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Steady compressible Navier–Stokes–Fourier system in two space dimensions

Petra Pecharová, Milan Pokorný

Abstract. We study steady flow of a compressible heat conducting viscous fluid in a bounded two-dimensional domain, described by the Navier–Stokes–Fourier system. We assume that the pressure is given by the constitutive equation $p(\rho,\theta) \sim \rho^\gamma + \rho\theta$, where ρ is the density and θ is the temperature. For $\gamma > 2$, we prove existence of a weak solution to these equations without any assumption on the smallness of the data. The proof uses special approximation of the original problem, which guarantees the pointwise boundedness of the density. Thus we get a solution with density in $L^\infty(\Omega)$ and temperature and velocity in $W^{1,q}(\Omega)$ for any $q < \infty$.

Keywords: steady compressible Navier–Stokes–Fourier equations, slip boundary condition, weak solutions, large data

Classification: 35Q30, 76N10

1. Introduction

The compressible Navier–Stokes–Fourier system of PDEs (a.k.a. the full Navier-Stokes system) describes flow of a compressible heat conducting newtonian fluid in a bounded domain Ω, and has therefore many applications in natural sciences and engineering (such as meteorology, geophysics, astrophysics, heat transfer in multi-phase flows in engineer models etc.). The effort to prove existence results for the problem comes from the need to justify the models as well as to ensure physical and thermodynamical properties of studied models.

We consider the steady flow of a newtonian compressible heat conducting fluid in a bounded domain $\Omega \subset \mathbb{R}^2$. It is described by

$$\operatorname{div}(\rho\boldsymbol{v}) = 0 \tag{1.1}$$

$$\operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathbf{S}(\boldsymbol{v}) + \nabla p(\rho,\theta) = \rho\boldsymbol{F} \tag{1.2}$$

$$\operatorname{div}(\rho e(\rho,\theta)\boldsymbol{v}) - \operatorname{div}(\kappa(\theta)\nabla\theta) = \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - p(\rho,\theta)\operatorname{div}\boldsymbol{v}, \tag{1.3}$$

where

$$\begin{aligned} \rho : \Omega &\to \mathbb{R}^+_0 \ \ldots \text{density of the fluid (sought)}\\ \boldsymbol{v} : \Omega &\to \mathbb{R}^2 \ \ldots \text{velocity field (sought)}\\ \theta : \Omega &\to \mathbb{R}^+ \ \ldots \text{temperature (sought)} \end{aligned}$$

$$
\begin{aligned}
p(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{R}^+ &\rightarrow \mathbb{R}_0^+ \ldots \text{pressure (given)}\\
\boldsymbol{F} : \Omega &\rightarrow \mathbb{R}^2 \ldots \text{external force (given)}\\
e(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{R}^+ &\rightarrow \mathbb{R}_0^+ \ldots \text{internal energy (given)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}(\boldsymbol{v}) &= 2\mu\mathbf{D}(\boldsymbol{v}) + \lambda(\operatorname{div}\boldsymbol{v})\mathbf{I} \ldots \text{the viscous part of the stress tensor}\\
\mathbf{D}(\boldsymbol{v}) &= \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T) \ldots \text{the symmetric part of the velocity gradient,}
\end{aligned}
$$

μ, λ are viscosity coefficients (constants) and $\kappa(\theta)$ is heat conductivity.

Note that in full generality equation (1.3) (conservation of internal energy) should be replaced by the conservation of total energy — but for the solution we are about to construct we have balance of kinetic energy as a consequence of the momentum equation[1].

We assume that the constitutive equation for the pressure takes form

$$p(\rho,\theta) = a_1\rho^\gamma + a_2\rho\theta, \ a_1, a_2 > 0, \tag{1.4}$$

i.e. the pressure has one part corresponding to the ideal fluid and a so called elastic part. The internal energy takes form

$$e(\rho,\theta) = a_3\theta + a_1\frac{\rho^{\gamma-1}}{\gamma-1}, \ a_3 > 0. \tag{1.5}$$

Next we specify the viscosity coefficients and heat conductivity. We assume μ, λ to be constants satisfying

$$\mu > 0, \ \lambda + \mu > 0 \tag{1.6}$$

and κ such that

$$\kappa(\theta) = a_4(1 + \theta^m), \ a_4, m > 0. \tag{1.7}$$

In what follows, to simplify the notation, we set $a_1 = a_2 = a_3 = a_4 = 1$.

Our solution will be such that $\rho \in L^\infty(\Omega)$ and $\boldsymbol{v} \in W^{1,p}(\Omega)$ for $p < \infty$. Hence we get ($\operatorname{div}(\rho\boldsymbol{v}) = 0$ in the weak sense)

$$\operatorname{div}\left(\frac{1}{\gamma-1}\rho^\gamma\boldsymbol{v}\right) = -\rho^\gamma \operatorname{div}\boldsymbol{v}$$

in the weak sense — so we are allowed to write

$$\operatorname{div}(\rho\theta\boldsymbol{v}) - \operatorname{div}(\kappa(\theta)\nabla\theta) = \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - \rho\theta\operatorname{div}\boldsymbol{v} \tag{1.8}$$

instead of equation (1.3).

[1]To be more specific, we will be able to test the momentum equation by $\boldsymbol{v}$, which gives us also conservation of kinetic energy.

Note that our choice of constitutive equations is compatible with the second law of thermodynamics, i.e. the pressure and the temperature fulfill the relation

$$\frac{1}{\rho^2}\Big(p - \theta\frac{\partial p}{\partial \theta}\Big) = \frac{\partial e}{\partial \rho}\,, \tag{1.9}$$

which is a straight consequence of the Maxwell relation and it guarantees the existence of entropy; the interested reader should consult the book [10].

Let the domain we are working with be sufficiently smooth, i.e. $\Omega \in \mathcal{C}^2$. For the velocity we consider slip boundary conditions

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{\tau}\cdot(\mathbf{T}(p,\boldsymbol{v})\boldsymbol{n}) + f\boldsymbol{v}\cdot\boldsymbol{\tau} = 0 \qquad \text{at } \partial\Omega, \tag{1.10}$$

where $\boldsymbol{\tau}$ stands for the tangent vector to $\partial\Omega$, $\boldsymbol{n}$ is the outer normal vector, $\mathbf{T}(p,\boldsymbol{v}) = -p\mathbf{I} + \mathbf{S}(\boldsymbol{v})$ is the stress tensor and coefficient f is a nonnegative constant. (In case $f = 0$ — perfect slip — we need to assume that Ω is not axially symmetric.[2])

