_k g_n = b_k \text{ for all } k \in \mathbb{N}, \text{ and } ag_n = a + b_n, n \in \mathbb{N}.$$

Clearly $[g_n, g_m] = 1$ for all $n, m \in \mathbb{N}$ and $G = \langle g_n \mid n \in \mathbb{N}t \rangle = \mathrm{Dr}_{n\in\mathbb{N}}\langle g_n \rangle$ is an infinite elementary abelian p-subgroup of $\mathrm{GL}(F, A)$. Put $B = \bigoplus_{n\in\mathbb{N}} b_n F$, then $\dim_F(A/B) = 1$ and $B = C_A(G)$. Let C be an arbitrary vector space of A. Then $\dim_F(C/(C \cap B)) = 1$ and the subspace $C \cap B$ is G-invariant. Hence every subspace of A is almost G-invariant. But the subspace aF is not G-invariant.

In the current article we consider linear groups G for which every subspace is G-invariant. Observe that the following result is some analog of the main result of [BLNSW].

Theorem 1. *Let F be a field, A be a vector space over F, and G be a periodic subgroup of $\mathrm{GL}(F, A)$. If every subspace of A is almost G-invariant, then A includes an FG-submodule B such that $\dim_F(A/B)$ is finite and every subspace of B is G-invariant.*

Corollary 1 of Theorem 1. *Let F be a field, A be a vector space over F, and G be a periodic subgroup of $\mathrm{GL}(F, A)$. If every subspace of A is almost G-invariant, then G has a series of normal subgroups $E \leq H \leq G$ where G/H is a subgroup of the multiplicative group of the field F, H/E is a locally finite and finite dimensional linear group, and E is abelian. Moreover, if $\mathrm{char}(F) = p > 0$, then E is elementary abelian p-subgroup; if $\mathrm{char}(F) = 0$, then $E = \langle 1 \rangle$.*

Corollary 2 of Theorem 1. *Let F be a field, A be a vector space over F, and G be a periodic locally soluble subgroup of* $\mathrm{GL}(F, A)$*. Suppose that every subspace of A is almost G-invariant.*

(i) *If* $\mathrm{char}(F) = p > 0$*, then G includes a normal nilpotent bounded p-subgroup P such that G/P is an abelian-by-finite group of finite special rank.*
(ii) *If* $\mathrm{char}(F) = 0$*, then G is an abelian-by-finite group of finite special rank.*

Preliminary results

We begin with the following result.

Lemma 1. *Let F be a field, A be a vector space over F, and G be a subgroup of* $\mathrm{GL}(F, A)$*. Suppose that* $\mathrm{char}(F) = p$ *is a prime. If every subspace of A is almost G-invariant, then for every p-element $g \in G$ the subspace $C_A(g)$ has finite codimension.*

Proof: Let m be the order of an element g. Then $m = p^n$ for some positive integer n. Then we can consider A as J-module where $J = F\langle x\rangle/((x^m - 1)F\langle x\rangle)$ where the action of x on A is defined by the rule $ax = ag$ for each $a \in A$. Then J is an algebra of finite representation type (see for example, [PR, 7.1, Lemma]). It follows that $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ where A_λ is a cyclic uniserial module (see, for example, [DK, Chapter X, Theorem 1.1]). In other words,

$$A_\lambda = a_{\lambda 1}F \oplus a_{\lambda 2}F \oplus \cdots \oplus a_{\lambda d(\lambda)}F$$

where $d(\lambda) \leq n$ and

$$a_{\lambda 1}(g-1) = 0, a_{\lambda 2}(g-1) = a_{\lambda 1}, \ldots, a_{\lambda d(\lambda)}(g-1) = a_{\lambda d(\lambda)-1}.$$

