Comment.Math.Univ.Carolin. 51,4 (2010)559–563

On finite commutative IP-loops with elementary abelian inner mapping groups of order p^4

Markku Niemenmaa

Abstract. We show that finite commutative inverse property loops with elementary abelian inner mapping groups of order p^4 are centrally nilpotent of class at most two.

Keywords: loop, inner mapping group

Classification: 20D10, 20N05

1. Introduction

If Q is a loop, then the two mappings $L_a(x) = ax$ (left translation) and $R_a(x) = xa$ (right translation) are permutations on Q for every $a \in Q$. The permutation group $M(Q) = \langle L_a, R_a : a \in Q\rangle$ is called the multiplication group of Q, the stabilizer of the neutral element e of Q is denoted by $I(Q)$ and we say that $I(Q)$ is the inner mapping group of Q. If Q is a group, then $I(Q) = \mathrm{Inn}(Q)$, the group of inner automorphisms of Q. Note that $I(Q) = 1$ if and only if Q is an abelian group.

The centre $Z(Q)$ of a loop Q contains all elements a which satisfy the conditions: $ax = xa$, $(ax)y = a(xy)$, $(xa)y = x(ay)$ and $(xy)a = x(ya)$ for every $x, y \in Q$. Clearly, $Z(Q)$ is an abelian group. If we write $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, then we obtain a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q and $Z_n = Q$, then Q is said to be centrally nilpotent of class n. Bruck [1] observed that if Q is centrally nilpotent of class at most two, then $I(Q)$ is an abelian group. Kepka and Niemenmaa [6], [7] managed to show that if Q is a finite loop and $I(Q)$ is abelian, then Q is a centrally nilpotent loop. However, they did not give any bound for the nilpotency class of Q. In 2007 Csörgö [3] showed that the converse of Bruck's result does not hold in general: she constructed (by using connected transversals) a loop of order 128 and nilpotency class three with an abelian inner mapping group. In Csörgö's example $I(Q)$ is an elementary abelian group of order 2^6. In 2008 Drápal and Vojtěchovský [4] gave several combinatorial constructions of loops of nilpotency class three with inner mapping groups which are elementary abelian of order 2^6. Earlier results by Csörgö, Kepka and Niemenmaa [2], [7] indicate that if $I(Q)$ is elementary abelian of order p^2 or p^3, then Q is centrally nilpotent of class at most two. What happens if $I(Q)$ is elementary abelian of order p^4 or p^5? The purpose of this paper is to investigate this question in the case that Q is a finite commutative IP-loop and $I(Q)$ is elementary abelian of order p^4.

A loop Q is said to be an inverse property loop (in short, IP-loop) if Q has a unique left and right inverse x^{-1} and $x^{-1}(xy) = y = (yx)x^{-1}$ for every $x, y \in Q$. The smallest IP-loop that is not a group is of order 7. In the main result of this paper we show that finite commutative IP-loops with elementary abelian inner mapping groups of order p^4 are centrally nilpotent of class at most two.

In the before mentioned construction Csörgö [3] was using the technique of connected transversals and we also formulate our results first in terms of group theory by using connected transversals. The loop theoretical results then follow as direct corollaries. Section 2 thus contains some basic facts about connected transversals and related group theoretical results. In Section 3 we prove our main result on the nilpotency class of finite commutative IP-loops by using connected transversals.

In this paper all groups and loops are finite. For basic facts about loop theory and its connections to group theory, the reader is advised to consult [1], [5] and [10].

2. Connected transversals

Let G be a group and $H \leq G$. By H_G we denote the core of H in G (the largest normal subgroup of G contained in H). If A and B are two left transversals to H in G and $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and for every $b \in B$, then we say that these two transversals are H-connected in G. In fact, if A and B are H-connected transversals, then A and B are both left and right transversals to H in G ([5, Lemma 2.1]). If $A = B$, then we say that A is a selfconnected transversal to H in G. In the sequel we assume that the subgroup H has connected transversals A and B in G.

Lemma 2.1. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

Lemma 2.2. *If H is abelian and $H_G = 1$, then the core of $HZ(G)$ in G contains $Z(G)$ as a proper subgroup.*

For the proofs, see [5, Proposition 2.7] and [9, Lemma 2.7].

Lemma 2.3. *If $H_G = 1$, then $AZ(G) \subseteq A$ (and $BZ(G) \subseteq B$).*

Proof: Let $az = bh$, where $a, b \in A$, $z \in Z(G)$ and $h \in H$. If $c \in B$, then $(az)^c = (bh)^c$ and thus $ak_1z = bk_2h^c$, where $k_1, k_2 \in H$. It follows that $ak_1z = azh^{-1}k_2h^c$ and $h^c = k_2{}^{-1}hk_1 \in H$. Thus $h \in \bigcap_{c\in B} H^{c^{-1}} = H_G$, which means that $h = 1$ and $az = b \in A$. $\square$

In the following three theorems we further assume that $G = \langle A, B\rangle$. As usual, p denotes a prime number and C_n denotes a cyclic group of order n.