For the temperature we assume

$$\kappa(\theta)\frac{\partial\theta}{\partial\boldsymbol{n}} + L(\theta)(\theta - \theta_0) = 0 \;\text{ at }\; \partial\Omega, \tag{1.11}$$

where $\theta_0 : \partial\Omega \to \mathbb{R}^+$ is a strictly positive sufficiently smooth given function (say $\theta_0 \in \mathcal{C}^2$), $0 < \theta_* \le \theta_0 \le \theta^* < \infty$ with $\theta_*, \theta^* \in \mathbb{R}^+$ and

$$L(\theta) = a_5(1+\theta^l), \qquad l \in \mathbb{R}^+_0. \tag{1.12}$$

We also prescribe the total mass of the fluid

$$\int_\Omega \rho\, dx = M > 0. \tag{1.13}$$

2. Main result

Definition 2.1. The triple $(\rho,\boldsymbol{v},\theta)$ is called a weak solution to problem (1.1)–(1.13) if $\rho \ge 0$ a.e. in Ω, $\rho \in L^{\max\{3,\gamma\}}(\Omega)$, $\boldsymbol{v} \in W^{1,2}(\Omega)$, $\theta > 0$ a.e. in Ω, $\theta \in W^{1,2}(\Omega)$ and $\theta^m\nabla\theta \in L^1(\Omega)$, $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ at $\partial\Omega$ in a trace sense, and

$$\int_\Omega \rho\boldsymbol{v}\cdot\nabla\eta = 0 \qquad \forall\, \eta \in \mathcal{C}^1(\overline{\Omega}) \tag{2.1}$$

$$\begin{aligned} &\int_\Omega \Big(-\rho\boldsymbol{v}\otimes\boldsymbol{v} : \nabla\boldsymbol{\varphi} + 2\mu\mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi}) + \lambda \operatorname{div}\boldsymbol{v}\operatorname{div}\boldsymbol{\varphi} - p(\rho,\theta)\operatorname{div}\boldsymbol{\varphi}\Big) dx \\ &\quad + f\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{\varphi}\, d\sigma = \int_\Omega \rho\boldsymbol{F}\cdot\boldsymbol{\varphi}\, dx \qquad \forall\, \boldsymbol{\varphi} \in \mathcal{C}^1(\overline{\Omega}); \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \;\text{ at }\; \partial\Omega \end{aligned} \tag{2.2}$$

[2]We need this condition to be fulfilled because of use of Korn's lemma at several moments of the proof; the Korn inequality requires also (1.6).

$$\begin{aligned} &\int_\Omega (\kappa(\theta)\nabla\theta\cdot\nabla\psi - \rho\theta\boldsymbol{v}\cdot\nabla\psi)\,dx + \int_{\partial\Omega} L(\theta)(\theta-\theta_0)\psi\,d\sigma \\ &= \int_\Omega (2\mu|\mathbf{D}(\boldsymbol{v})|^2\psi + \lambda(\operatorname{div}\boldsymbol{v})^2\psi - \rho\theta\operatorname{div}\boldsymbol{v}\psi)\,dx \qquad \forall\,\psi\in\mathcal{C}^1(\overline{\Omega}). \end{aligned} \tag{2.3}$$

Now we are ready to state our main result:

Theorem 2.1. *Let $\Omega\in\mathcal{C}^2$ be a bounded domain, $\Omega\subset\mathbb{R}^2$. Let $\boldsymbol{F}\in L^\infty(\Omega)$, $m=l+1$ and*

$$\gamma>2, \qquad m>\frac{\gamma-1}{\gamma-2}\,.$$

Then there exists a weak solution to (1.1)–(1.13) *such that $\rho\in L^\infty(\Omega)$, $\boldsymbol{v}\in W^{1,q}(\Omega)$ and $\theta\in W^{1,q}(\Omega)$ for all $1\le q<\infty$.*

Note that Theorem 2.1 could be proved also for $m\neq l+1$ and $\boldsymbol{F}\in L^p(\Omega)$ for $p<\infty$ (under suitable assumptions on the relations between m, l, p and γ); however, the details of the proof would be much more complicated than in our "simple" case. More details can be found in [11].

The aim of this paper is to prove Theorem 2.1. We will continue in work of Mucha and Pokorný ([6] and [13] — 2D and 3D Navier–Stokes equations in barotropic regime, no temperature or internal energy is considered here; [7] and [8] — 3D Navier–Stokes–Fourier equations, temperature and equation for internal or total energy included). The authors were able to prove the existence for $\gamma>3$ and $m>\frac{3\gamma-1}{3\gamma-7}$ in [7] and $\gamma>\frac{7}{3}$ and $m>\frac{3\gamma-1}{3\gamma-7}$ in [8] (in both $m=l+1$); as announced above, we get the existence for $\gamma>2$ and $m>\frac{\gamma-1}{\gamma-2}$, which is, in some way, better result than in 3D, but far from the case in 2D without temperature ([6]).

One of the possible approaches to problem (1.1)–(1.13) was introduced in [4]; however, to overcome the difficulties with the lack of a priori estimates, the author considered $\int_\Omega \rho^p = M^p$ for sufficiently large p instead of (1.13), which is not really acceptable from the physical point of view. Concerning the barotropic case, the first existence result for large data is due to P.L. Lions. The existence of a weak solution was proved in [4] for $\gamma>1$ (2D) and $\gamma>\frac{5}{3}$ (3D). Novo and Novotný in [9] improved the result by getting $\gamma>\frac{3}{2}$, but only for potential force with small nonpotential perturbation. Recently (see [2], [3]) it was shown that the solution exists also for $\gamma=1$ in 2D and for $\gamma>\frac{4}{3}$ in 3D. The main idea of the improvement goes back to [12], however, therein, the authors were not able to work with the Navier–Stokes equations and considered only its modification. Although the results in [2] and [3] are for the Dirichlet boundary conditions, it can be easily modified also for the slip boundary conditions.

To prove Theorem 2.1, we have to construct an approximation of the original problem which we implement in Section 3. We prove a priori estimates for the approximative system from which we conclude the existence of a solution to the approximative system (Section 4).

In the next sections we have to show that the solution to the approximative problem we constructed in the previous section converges to the solution of the original system. To this purpose, we prove that the temperature and the velocity cause no problems for convergence. Then we introduce quantity known as effective viscous flux, which plays key role in the proof of strong convergence of density. The last difficulty remains velocity gradient, but once the density converges strongly, we may pass to the limit also in the internal energy balance.

3. Approximation

We are going to construct the approximation of the original problem in the same way as in [7]. For $k > 0$ we consider a function $K(y) \in \mathcal{C}^\infty(\mathbb{R})$ with

$$K(y) = \begin{cases} 1 & \text{for } \ y \le k \\ \in (0,1) & \text{for } \ k < y < k+1 \\ 0 & \text{for } \ y \ge k+1, \end{cases} \tag{3.1}$$

$k > 0$. Moreover, we assume that $K'(y) < 0$ for $t \in (k, k+1)$.

We take $\varepsilon > 0$ and consider in Ω

$$\varepsilon\rho + \operatorname{div}(K(\rho)\rho\boldsymbol{v}) - \varepsilon\Delta\rho = \varepsilon h K(\rho) \tag{3.2}$$

$$\frac{1}{2}\operatorname{div}(K(\rho)\rho\boldsymbol{v}\otimes\boldsymbol{v}) + \frac{1}{2}K(\rho)\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \operatorname{div}\mathbf{S}(\boldsymbol{v}) + \nabla P(\rho,\theta) = K(\rho)\rho\boldsymbol{F} \tag{3.3}$$

$$\begin{aligned} &-\operatorname{div}\left((1+\theta^m)\frac{\varepsilon+\theta}{\theta}\nabla\theta\right) + \operatorname{div}\left(\boldsymbol{v}\int_0^\rho K(y)\,dy\right)\theta + \operatorname{div}\left(K(\rho)\rho\boldsymbol{v}\right)\theta \\ &+ K(\rho)\rho\boldsymbol{v}\cdot\nabla\theta - \theta K(\rho)\boldsymbol{v}\cdot\nabla\rho = \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v}, \end{aligned} \tag{3.4}$$

where

$$P(\rho,\theta) = \int_0^\rho \gamma y^{\gamma-1}K(y)\,dy + \theta\int_0^\rho K(y)\,dy = P_b(\rho) + \theta\int_0^\rho K(y)\,dy, \tag{3.5}$$

and $h = \frac{M}{|\Omega|}$.

Note that when we pass with $\varepsilon \to 0^+$ we will have to show that for the limit density $K(\rho) \equiv 1$. This causes additional difficulties; on the other hand, the positive features of our approximation are more significant.