Let

$$M = \{\lambda | \lambda \in \Lambda \text{ and } a_{\lambda 2} \neq 0\}.$$

Suppose that the subset M is infinite. Put $B = \bigoplus_{\lambda\in M} a_{\lambda 1}F$, $C = \bigoplus_{\lambda\in M} a_{\lambda 2}F$, and $D = B \oplus C$. The subspace C is almost G-invariant. Let $E = \mathrm{Core}_G(C)$. Then $\dim_F(C/E)$ is finite, in particular, E is non-zero. If a is a non-zero element of E, then

$$a = \alpha_1 a_{\lambda(1)\ 2} + \alpha_2 a_{\lambda(2)\ 2} + \cdots + \alpha_t a_{\lambda(t)\ 2}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_t \in F, \lambda(1), \lambda(2), \ldots, \lambda(t) \in M$. Observe that

$$a(g-1) = \alpha_1 a_{\lambda(1)\ 1} + \alpha_2 a_{\lambda(2)\ 1} + \cdots + \alpha_t a_{\lambda(t)\ 1} \neq 0$$

is a non-zero element of B. On the other hand, $a(g-1) \in E \leq C$, so that $a(g-1) \in B \cap C = \langle 0 \rangle$. Thus C cannot be almost G-invariant. This contradiction shows that the subset M is finite. Then

$$A = \bigoplus_{\lambda \in M} A_\lambda \oplus \bigoplus_{\lambda \in \Lambda \setminus M} A_\lambda.$$

It follows that the subspace $\bigoplus_{\lambda \in \Lambda \setminus M} A_\lambda$ has a finite codimension. Now we can observe that $\bigoplus_{\lambda \in \Lambda \setminus M} a_\lambda J \leq C_A(g)$. □

Lemma 2. *Let F be a field, A be a vector space over F, and G be a subgroup of $\mathrm{GL}(F, A)$. Let g be an element of G having finite order. Suppose that if $\mathrm{char}(F) = p > 0$, then g is a p'-element. If every subspace of A is almost G-invariant, then A includes an $F\langle g \rangle$-submodule B such that $\dim_F(A/B)$ is finite and every subspace of B is $\langle g \rangle$-invariant.*

Proof: We can consider A as an $F\langle g \rangle$-module. Then this module is semisimple (see, for example, [KOS, Corollary 5.15]). In other words, $A = \bigoplus_{\lambda \in \Lambda} B_\lambda$ where B_λ is a simple $F\langle g \rangle$-submodule for every $\lambda \in \Lambda$. Put $M = \{\lambda \in \Lambda \mid \dim_F(B_\lambda) > 1\}$. Lemma 2.1 of the paper [KSaSu] proves that the set M is finite. Then $\dim_F(B_\lambda) = 1$ for every $\lambda \in \Lambda \setminus M$, and therefore we can choose elements b_λ with the property $B_\lambda = b_\lambda F$ for every $\lambda \in \Lambda \setminus M$. Suppose now that there are two infinite subsets Δ, Σ of Λ such that $b_\lambda g = \gamma b_\lambda$ for every $\lambda \in \Delta$, $b_\lambda g = \eta b_\lambda$ for every $\lambda \in \Sigma$ and $\gamma \neq \eta$. Choose in Δ (respectively, in Σ) a countable subset $\{\delta(n) \mid n \in \mathbb{N}\}$ (respectively, $\{\sigma(n) \mid n \in \mathbb{N}\}$). Put $D_n = b_{\delta(n)}F \oplus b_{\sigma(n)}F$, $c_n = b_{\delta(n)} + b_{\sigma(n)}$, $n \in \mathbb{N}$. Then $c_n g = \lambda b_{\delta(n)} + \eta b_{\sigma(n)} \notin c_n F$ for all $n \in \mathbb{N}$. Lemma 2.4 of the paper [KSaSu] shows that the subspace $\bigoplus_{n \in \mathbb{N}} D_n$ includes a not almost G-invariant subspace. This contradiction shows that there exists a subset $\Xi \subseteq \Lambda \setminus M$ such that $(\Lambda \setminus M) \setminus \Xi$ is finite and $B_\lambda \cong_{F\langle g \rangle} B_\mu$ for all $\lambda, \mu \in \Xi$. Then $\Lambda \setminus \Xi$ is finite and every subspace of $B = \bigoplus_{\lambda \in \not\leq} B_\lambda$ is $\langle g \rangle$-invariant. The finiteness of $\Lambda \setminus \Xi$ implies that $\dim_F(A/B)$ is finite. □