Theorem 2.4. *If H is cyclic, then $G' \leq H$.*

Theorem 2.5. *If $H \cong C_p \times C_p$ or $H \cong C_p \times C_p \times C_p$, then $G' \leq N_G(H)$.*

Theorem 2.6. *If H is an abelian p-group and $H_G = 1$, then $Z(G) > 1$.*

For the proofs, see [5, Theorem 3.5], [7, Lemma 4.2] and [2, Theorem 3.7] and [10, Theorem 3.1].

Connected transversals appear in loop theory in the following way: If $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$, then A and B are $I(Q)$-connected transversals in $M(Q)$. As $M(Q)$ is transitive on Q, it follows that the core of $I(Q)$ in $M(Q)$ is trivial. The following characterization theorem was proved by Kepka and Niemenmaa [5, Theorem 4.1] in 1990.

Theorem 2.7. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H of G satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

It should be noted that in the role of the loop elements we have the left cosets of H and the subgroup H is isomorphic to the inner mapping group of the loop. Many of the results mentioned in the introduction (on the relation between the structures of $M(Q)$, $I(Q)$ and Q) were proved by using group theoretical arguments and connected transversals. If Q is a commutative loop, then $A = B$ and if, in addition, Q is an IP-loop, then ${L_a}^{-1} = L_{a^{-1}}$ for every $a \in Q$, hence $A = A^{-1}$.

3. Main results

We shall now consider the situation where $A = B$, $A = A^{-1}$ and H is elementary abelian of order p^4.

Theorem 3.1. *Let H be an elementary abelian subgroup of order p^4 of a finite group G and let A be a selfconnected transversal to H in G. If $G = \langle A\rangle$ and $A = A^{-1}$, then $G' \leq N_G(H)$.*

Proof: We proceed by induction on the order of G. If $H_G > 1$, then we consider G/H_G, H/H_G and AH_G/H_G and the claim follows from Theorems 2.4 and 2.5. Thus we may assume that $H_G = 1$. By Theorem 2.6, it follows that $Z(G) > 1$ and from Lemma 2.1 we get $N_G(H) = H \times Z(G)$. Furthermore, from Lemma 2.2 it follows that the core of $HZ(G)$ in G is equal to $KZ(G)$, where $1 < K \leq H$. If $|K| \geq p^3$, then we consider $G/KZ(G)$, $HZ(G)/KZ(G)$ and $AKZ(G)/KZ(G)$. From Theorem 2.4 it follows that $(G/KZ(G))' \leq HZ(G)/KZ(G)$, hence $G' \leq HZ(G) = N_G(H)$. Thus we may assume that $|K| = p$ or $|K| = p^2$. By applying Theorem 2.5 and Lemma 2.1 on $G/KZ(G)$, $HZ(G)/KZ(G)$ and $AKZ(G)/KZ(G)$, we see that $(G/KZ(G))' \leq N_{G/KZ(G)}(HZ(G)/KZ(G)) = HZ(G)/KZ(G) \times Z(G/KZ(G))$. If we write $Z(G/KZ(G)) = M/KZ(G)$, then $G' \leq N_G(HZ(G)) = HM$. Here M is a normal subgroup of G and $M \cap HZ(G) = KZ(G)$. Thus

$M = (A \cap M)KZ(G) = (A \cap M)K$ (see Lemma 2.3). We now divide the proof into two parts:

1) Assume first that $K \cong C_p$. Let $a \in A \cap M$ and $b \in A$ and write $ab = ch$, where $c \in A$ and $h \in H$. If $t \in A$, then $a^t b^t = c^t h^t$. Thus $akbu = cwh^t = abh^{-1}wh^t$, where $k \in K$ and $u, w \in H$. It follows that $h^t = w^{-1}hk^b u$. As $KZ(G)$ is normal in G, it follows that $k^b \in KZ(G)$, hence $h^t = w^{-1}hk^b u \in HZ(G)$. We conclude that $h^t \in HZ(G)$ for every $t \in A$. This means that $h = c^{-1}ab$ belongs to the core of $HZ(G)$ in G which is equal to $KZ(G)$ and thus $h \in K \cong C_p$. Now $h^{b^{-1}} = (c^{-1}ab)^{b^{-1}} = bc^{-1}b^{-1}cc^{-1}ba = bc^{-1}b^{-1}cc^{-1}abl$, where $l \in H$. Since $A = A^{-1}$, it follows that $h^{b^{-1}} \in H \cap KZ(G) = K \cong C_p$. If $h \neq 1$, then $b \in N_G(K)$ and $h^t = w^{-1}hk^b u \in H$. As $h^t \in H$ for every $t \in A$, we get $h \in H_G = 1$, a contradiction.

Thus we know that $h = 1$ and $ab = c \in A$ for every $a \in A \cap M$ and for every $b \in A$. As $A = A^{-1}$, we also have $a^{-1}b^{-1} = d \in A$. Then $a^{-1}b^{-1}ab = dc \in H$, which means that $c \in d^{-1}H$, hence $c = d^{-1} \in A$. But then $a^{-1}b^{-1}ab = 1$ and we conclude that $a \in Z(\langle A \rangle) = Z(G)$. Thus $N_G(HZ(G)) = HM = HZ(G)$ and $G' \leq HZ(G) = N_G(H)$.