We also define quantity s as

$$s = \ln\theta.$$

Even though the physical entropy for our problem is $\eta = \ln\theta - \ln\rho$, the quantity s is more useful for us. If we consider θ sufficiently smooth and positive, we may rewrite equation (3.4) as

$$
\begin{aligned}
&-\operatorname{div}\left((1+e^{sm})\frac{\varepsilon+e^s}{e^s}\nabla s\right)+K(\rho)\rho\boldsymbol{v}\cdot\nabla s-K(\rho)\boldsymbol{v}\cdot\nabla\rho\\
&+\operatorname{div}\left(\boldsymbol{v}\int_0^\rho K(y)\,dy\right)+\operatorname{div}(K(\rho)\rho\boldsymbol{v})\\
&\qquad\qquad=\frac{\mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}}{e^s}+\frac{(1+e^{sm})(\varepsilon+e^s)}{e^s}|\nabla s|^2
\end{aligned}
\tag{3.6}
$$

in Ω. We will appreciate this form of internal energy equation (or rather entropy equation) later — we will need to control the positiveness of temperature, which does not work very well with equation (3.4).

At $\partial\Omega$ we consider the following boundary conditions (see Section 1 for the original ones)

$$(1+\theta^m)(\varepsilon+\theta)\frac{\partial s}{\partial\boldsymbol{n}}+L(\theta)(\theta-\theta_0)+\varepsilon s = 0 \tag{3.7}$$

$$\boldsymbol{v}\cdot\boldsymbol{n}=0,\qquad \boldsymbol{\tau}\cdot(\mathbf{T}(p,\boldsymbol{v})\boldsymbol{n})+f\boldsymbol{v}\cdot\boldsymbol{\tau} = 0 \tag{3.8}$$

$$\frac{\partial\rho}{\partial\boldsymbol{n}} = 0. \tag{3.9}$$

4. Existence for the approximative system

Theorem 4.1. *Let the assumptions of Theorem* 2.1 *be satisfied. Let* $\varepsilon>0$ *and* $k>0$. *Then there exists a strong solution* $(\rho,\boldsymbol{v},s)$ *to* (3.2)–(3.4) *such that*

$$\rho\in W^{2,p}(\Omega),\quad \boldsymbol{v}\in W^{2,p}(\Omega)\ \text{ and }\ \theta\in W^{2,p}(\Omega)\qquad \text{for }\ 1\le p<\infty. \tag{4.1}$$

Moreover $0\le\rho\le k+1$ *in* Ω, $\int_\Omega\rho\,dx\le M$, $\theta>0$ *and*

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{1,q}+\|\int_0^\rho K(y)\,dy\|_{2\gamma}+\|K(\rho)\rho\|_{2\gamma}+\|\nabla\theta\|_r+\|\theta\|_q\le C(k),\\
&\|\boldsymbol{v}\|_{1,2}+\|\int_0^\rho K(y)\,dy\|_{2\gamma}+\|K(\rho)\rho\|_{2\gamma}+\|\nabla\theta\|_r+\|\theta\|_q\le C,
\end{aligned}
\tag{4.2}
$$

where $r=2$ *if* $m\ge2$ *and* $r=1+\delta$, $0\le\delta<1$ *is arbitrary for* $m<2$, $1\le q<\infty$ *arbitrary.*

The proof of this theorem will be split into several lemmae.

Let us denote (for $p\in[1,\infty]$)

$$M_p=\{\boldsymbol{u}\in W^{1,p}(\Omega);\boldsymbol{u}\cdot\boldsymbol{n}=0\ \text{ at }\ \partial\Omega\}$$

and define operator

$$S:M_p\to W^{2,p}(\Omega)\ \text{ for }\ 1\le p<\infty$$

such that $\rho = S(\boldsymbol{v})$, where ρ solves the following problem

$$
\begin{aligned}
\varepsilon\rho - \varepsilon\Delta\rho &= \varepsilon h K(\rho) - \operatorname{div}(K(\rho)\rho\boldsymbol{v}) \qquad \text{in } \ \Omega \\
\frac{\partial\rho}{\partial\boldsymbol{n}} &= 0 \qquad \text{at } \ \partial\Omega
\end{aligned}
\tag{4.3}
$$

(see the continuity equation). With this notation we have

Lemma 4.1. *Operator S, defined by* (4.3), *is a well-defined compact continuous operator from M_p to $W^{2,p}(\Omega)$, $1 \le p < \infty$; in particular, the solution to* (4.3) *is unique and satisfies $0 \le \rho \le k+1$. Moreover, for $p > 2$, we have*

$$
\|\rho\|_{2,p} \le C(k,\varepsilon)(\|\boldsymbol{v}\|_{1,p}^2 + 1). \tag{4.4}
$$

Proof: See [6, Proposition 3.1], and [10, Proposition 4.22]; the only difference in our case is the estimate on $\|\rho\|_{2,p}$. We state the idea of the proof here: we take the definition of operator S (4.3) and estimate $\|\nabla\rho\|_p$ from there. The worst term is $\operatorname{div}(K(\rho)\rho\boldsymbol{v})$, so using standard elliptic regularity we get

$$
\varepsilon\|\rho\|_{1,p} \le C(1 + \|K(\rho)\rho\boldsymbol{v}\|_p) \le C(k)(1 + \|\boldsymbol{v}\|_p)
$$

and therefore, due to the same argument,

$$
\begin{aligned}
\varepsilon\|\nabla^2\rho\|_p &\le C(1 + \|\nabla\rho\cdot\boldsymbol{v}\|_p + \|\rho\operatorname{div}\boldsymbol{v}\|_p) \\
&\le C(1 + \|\boldsymbol{v}\|_\infty\|\nabla\rho\|_p + (k+1)\|\operatorname{div}\boldsymbol{v}\|_p) \le C(k)(1 + \|\boldsymbol{v}\|_{1,p}^2)
\end{aligned}
$$

for $p > 2$. □

Now we define another operator, this time connected with the momentum and energy equations:

$$
T : M_p \times W^{2,p}(\Omega) \to M_p \times W^{2,p}(\Omega) \quad \text{s.t.} \quad T(\boldsymbol{v}, s) = (\boldsymbol{w}, z),
$$

where $(\boldsymbol{w}, z)$ is given as a solution to

$$
\begin{aligned}
-\operatorname{div}\mathbf{S}(\boldsymbol{w}) &= -\frac{1}{2}\operatorname{div}(K(\rho)\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \frac{1}{2}K(\rho)\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} \\
&\qquad - \nabla P(\rho, e^s) + K(\rho)\rho\boldsymbol{F} \quad \text{in } \ \Omega, \\
-\operatorname{div}((1+e^{ms})(\varepsilon + e^s)\nabla z) &= \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - \operatorname{div}\left(\boldsymbol{v}\int_0^\rho K(y)\,dy\right)e^s \\
&\qquad - \operatorname{div}(K(\rho)\rho\boldsymbol{v})\nabla s - e^s K(\rho)\rho\boldsymbol{v}\cdot\nabla s + e^s K(\rho)\boldsymbol{v}\cdot\nabla\rho \quad \text{in } \ \Omega, \\
\boldsymbol{w}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{n}\cdot\mathbf{S}(\boldsymbol{w})\cdot\boldsymbol{\tau} &+ f\boldsymbol{w}\cdot\boldsymbol{\tau} = 0 \qquad \text{at } \ \partial\Omega, \\
(1+e^{ms})(\varepsilon+e^s)\nabla z + \varepsilon z &= -L(e^s)(e^s - \theta_0) \quad \text{at } \ \partial\Omega,
\end{aligned}
\tag{4.5}
$$

where $\rho = S(\boldsymbol{v})$ is given by (4.3) and Lemma 4.1. Note that $(4.5)_2$ is just (3.4) rewritten for our "entropy" s.