Corollary 1 of Lemma 2. *Let F be a field, A be a vector space over F, and G be a periodic subgroup of $\mathrm{GL}(F, A)$. Suppose that g is an arbitrary element of G. If every subspace of A is almost G-invariant, then A includes an $F\langle g \rangle$-submodule B such that $\dim_F(A/B)$ is finite and every subspace of B is $\langle g \rangle$-invariant.*

Proof: If $\mathrm{char}(F) = 0$, then this assertion follows directly from Lemma 2. Suppose that $\mathrm{char}(F) = p$ is a prime. We have $g = xy$ where $[x, y] = 1$, x is a p-element and y is a p'-element. Put $C = C_A(x)$. Then C is $\langle g \rangle$-invariant. By Lemma 1, $\dim_F(A/C)$ is finite. Since C is almost G-invariant, C includes an FG-submodule E such that $\dim_F(C/E)$ is finite. Then $\dim_F(A/E)$ is also finite. By Lemma 2, E includes an $F\langle g \rangle$-submodule B such that $\dim_F(E/B)$ is finite and every subspace of B is $\langle y \rangle$-invariant. The inclusion $E \leq C_A(x)$ implies that every subspace of B is $\langle g \rangle$-invariant. Clearly $\dim_F(A/B)$ is finite. □

Corollary 2 of Lemma 2. *Let F be a field, A be a vector space over F and G be a periodic subgroup of $\mathrm{GL}(F, A)$. If every subspace of A is almost G-invariant, then G is locally finite.*

Proof: Let K be an arbitrary finitely generated subgroup of G, $K = \langle g_1, \dots, g_n\rangle$. Corollary 1 of Lemma 2 shows that for every element g_j the space A includes a subspace B_j such that $\dim_F(A/B_j)$ is finite and every subspace of B_j is $\langle g_j\rangle$-invariant, $1 \le j \le n$. Put

$$B = B_1 \cap \dots \cap B_n.$$

Then $\dim_F(A/B)$ is finite, and every subspace of B is K-invariant. By Lemma 3.4 of the paper [KSaSu], $K/C_K(B)$ is isomorphic to a subgroup of $\mathbf{U}(F)$. In particular, $K/C_K(B)$ is abelian. Being periodic and finitely generated, $K/C_K(B)$ is finite. Put $H = C_K(B)$. We remind that every subgroup of finitely generated group having finite index is also finitely generated (see, for example, [RD, Theorem 1.41]). Therefore H is also finitely generated, and $B \le C_A(H)$, so that $\dim_F(A/C_A(H)) = m$ is finite. Thus we can consider $H/C_H(A/C_A(H))$ as a subgroup of $\mathrm{GL}_m(F)$. We observe now that the periodic subgroup of $\mathrm{GL}_m(F)$ is locally finite (see, for example, [WB, Corollary 4.8]). If follows that $H/C_H(A/C_A(H))$ is finite, so that $C_H(A/C_A(H))$ is finitely generated. It is not hard to prove that this subgroup is abelian, and therefore it is finite. Hence, the entire subgroup K is finite. □

Proof the main theorem and its corollaries

Proof of Theorem 1: If $ag \in aF$ for every elements $g \in G, a \in A$, then all is proved. Suppose that there exist the elements $g_1 \in G$ and $a_1 \in A$ such that $a_1(g_1 - 1) = b_1 \notin a_1F$. If follows that $a_1F + b_1F = a_1F \oplus b_1F$, so that $\dim_F(a_1F + b_1F) = 2$.