2) Then assume that $K \cong C_p \times C_p$. Let $a, b \in A$ and $ab = ch$, where $c \in A$ and $h \in H$. If $d \in A$, then $h^d = (c^{-1}ab)^d = h_1 c^{-1} a h_2 b h_3 = h_1 h b^{-1} h_2 b h_3 \in HH^bH$ (here $h_1, h_2, h_3 \in H$). Now $HZ(G)$ is normal in HM and as $H^b \leq HM$, we get $HH^bH \subseteq HZ(G)H^b \leq G$. Thus $h \in (HZ(G)H^b)^{d^{-1}}$ for every $d \in A$, hence $h \in \bigcap_{g \in G} [HZ(G)H^b]^g$. This intersection is a normal subgroup of G and we denote it by $N(b)$. Clearly, $N(b) \geq KZ(G)$ for every $b \in A$ and $ab \in A[N(b) \cap H]$. If $N(b) \cap H = K$ for every $b \in A$, then $A^2 \subseteq AK$. By Lemma 2.3, $AKZ(G) = AZ(G)K = AK$ and we may conclude that AK is a proper subgroup of G. But then $\langle A \rangle \leq AK < G$, a contradiction.

Thus we may assume that there exists $d \in A$ such that $N(d) \cap H > K$. As $K \cong C_p \times C_p$, we have $|N(d) \cap H| \geq p^3$. We now consider $G/N(d)$, $HN(d)/N(d)$ and $AN(d)/N(d)$. From Theorem 2.4 it follows that $G' \leq HN(d) \leq HZ(G)H^d$. This means that $HZ(G)H^d$ and $Z(HZ(G)H^d)$ are normal subgroups of G. As $Z(HZ(G)H^d)$ is a subgroup of $N_G(H) = HZ(G)$, we conclude that $Z(HZ(G)H^d) \leq KZ(G)$. If $a \in A$, then $da^{-1}d^{-1}a \in H$, since $A = A^{-1}$. Thus $a^{-1}d^{-1}ad \in H \cap H^d$ and therefore $a^{-1}d^{-1}ad \in Z(HZ(G)H^d) \leq KZ(G)$ for every $a \in A$. As $G = \langle A \rangle$, it follows that $dKZ(G) \in Z(G/KZ(G))$, hence $H^d \leq HZ(G)$. But then $G' \leq HZ(G)H^d = HZ(G) = N_G(H)$ and the proof is complete. $\square$

If Q is a loop and $M(Q)' \leq N_{M(Q)}(I(Q)) = I(Q) \times Z(M(Q))$, then Q is centrally nilpotent of class at most two (see [1], also [10, Theorem 6.5]). Thus by combining Theorem 3.1 with Theorem 2.7 we get

Theorem 3.2. *Let Q be a finite commutative IP-loop and let $I(Q)$ be an elementary abelian group of order p^4. Then Q is centrally nilpotent of class at most two.*

Remark. If Q is a commutative loop and $x(xy) = y$ for every $x, y \in Q$, then Q is a Steiner loop. It is not hard to show that Steiner loops are precisely IP-loops of exponent two. Kinyon [8] has informed the author of this paper that Steiner loops with abelian inner mapping groups are centrally nilpotent of class at most two.

References

[1] Bruck R.H., *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.
[2] Csörgö P., *On connected transversals to abelian subgroups and loop theoretical consequences*, Arch. Math. (Basel) **86** (2006), no. 6, 499–516.
[3] Csörgö P., *Abelian inner mappings and nilpotency class greater than two*, European J. Combin. **28** (2007), no. 3, 858–867.
[4] Drápal A., Vojtěchovský P., *Explicit constructions of loops with commuting inner mappings*, European J. Combin. **29** (2008), no. 7, 1662–1681.
[5] Kepka T., Niemenmaa M., *On multiplication groups of loops*, J. Algebra **135** (1990), 112–122.
[6] Kepka T., Niemenmaa M., *On connected transversals to abelian subgroups in finite groups*, Bull. London Math. Soc. **24** (1992), 343–346.
[7] Kepka T., Niemenmaa M., *On connected transversals to abelian subgroups*, Bull. Australian Math. Soc. **49** (1994), 121–128.
[8] Kinyon M., *private communication*, 2009.
[9] Niemenmaa M., *On finite loops whose inner mapping groups are abelian II*, Bull. Australian Math. Soc. **71** (2005), 487–492.
[10] Niemenmaa M., Rytty M., *Connected transversals and multiplication groups of loops*, Quasigroups and Related Systems **15** (2007), 95–107.

Department of Mathematical Sciences, University of Oulu, PL 3000, 90014 Oulu, Finland

E-mail: mniemen@cc.oulu.fi

(*Received* December 2, 2009, *revised* August 24, 2010)