To apply the Schaeffer fixed point theorem (see e.g. [1]) we need to verify that T is a continuous and compact mapping from $M_p \times W^{2,p}(\Omega)$ to $M_p \times W^{2,p}(\Omega)$ and that all solutions satisfying

$$tT(\boldsymbol{w}, z) = (\boldsymbol{w}, z), \qquad t \in [0,1] \tag{4.6}$$

are bounded in $M_p \times W^{2,p}(\Omega)$.

Lemma 4.2. *Let $p > 2$; let all assumptions of Theorem 4.1 be satisfied. Then T is a continuous and compact operator from $M_p \times W^{2,p}(\Omega)$ to $M_p \times W^{2,p}(\Omega)$.*

Proof: We are going to use two facts: first, $\forall\, \varepsilon > 0$ system (4.5) is strictly elliptic; second, for $p > 2$ the $W^{1,p}(\Omega)$ space is an algebra. Due to the second fact the RHS of (4.5) belongs to $L^p(\Omega)$ and the boundary terms belong to $W^{1-1/p,p}(\partial\Omega)$.

The coefficients on the LHS of the second equation of (4.5) are of $C^{1+\alpha}(\overline{\Omega})$. The standard elliptic theory gives us information about existence of solution to (4.5) in $M_p \times W^{2,p}(\Omega)$ with the following bound

$$\begin{aligned}\|\boldsymbol{w}\|_{2,p} + \|z\|_{2,p} \le C(\|e^s\|_{C^{1+\alpha}(\Omega)})(&\|RHS \ \text{ of } \ (4.5)_1\|_p \\ &+ \|RHS \ \text{ of } \ (4.5)_2\|_p + \|RHS \ \text{ of } \ (4.5)_4\|_{W^{1-1/p,p}(\partial\Omega)}).\end{aligned}$$

Moreover, this also implies uniqueness of the solution and continuous dependence on the data.

The RHS of (4.5) is at most of the first order of sought functions, which implies the compactness of the operator T. □

Lemma 4.3. *All solutions to (4.6) in $M_p \times W^{2,p}(\Omega)$ satisfy the following bounds*

$$\begin{gathered} 0 \le \rho \le k+1 \\ \|\boldsymbol{w}\|_{1,2} + \|\theta\|_q + \|\nabla\theta\|_{1+\delta} + \sqrt{\varepsilon}\|\nabla\rho\|_2 + \|K(\rho)\rho\|_{2\gamma} \\ + \Big\|\int_0^\rho K(y)\,dy\Big\|_{2\gamma} \le C(\boldsymbol{F}, M, q), \end{gathered} \tag{4.7}$$

where $\theta = e^z$, $\delta = 1$ for $m \ge 2$ and $0 \le \delta < 1$, arbitrary, for $m < 2$, and $C(\boldsymbol{F}, M, q)$ is independent of ε, k and $t \in [0,1]$. Moreover,

$$\|\boldsymbol{w}\|_{1,q} \le C(k,q), \qquad 2 < q < \infty.$$

Proof: We need to get a priori estimate for system (4.6) independent of t.

First of all we multiply the first equation from (4.5)[3] by $\boldsymbol{w}$ and integrate over Ω to get

$$\begin{aligned}\int_\Omega \mathbf{S}(\boldsymbol{w}) : \nabla\boldsymbol{w}\,dx + \int_{\partial\Omega} f|\boldsymbol{w}|^2\,d\sigma = t\Big(&-\int_\Omega \boldsymbol{w}\cdot\nabla P_b(\rho,\theta)\,dx \\ &+ \int_\Omega K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}\,dx + \int_\Omega \Big(\int_0^\rho K(y)\,dy\Big)\theta\,\mathrm{div}\,\boldsymbol{w}\,dx\Big).\end{aligned} \tag{4.8}$$

[3] The terms on the RHS are multiplied by t and $(\boldsymbol{v}, s)$ is replaced by $(\boldsymbol{w}, z)$.

To get the information about the second term on the RHS of this equation we use the approximative continuity equation — i.e. equation $(4.3)_1$

$$\begin{aligned}\int_\Omega \boldsymbol{w}\cdot\nabla P_b(\rho)\,dx &= \frac{\gamma}{\gamma-1}\int_\Omega K(\rho)\rho\boldsymbol{w}\cdot\nabla\rho^{\gamma-1}\,dx\\ &= -\frac{\gamma}{\gamma-1}\int_\Omega (\varepsilon\Delta\rho+\varepsilon hK(\rho)-\varepsilon\rho)\,\rho^{\gamma-1}\,dx\\ &= \frac{\varepsilon\gamma}{\gamma-1}\int_\Omega(\rho-hK(\rho))\rho^{\gamma-1}\,dx+\varepsilon\gamma\int_\Omega \rho^{\gamma-2}|\nabla\rho|^2\,dx.\end{aligned}$$

Using this fact we get

$$\begin{aligned}&\int_\Omega \mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}\,dx+\int_{\partial\Omega} f|\boldsymbol{w}|^2\,d\sigma+t\varepsilon\gamma\int_\Omega\rho^{\gamma-2}|\nabla\rho|^2\,dx+t\frac{\varepsilon\gamma}{\gamma-1}\int_\Omega\rho^\gamma\,dx\\ &\quad -t\int_\Omega\Big(\int_0^\rho K(y)\,dy\Big)\theta\,\mathrm{div}\,\boldsymbol{w}\,dx\le Ct\Big(1+\int_\Omega|K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}|\,dx\Big).\end{aligned}\tag{4.9}$$

Next we integrate the second equation in (4.5) and using the boundary condition we get

$$\int_{\partial\Omega}\big(tL(\theta)(\theta-\theta_0)+\varepsilon z\big)\,d\sigma=t\int_\Omega\Big(\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}-\Big(\int_0^\rho K(y)\,dy\Big)\theta\,\mathrm{div}\,\boldsymbol{w}\Big)dx,\tag{4.10}$$

since the integration by parts gives the following identity

$$\begin{aligned}&\int_\Omega\Big(K(\rho)\rho\boldsymbol{w}\cdot\nabla\theta-\theta K(\rho)\boldsymbol{w}\cdot\nabla\rho+\mathrm{div}\,\Big(\boldsymbol{w}\int_0^\rho K(y)\,dy\Big)\theta\\ &\quad+\mathrm{div}\,\big(K(\rho)\rho\boldsymbol{w}\big)\theta\Big)dx=\int_\Omega\Big(\int_0^\rho K(y)\,dy\Big)\theta\,\mathrm{div}\,\boldsymbol{w}\,dx.\end{aligned}\tag{4.11}$$

Summing up (4.9) and (4.10) we get

$$\begin{aligned}&\int_{\partial\Omega}\big(tL(\theta)(\theta-\theta_0)+\varepsilon z^+\big)d\sigma+\int_{\partial\Omega}f|\boldsymbol{w}|^2\,d\sigma+t\varepsilon\gamma\int_\Omega\rho^{\gamma-2}|\nabla\rho|^2\,dx\\ &\quad+(1-t)\int_\Omega\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}\,dx+t\frac{\varepsilon\gamma}{\gamma-1}\int_\Omega\rho^\gamma\,dx\\ &\le Ct\Big(1+\int_\Omega|K(\rho)\rho\boldsymbol{v}\cdot\boldsymbol{F}|\,dx\Big)+C\varepsilon\int_{\partial\Omega}z^-\,d\sigma,\end{aligned}\tag{4.12}$$

where z^+ and z^- are the positive and negative parts of the entropy ($z=z^+-z^-$). Now we take care of the first term of the RHS of (4.12). Note that the control of the negative part of entropy z is not immediate.