By Corollary 1 of Lemma 2, A includes an $F\langle g_1\rangle$-submodule E_1 such that $\dim_F(A/E_1)$ is finite and every subspace of E_1 is $\langle g_1\rangle$-invariant. Without loss of generality, we can suppose that $(a_1F + b_1F) \cap E_1 = \{0\}$. Being almost G-invariant, E_1 includes an FG-submodule L_1 such that $\dim_F(E_1/L_1)$ is finite. Then $\dim_F(A/L_1)$ is finite and every subspace of L_1 is $\langle g_1\rangle$-invariant.

If $ag \in aF$ for every elements $g \in G, a \in L_1$, then we put $B = L_1$ and all is proved. Suppose that there exist elements $g_2 \in G$ and $a_2 \in L_1$ such that $a_2(g_2 - 1) = b_2 \notin a_2F$. If follows that $a_2F + b_2F = a_2F \oplus b_2F$, so that $\dim_F(a_2F + b_2F) = 2$. As above, we can choose an FG-submodule L_2 of L_1 such that $\dim_F(A/L_2)$ is finite, $(a_2F + b_2F) \cap L_2 = \{0\}$, and every subspace of L_2 is $\langle g_2\rangle$-invariant. By the choice of L_1, every subspace of L_2 is also $\langle g_1\rangle$-invariant.

If $ag \in aF$ for every elements $g \in G$, $a \in L_2$, then we put $B = L_2$ and all is proved. If not, we will continue this process. We have two possibilities: (i) this process will finish after finitely many steps; and (ii) this process is infinite. In the first case we obtain an FG-submodule B such that $\dim_F(A/B)$ is finite and every subspace of B is G-invariant. Consider the second case. Then we obtain an infinite subset $\{g_n \mid n \in \mathbb{N}\}$ of elements of G and the infinite subset $\{a_n \mid n \in \mathbb{N}\}$ of elements of A satisfying the following conditions:

(i) $a_n(g_n - 1) = b_n$;
(ii) $a_nF + b_nF = a_nF \oplus b_nF$;
(iii) $(a_nF \oplus b_nF) \cap \bigoplus_{1 \le k \le n-1}(a_kF \oplus b_kF) = \{0\}$;
(iv) $a_ng_k \in a_nF$, $b_ng_k \in b_nF$ whenever $k < n$, $n, k \in \mathbb{N}$.

Let $C = \bigoplus_{j \in \mathbb{N}} a_jF$, $D = \bigoplus_{j \in \mathbb{N}} b_jF$. Then $C \cap D = \langle 0 \rangle$. Let $Z = \mathrm{Core}_G(C)$. Then $\dim_F(C/Z)$ is finite, in particular, Z is non-zero. The inclusion $Z \le C$ implies that $Z \cap D = \langle 0 \rangle$. Let a be a non-zero element of Z. Then $a = \alpha_1 a_{k(1)} + \alpha_2 a_{k(2)} + \cdots + \alpha_t a_{k(t)}$ for some positive integers $k(1) < \cdots < k(t)$, and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are the non-zero elements of F. We have

$$\begin{aligned} a(g_{k(1)} - 1) &= (\alpha_1 a_{k(1)} + \alpha_2 a_{k(2)} + \cdots + \alpha_t a_{k(t)})(g_{k(1)} - 1) \\ &= \alpha_1 a_{k(1)}(g_{k(1)} - 1) + \cdots + \alpha_t a_{k(t)}(g_{k(1)} - 1) \\ &= \alpha_1 b_{k(1)} + \beta_2 a_{k(2)} + \cdots + \beta_t a_{k(t)}. \end{aligned}$$

Since $\alpha_1 \ne 0$, $\alpha_1 b_{k(1)}$ is a non-zero element of D. On the other hand, $\beta_2 a_{k(2)} + \cdots + \beta_t a_{k(t)} \in C$, so that $\alpha_1 b_{k(1)} + \beta_2 a_{k(2)} + \cdots + \beta_t a_{k(t)} \notin C \ge Z$. Hence in the case (ii) we obtain a contradiction, which proves the result. □