We integrate the entropy equation (3.6) over Ω getting

$$\begin{aligned}
&\int_{\partial\Omega}\Big(\frac{tL(\theta)(\theta-\theta_0)}{\theta}+t\varepsilon z e^{-z}\Big)\,d\sigma+t\int_\Omega\Big(K(\rho)\rho\frac{\boldsymbol{v}\cdot\nabla\theta}{\theta}-K(\rho)\boldsymbol{v}\cdot\nabla\rho\Big)dx\\
&=t\int_\Omega\Big(\frac{\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}}{\theta}+\frac{(1+\theta^m)(\varepsilon+\theta)}{\theta}|\nabla z|^2\Big)\,dx.
\end{aligned}\tag{4.13}$$

So

$$\begin{aligned}
&(1-t)\int_\Omega\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}\,dx+\int_{\partial\Omega}f|\boldsymbol{w}|^2\,d\sigma+t\int_\Omega\frac{(1+\theta^m)(\varepsilon+\theta)}{\theta}|\nabla z|^2\,dx\\
&\quad+t\int_\Omega\Big(\frac{\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}}{\theta}+\varepsilon\gamma\rho^{\gamma-2}|\nabla\rho|^2+\frac{\varepsilon\gamma}{\gamma-1}\rho^\gamma\Big)dx\\
&\quad+\varepsilon\int_{\partial\Omega}\big(z_+(1-e^{-z_+})+|z_-|(e^{|z_-|}-1)\big)\,d\sigma\\
&\quad+t\int_{\partial\Omega}\Big(L(\theta)\theta-L(\theta)\theta_0+\frac{L(\theta)\theta_0}{\theta}-L(\theta)\Big)d\sigma\\
&\le t\int_\Omega\Big(K(\rho)\rho\boldsymbol{w}\cdot\nabla z-K(\rho)\boldsymbol{w}\cdot\nabla\rho\Big)dx+tC\Big(1+\int_\Omega|K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}|\,dx\Big),
\end{aligned}\tag{4.14}$$

where $\rho=S(\boldsymbol{w})$. From the first term in the RHS of (4.14) we have

$$\int_\Omega K(\rho)\rho\boldsymbol{w}\cdot\nabla(z-\ln\rho)\,dx=-\int_\Omega K(\rho)\rho\boldsymbol{w}\cdot\nabla\ln\rho\,dx+\int_\Omega K(\rho)\rho\boldsymbol{w}\cdot\nabla z\,dx\tag{4.15}$$

and with help of (4.3) we get for the first integral in (4.15)

$$\begin{aligned}
-\int_\Omega K(\rho)\rho\boldsymbol{w}\cdot\nabla\ln\rho\,dx&=\int_\Omega\operatorname{div}(K(\rho)\rho\boldsymbol{w})\ln\rho\,dx\\
&=-\int_\Omega\big(-\varepsilon\Delta\rho+\varepsilon\rho-\varepsilon hK(\rho)\big)\ln\rho\,dx\\
&=-\int_\Omega\Big(\varepsilon\frac{|\nabla\rho|^2}{\rho}-\varepsilon hK(\rho)\ln\rho+\varepsilon\rho\ln\rho\Big)dx.
\end{aligned}\tag{4.16}$$

The first term has a good sign (considered on the LHS), the second term has a good sign for $\rho\le 1$, too, and for $\rho\ge 1$ is easily bounded by $\varepsilon h\rho$. Similarly, the last term can be controlled by the term $\varepsilon\int_\Omega\rho^\gamma\,dx$. The proof was rather formal, as we do not know whether $\rho>0$ in Ω. However, we may write $K(\rho)\boldsymbol{v}\cdot\nabla(\rho+\delta)$ in (4.15) with $\delta>0$ and find an analogue of (4.16) with $\ln(\rho+\delta)$. Finally we pass with $\delta\to 0^+$ and get precisely the same information as above.

Next, for the second integral in (4.15)

$$
\begin{aligned}
\int_\Omega K(\rho)\rho\boldsymbol{w}\cdot\nabla z\,dx &= -\int_\Omega \big(\varepsilon\Delta\rho-\varepsilon\rho+\varepsilon hK(\rho)\big)z\,dx \\
&= \int_\Omega \big(\varepsilon\nabla\rho\cdot\nabla z+\varepsilon\rho\ln\theta-\varepsilon hK(\rho)\ln\theta\big)\,dx.
\end{aligned}
\tag{4.17}
$$

Considering the RHS of (4.17), we have

$$
\begin{aligned}
\Big|\varepsilon\int_\Omega \nabla\rho\cdot\nabla z\,dx\Big| &\le \varepsilon\|\nabla\rho\|_2\|\nabla z\|_2 \\
&\le \frac14\varepsilon\Big(\int_\Omega\frac{|\nabla\rho|^2}{\rho}\,dx+\int_\Omega|\nabla\rho|^2\rho^{\gamma-2}dx\Big)+\frac14\|\nabla z\|_2^2.
\end{aligned}
\tag{4.18}
$$

Moreover, $-\int_\Omega \varepsilon\rho\ln\theta\,dx$ has a good sign for $\theta\le 1$ (again, considered on the LHS); for $\theta>1$ we have

$$
\begin{aligned}
\int_\Omega \varepsilon\rho(\ln\theta)^+\,dx &\le \varepsilon\|\rho\|_2\|z^+\|_2 \\
&\le \frac{\varepsilon}{4}\big(\|(\ln\theta)^+\|_{L_1(\partial\Omega)}+\|\nabla z\|_2\big)^2+\frac{\varepsilon}{4}\|\rho^\gamma\|_1+C \\
&\le \frac{\varepsilon}{4}\int_{\partial\Omega}L(\theta)\theta\,d\sigma+\frac{\varepsilon}{4}\|\nabla z\|_2^2+\frac{\varepsilon}{4}\|\rho^\gamma\|_1+C.
\end{aligned}
\tag{4.19}
$$

The last term of (4.17) can be treated as follows (one part has again a good sign)

$$
\begin{aligned}
\int_\Omega \varepsilon hK(\rho)|(\ln\theta)^-|\,dx &\le C\varepsilon\int_\Omega|z^-|\,dx \\
&\le C+\frac12\int_{\partial\Omega}\varepsilon|z^-|e^{|z^-|}\,d\sigma+\frac14||\nabla z||_{L_2(\Omega)}.
\end{aligned}
\tag{4.20}
$$

Then combining (4.14) with inequality (4.12) and with (4.16)–(4.20) we obtain

$$
\begin{aligned}
&t\int_\Omega\Big(\frac{\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}}{\theta}+\frac{1+\theta^m}{\theta^2}|\nabla\theta|^2\Big)\,dx \\
&\quad+\int_{\partial\Omega}\Big(tL(\theta)\theta+t\frac{L(\theta)\theta_0}{\theta}+\varepsilon|z|\Big)\,d\sigma\le tH,
\end{aligned}
\tag{4.21}
$$

where

$$
H=C\Big(1+\int_\Omega|K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}|\,dx\Big).
$$

From the growth conditions we deduce

$$
\Big(\int_{\partial\Omega}\theta^{l+1}\,d\sigma\Big)^{1/(l+1)}\le H^{1/(l+1)},\qquad \Big(\int_\Omega|\nabla\big(\theta^{m/2}\big)|^2\Big)^{1/m}\le tH^{1/m}.
$$

To get bounds for the temperature we use the following Poincaré type inequality

$$\|\theta^{m/2}\|_2 \le C(\Omega)\Big(\|\nabla\big(\theta^{m/2}\big)\|_2 + \|\theta\|^{m/2}_{l+1,\partial\Omega}\Big).$$

The imbedding theorem leads to the bound

$$\|\theta\|_q \le C\left(tH^{1/m} + H^{1/(l+1)}\right) \tag{4.22}$$

for any $q < \infty$; note that $C = C(q)$ and $C(q) \to \infty$ for $q \to \infty$. In this very moment we set (for the sake of simplicity) $m = l+1$. It is possible to consider $m \neq l+1$ or even $\boldsymbol{F} \in L^p(\Omega)$ for $p < \infty$ but all related calculations become too technical.

We now return to (4.9). Hölder's inequality with help of Korn's inequality[4] yields

$$\begin{aligned} &\|\boldsymbol{w}\|^2_{1,2} + t\varepsilon\gamma\int_\Omega |\nabla\rho|^2\rho^{\gamma-2}\,dx + t\frac{\varepsilon\gamma}{\gamma-1}\int_\Omega \rho^\gamma\,dx \\ &\le Ct\left(1 + \int_\Omega |K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}|\,dx + \int_\Omega |\theta\int_0^\rho K(y)\,dy|^2\,dx\right). \end{aligned} \tag{4.23}$$

In what follows we use the imbedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1,\infty)$; in the following q is always this number taken from the imbedding. (Note that our aim is to use q as large as possible.)

The first term on the RHS of this equation is relatively simple; assuming $\frac{q}{q-1} \in (1, 2\gamma)$ we have

$$\begin{aligned} \int_\Omega |K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}|\,dx &\le C\|K(\rho)\rho\|_{\frac{q}{q-1}}\|\boldsymbol{w}\|_q\|\boldsymbol{F}\|_\infty \\ &\le C\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma}{q(2\gamma-1)}}\|\boldsymbol{w}\|_{1,2}. \end{aligned} \tag{4.24}$$

However, to proceed, we need to get a bound of $P_b(\rho)$. To find it we use the so-called Bogovskii operator. We introduce $\boldsymbol{\Phi} : \Omega \to \mathbb{R}^2$ defined as a special solution to the problem

$$\begin{aligned} \operatorname{div}\boldsymbol{\Phi} &= P_b(\rho) - \frac{1}{|\Omega|}\int_\Omega P_b(\rho)\,dx \quad \text{in } \Omega \\ \boldsymbol{\Phi} &= \boldsymbol{0} \quad \text{at } \partial\Omega, \end{aligned} \tag{4.25}$$

satisfying

$$\|\boldsymbol{\Phi}\|_{1,2} \le C\|P_b(\rho)\|_2, \tag{4.26}$$

[4] i.e. for $f = 0$ we require that Ω is not rotationally symmetric, for more details see [10].

see e.g. [10]. Using structure of $P_b(\rho)$, the fact that $\int_\Omega \rho\,dx \le M$ and interpolation inequality, we have

$$\int_\Omega P_b(\rho)\,dx \le \delta \|P_b(\rho)\|_2 + C(\delta, M) \qquad \text{for all} \quad \delta > 0.$$

Now, we are ready to multiply the momentum equation (first equation in (4.5)) by $\mathbf{\Phi}$, use (4.23) and (4.26) together with standard estimates on the RHS to (4.5), and get

$$\begin{aligned} &t\|P_b(\rho)\|_2^2 \\ &\le C\left(t + \|\nabla \boldsymbol{w}\|_2^2 + t\int_\Omega |K(\rho)\rho\boldsymbol{w}\otimes\boldsymbol{w}|^2\,dx + t\int_\Omega |\theta\int_0^\rho K(y)\,dy|^2\,dx\right). \end{aligned} \tag{4.27}$$

We have

$$\|P_b(\rho)\|_2^2 \ge C\Big(\int_\Omega (K(\rho)\rho)^{2\gamma}\,dx + \int_\Omega \Big(\int_0^\rho K(y)\,dy\Big)^{2\gamma}\Big); \tag{4.28}$$

therefore we estimate the first integral in the RHS of (4.27) as

$$\int_\Omega |K(\rho)\rho\boldsymbol{w}\otimes\boldsymbol{w}|^2\,dx \le C\|K(\rho)\rho\|^2_{\frac{2q}{q-4}}\|\boldsymbol{w}\|_q^4 \le C\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(q+4)}{q(2\gamma-1)}}\|\boldsymbol{w}\|_q^4 \tag{4.29}$$

for any sufficiently large $q < \infty$. We now take care of the second integral in (4.27) with the help of Hölder's inequality; we get $\|\int_0^\rho K(t)\,dt\|_{\frac{2q}{q-2}}$ and then we use interpolation between L^1 and $L^{2\gamma}$

$$\|\theta\int_0^\rho K(t)\,dt\|_2^2 \le \|\theta\|_q^2\|\int_0^\rho K(y)\,dy\|^2_{\frac{2q}{q-2}} \le \|\theta\|_q^2\|\int_0^\rho K(y)\,dy\|_{2\gamma}^{\frac{2\gamma(q+2)}{q(2\gamma-1)}}. \tag{4.30}$$

Apart from that, we have (4.22) and (4.24), which gives us the following bound

$$\|\theta\|_q \le Ct\big(\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma}{q(2\gamma-1)}}\|\boldsymbol{w}\|_{1,2}\big)^{\frac{1}{m}}. \tag{4.31}$$

Therefore we get

$$\|\theta\int_0^\rho K(y)\,dy\|_2^2 \le Ct^2\|\int_0^\rho K(y)\,dy\|_{2\gamma}^{\frac{2\gamma(q+2)}{q(2\gamma-1)}}\big(\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma}{q(2\gamma-1)}}\|\boldsymbol{w}\|_{1,2}\big)^{\frac{2}{m}}. \tag{4.32}$$

From the inequalities above we see that

$$\begin{aligned} t\|K(\rho)\rho\|_{2\gamma}^{2\gamma} \le C\big(&t\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(q+4)}{q(2\gamma-1)}}\|\boldsymbol{w}\|_{1,2}^4 \\ &+ t\|K(\rho)\rho\|_{2\gamma}^{2\gamma\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{m}\frac{1}{q(2\gamma-1)}\right)}\|\boldsymbol{w}\|_{1,2}^{\frac{2}{m}} + \|\nabla\boldsymbol{w}\|_2^2 + t\big). \end{aligned} \tag{4.33}$$

Young's inequality yields

$$t\|K(\rho)\rho\|_{2\gamma}^{2\gamma} \le Ct\big(\|\boldsymbol{w}\|_{1,2}^{\frac{2q(2\gamma-1)}{q(\gamma-1)-2}} + \|\boldsymbol{w}\|_{1,2}^{\frac{q(2\gamma-1)}{mq(\gamma-1)-m-1}}\big) + C\|\boldsymbol{w}\|_{1,2}^2 \tag{4.34}$$

which, together with (4.21) and a little help of Young's inequality, leads us to the final bound

$$\begin{aligned}\|\boldsymbol{w}\|_{1,2}^2 \le Ct\Big(&\|\boldsymbol{w}\|_{1,2}^{1+\frac{1}{q(2\gamma-1)}\frac{2q(2\gamma-1)}{q(\gamma-1)-2}} + \|\boldsymbol{w}\|_{1,2}^{1+\frac{1}{q(2\gamma-1)}\frac{q(2\gamma-1)}{mq(\gamma-1)-m-1}}\\ &+ \|\boldsymbol{w}\|_{1,2}^{\frac{2q(2\gamma-1)}{q(\gamma-1)-2}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{m}\frac{1}{q(2\gamma-1)}\right)+\frac{2}{m}}\\ &+ \|\boldsymbol{w}\|_{1,2}^{\frac{q(2\gamma-1)}{mq(\gamma-1)-m-1}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{m}\frac{1}{q(2\gamma-1)}\right)+\frac{2}{m}}\Big) + C.\end{aligned} \tag{4.35}$$