Proof of Corollary 1 of Theorem 1: By Theorem 1, A includes an FG-submodule B of finite codimension such that every subspace of B is G-invariant. By Lemma 3.4 of the paper [KSaSu], $G/C_G(B)$ is isomorphic to a subgroup of $\mathbf{U}(F)$. Put $H = C_G(B)$. Then $B \le C_A(H)$, so that $\dim_F(A/C_A(H)) = m$ is finite. Thus we can consider $H/C_H(A/C_A(H))$ as a subgroup of $\mathrm{GL}_m(F)$. We observe that the periodic subgroup of $\mathrm{GL}_m(F)$ are locally finite (see, for example, [WB, Corollary 4.8]), so that $H/C_H(A/C_A(H))$ is locally finite. Put $E = C_H(A/C_A(H))$. It is not hard to prove that this subgroup is abelian, and moreover, if $\mathrm{char}(F) = p > 0$, then E is an elementary abelian p-subgroup; if $\mathrm{char}(F) = 0$, then $E = \langle 1 \rangle$. □

For the case when a group G is locally soluble, we can obtain a significantly more detail description of the structure of G.

Let G be a group. We recall that G *has finite special rank* $r(G) = r$, if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property.

Proof of Corollary 2 of Theorem 1: By Theorem 1, A includes an FG-submodule B of finite codimension such that every subspace of B is G-invariant. Then A has a finite series of an FG-submodules

$$\langle 0 \rangle = B_0 \le B = B_1 \le B_2 \le \cdots \le B_n = A$$

such that B_{j+1}/B_j are finite dimensional simple FG-modules, $1 \le j \le n-1$. By Lemma 3.4 of the paper [KSaSu], $G/C_G(B)$ is isomorphic to a subgroup of

$\mathbf{U}(F)$. By Lemma 3.5 of the paper [KSaSu], $G/C_G(B_{j+1}/B_j)$ is an abelian-by-finite group of finite special rank. Let $P = \bigcap_{0\leq j\leq n} C_G(B_{j+1}/B_j)$. By Remak's Theorem (see, for example, [KM, Theorem 4.3.9]), we obtain an embedding

$$G/P \hookrightarrow G/C_G(B_1/B_0) \times \cdots \times G/C_G(B_n/B_{n-1}),$$

which shows that G/P is an abelian-by-finite group of finite special rank. Finally, every element of P acts trivially on every factor B_{j+1}/B_j, so that P is a nilpotent subgroup, and moreover, it is a bounded p-subgroup if $\mathrm{char}(F) = p > 0$, and P is torsion-free if $\mathrm{char}(F) = 0$ (see, for example, [KW, Theorem 1.C.1 and Proposition 1.C.3] and [FL, Section 8]). Since G is periodic, in the last case, $P = \langle 1\rangle$, and all is proved. □

Finally we note that the condition

(A) *every subspace of a vector space A is almost G-invariant*

is equivalent to the condition

(B) *every subspace of A having infinite dimension, includes a non-zero G-invariant subspace.*

Indeed, it is clear that (A) implies (B). Conversely, assume that a vector space A satisfies (B). Suppose that A includes a subspace D which is not almost G-invariant. Clearly $\dim_F(D)$ is infinite. Let $K = \mathrm{Core}_G(D)$. By (B), K is non-zero. Since D is not almost G-invariant, $\dim_F(D/K)$ is infinite. There exists a subspace L such that $D = K \oplus L$. Since L has infinite dimension, L includes a non-zero G-invariant subspace T. Then $K + T$ is a G-invariant subspace of D, and $K + T \neq K$. So we obtain a contradiction with the choice of K. This contradiction proves that every subspace of A is almost G-invariant.

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