To get a reasonable bound on $\|\boldsymbol{w}\|_{1,2}$ we need all the exponents in (4.35) to be less than 2. After some algebra we get

$$\begin{aligned}&\gamma > 2\\ &\gamma > \frac{4+q}{q}\\ &m > \frac{2}{q(\gamma-1)-1}\\ &m > \frac{q(2\gamma-1)}{2q(\gamma-1)}\\ &m > \frac{q(\gamma-1)}{q(\gamma-2)-4}\,;\end{aligned} \tag{4.36}$$

hence, recalling that q can be arbitrarily large,

$$\gamma > 2 \qquad \& \qquad m > \frac{\gamma-1}{\gamma-2}\,. \tag{4.37}$$

Now we are almost done: we have

$$\|\boldsymbol{w}\|_{1,2}^2 \le C(\|\boldsymbol{F}\|_\infty, M). \tag{4.38}$$

This fact together with estimates for temperature ((4.12) and (4.22)) gives us (4.7), except for one term.

To finish the proof we multiply the approximative momentum equation (3.3) by ρ and integrate by parts to get

$$\varepsilon\int_\Omega (|\nabla\rho^2| + \rho^2)\,dx \le \varepsilon\int_\Omega hK(\rho)\rho\,dx + \int_\Omega\Big(\int_0^\rho K(y)y\,dy\Big)|\operatorname{div}\boldsymbol{w}|\,dx,$$

from where we extract the bound for $\sqrt{\varepsilon}\|\nabla\rho\|_2$. □

At last we want to verify the bounds on $\boldsymbol{w}$ and z, i.e. to make sure our operator T maps $M_p \times W^{2,p}(\Omega)$ to $M_p \times W^{2,p}(\Omega)$. We apply the bootstrap method to the system

$$
\begin{aligned}
-\operatorname{div}\mathbf{S}(\boldsymbol{w}) = t\Big(&-\frac{1}{2}\operatorname{div}(K(\rho)\rho\boldsymbol{w}\otimes\boldsymbol{w}) - \frac{1}{2}K(\rho)\rho\boldsymbol{w}\cdot\nabla\boldsymbol{w} \\
&- \nabla P(\rho, e^z) + K(\rho)\rho\boldsymbol{F}\Big) \qquad \text{in } \Omega
\end{aligned}
\tag{4.39}
$$

$$
\begin{aligned}
-\operatorname{div}\big((1+e^{mz})(\varepsilon+e^z)\nabla z\big) &= t\Big(\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w} - \operatorname{div}\Big(\boldsymbol{w}\int_0^\rho K(y)\,dy\Big)e^z \\
&- \operatorname{div}\big(K(\rho)\rho\boldsymbol{w}\big)e^z - e^zK(\rho)\rho\boldsymbol{w}\cdot\nabla z + e^zK(\rho)\boldsymbol{w}\cdot\nabla\rho\Big) \qquad \text{in } \Omega
\end{aligned}
\tag{4.40}
$$

with $\rho = S(\boldsymbol{v})$; the boundary conditions are as follows

$$
\begin{aligned}
&\boldsymbol{w}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{n}\cdot\mathbf{S}(\boldsymbol{w})\cdot\boldsymbol{\tau} + f\boldsymbol{w}\cdot\boldsymbol{\tau} = 0 \\
&(1+e^{mz})(\varepsilon+e^z)\nabla z + \varepsilon z = -tL(e^z)(e^z-\theta_0).
\end{aligned}
\tag{4.41}
$$

First knowledge (and the one we have for free) is that $\boldsymbol{w}\in W^{1,2}(\Omega)\hookrightarrow L^q(\Omega)$ for any $q\in[1,\infty)$. Thus, when deducing bounds to the RHS of (4.39), the most restrictive term is $\nabla P(\rho,e^z)$ — but we have $\theta\in L^q(\Omega)$ and $\rho\in L^\infty(\Omega)$ and therefore we have also (RHS of (4.39)) $\in (W^{1,q'}(\Omega))^* \Rightarrow \boldsymbol{w}\in W^{1,q}(\Omega)$. As a consequence, from the continuity equation (3.2) we get $\rho\in W^{2,q}(\Omega)$. Note, however, that $\|\boldsymbol{w}\|_{1,q}\le C(k)$.

Next step is to rewrite equation (4.40):

$$
\begin{aligned}
-\Delta\Phi(z) = t\Big(&\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w} - e^zK(\rho)\rho\boldsymbol{w}\cdot\nabla z + e^zK(\rho)\boldsymbol{w}\cdot\nabla\rho \\
&- \operatorname{div}\Big(\boldsymbol{w}\int_0^\rho K(y)\,dy\Big)e^z - \operatorname{div}\big(K(\rho)\rho\boldsymbol{w}\big)e^z\Big) \qquad \text{in } \Omega, \\
\frac{\partial\Phi(z)}{\partial\boldsymbol{n}} = -\varepsilon z &- tL(e^z)(e^z-\theta_0) \qquad \text{at } \partial\Omega,
\end{aligned}
\tag{4.42}
$$

where

$$
\Phi(z) = \int_0^z (1+e^{m\tau})(\varepsilon+e^\tau)\,d\tau.
$$

We need to verify that Φ is bounded. We multiply (4.39) by Φ and integrate over Ω. It leads to

$$
\|\nabla\Phi\|_2^2 + \int_{\partial\Omega}(tL(e^z)(e^z-\theta_0)\Phi + \varepsilon z\Phi)\,d\sigma \le C\|\text{RHS of (4.39)}\|_{\frac{q}{q-1}}\|\Phi\|_q.
$$

Now we realize that $\Phi(s)\sim\varepsilon s$ for $s\to-\infty$ and $\Phi(s)\sim e^{(m+1)s}$ for $s\to\infty$; from this fact we deduce that

$$
\begin{aligned}
&\int_{\partial\Omega}(tL(e^z)(\theta-\theta_0)\Phi(z) + \varepsilon z\Phi(z))1_{\{\Phi\le 0\}}\,d\sigma \ge \varepsilon^2\|\Phi(z)\|_2^2 - C \\
&\int_{\partial\Omega}(tL(e^z)(\theta-\theta_0)\Phi(z) + \varepsilon z\Phi(z))1_{\{\Phi\ge 0\}}\,d\sigma \ge \varepsilon\|\Phi(z)\|_1 - C,
\end{aligned}
$$

where 1_A is the characteristic function of a set A; thus $\|\Phi(z)\|_{1,2} \leq C$ with C independent of t. This fact implies

$$\nabla\theta = e^z \nabla z \in L^2(\Omega).$$

Consequently we have (using imbedding theorems) $\Phi \in W^{1,\widetilde{q}}(\Omega)$ for arbitrary $\widetilde{q} < \infty$. Now, we are ready to show that $\Phi \in W^{2,\widetilde{q}}(\Omega)$ (the method is just the same as in previous case, we use standard elliptic theory) and from this fact we conclude that

$$z \in W^{2,\widetilde{q}}(\Omega) \hookrightarrow L^\infty(\Omega), \qquad \nabla z \in W^{1,\widetilde{q}}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Using all these facts and equation (4.39) we also get

$$\boldsymbol{w} \in W^{2,q}(\Omega).$$

This finishes the proof of Theorem 4.1, as

$$\|\boldsymbol{w}\|_{2,r} + \|z\|_{2,r} + \|\theta\|_{2,r} \leq C, \qquad 1 \leq r < \infty,$$

where the constant C does not depend on t. Moreover, $\theta = e^z$, $z \in L^\infty(\Omega)$, therefore we have $\theta \geq c(\varepsilon) > 0$.

5. Convergence

Using estimates from Theorem 4.1 we know that there exists a subsequence $\varepsilon_n \to 0^+$ such that:

$$\begin{aligned}
\boldsymbol{v}_{\varepsilon_n} &\rightharpoonup \boldsymbol{v} && \text{in } W^{1,q}(\Omega),\\
\boldsymbol{v}_{\varepsilon_n} &\to \boldsymbol{v} && \text{in } L^\infty(\Omega),\\
\rho_{\varepsilon_n} &\rightharpoonup^* \rho && \text{in } L^\infty(\Omega),\\
P_b(\rho_{\varepsilon_n}) &\rightharpoonup^* \overline{P_b(\rho)} && \text{in } L^\infty(\Omega),\\
K(\rho_{\varepsilon_n})\rho_{\varepsilon_n} &\rightharpoonup^* \overline{K(\rho)\rho} && \text{in } L^\infty(\Omega),\\
K(\rho_{\varepsilon_n}) &\rightharpoonup^* \overline{K(\rho)} && \text{in } L^\infty(\Omega),\\
\int_0^{\rho_{\varepsilon_n}} K(y)\,dy &\rightharpoonup^* \overline{\int_0^{\rho} K(y)\,dy} && \text{in } L^\infty(\Omega),\\
\theta_{\varepsilon_n} &\rightharpoonup \theta && \text{in } W^{1,1+\delta}(\Omega), \quad 0<\delta<1 \text{ arbitrary},\\
\theta_{\varepsilon_n} &\to \theta && \text{in } L^q(\Omega),
\end{aligned} \tag{5.1}$$

where the bar over a quantity denotes its weak limit for $\varepsilon_n \to 0^+$. From now on, we denote the sequences again $(\boldsymbol{v}_\varepsilon, \rho_\varepsilon, \theta_\varepsilon)$.

With this knowledge the limit of our problem looks as follows

$$\operatorname{div}(\overline{K(\rho)\rho}\boldsymbol{v}) = 0 \tag{5.2}$$

$$
\begin{aligned}
&- \operatorname{div}\Big(2\mu\mathbf{D}(\boldsymbol{v}) + \nu(\operatorname{div}\boldsymbol{v})\mathbf{I} - \overline{P_b(\rho)}\mathbf{I} - \theta\big(\overline{\int_0^\rho K(y)\,dy}\big)\mathbf{I}\Big) \\
&+ \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \overline{K(\rho)\rho}\boldsymbol{F}
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&\operatorname{div}((1+\theta^m)\nabla\theta) + \theta\overline{\big(\operatorname{div}\boldsymbol{v}\int_0^\rho K(y)\,dy\big)} + \operatorname{div}(\overline{K(\rho)\rho}\theta\boldsymbol{v}) \\
&= 2\mu\overline{|\mathbf{D}(\boldsymbol{v})|^2} + \nu\overline{(\operatorname{div}\boldsymbol{v})^2}
\end{aligned} \tag{5.4}
$$

together with the boundary conditions (1.10) and (1.11); all the equations are fulfilled in the weak sense. We will comment the less trivial limit passage in the internal energy balance, including the boundary terms, at the end of this paper.

Note that in equation (5.3), especially in the last term on the LHS, we used the fact that $\operatorname{div}(\overline{K(\rho)\rho}\boldsymbol{v}) = 0$.

Lemma 5.1. *Under the assumptions of Theorem* 2.1 *and Theorem* 4.1, *we have for* $q > 2$

$$
\|\rho_\varepsilon\|_\infty \le k+1 \qquad \text{and} \qquad \|\boldsymbol{v}_\varepsilon\|_{1,q} \le C(1 + k^{\gamma\frac{q-2}{q}}). \tag{5.5}
$$

Proof: The bound on the density follows directly from Theorem 4.1 — therefore we are going to estimate the velocity. If we write the approximative momentum equation (3.3) in the form

$$
\begin{aligned}
-\operatorname{div}\mathbf{S}(\boldsymbol{v}) = &-\nabla\Big(P_b(\rho_\varepsilon) + \theta_\varepsilon\big(\int_0^{\rho_\varepsilon} K(t)\,dt\big)\Big) + K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{F} \\
&- \frac{1}{2}\operatorname{div}\big(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon\big) - \frac{1}{2}K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon,
\end{aligned}
$$

we can notice that

$$
\begin{aligned}
\|\boldsymbol{v}_\varepsilon\|_{1,q} \le C\Big(&\|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon\|_q + \|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon\|_{\frac{2q}{q+2}} \\
&+ \|P_b(\rho_\varepsilon)\|_q + \|\theta_\varepsilon\big(\int_0^{\rho_\varepsilon} K(t)\,dt\big)\|_q + \|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{F}\|_{\frac{2q}{q+2}}\Big).
\end{aligned} \tag{5.6}
$$

The bounds on the density and temperature yield for q sufficiently large

$$
\|P_b(\rho_\varepsilon)\|_q \le \|P_b(\rho_\varepsilon)\|_2^{\frac{2}{q}}\|P_b(\rho_\varepsilon)\|_\infty^{\frac{q-2}{q}} \le C(1 + k^{\gamma\frac{q-2}{q}}) \tag{5.7}
$$

and for $q \ge 2\gamma$

$$
\|\theta_\varepsilon\big(\int_0^{\rho_\varepsilon} K(y)\,dy\big)\|_q \le C\|\rho\|_{q+\delta}\|\theta\|_{p(q,\delta)} \le C(1 + k^{1-\frac{2\gamma}{q+\delta}}). \tag{5.8}
$$

The only thing that remains is to estimate the convective term

$$
\|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon\|_q + \|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon\|_{\frac{2q}{q+2}} \le C\|\boldsymbol{v}\|_{1,2}^2\|P_b(\rho)\|_{\frac{q+\delta'}{\gamma}}^{\frac{1}{\gamma}},
$$

from where we directly deduce (using equations (5.6), (5.7) and (5.8) and also the fact that $\gamma > 2$) the bound for $q > 1$

$$\|\boldsymbol{v}_\varepsilon\|_{1,q} \le C(1 + k^{\gamma\frac{q-2}{q}}). \qquad \square$$

6. Convergence of the temperature

Lemma 6.1. *There exists a subsequence $\{s_\varepsilon\}$ such that*

$$s_\varepsilon \to s \qquad \text{in} \;\; L^2(\Omega).$$

Consequently

$$\theta_\varepsilon \to \theta \qquad \text{in} \;\; L^q(\Omega) \;\; \text{for} \;\; q < \infty,$$

where $\theta > 0$ a.e. in Ω.

Proof: From the previous sections we have

$$\int_\Omega |\nabla s_\varepsilon|^2 \, dx + \int_{\partial\Omega} (e^{s_\varepsilon} + e^{-s_\varepsilon}) \, d\sigma < C,$$

especially

$$\int_\Omega |\nabla s_\varepsilon|^2 dx + \int_{\partial\Omega} |s_\varepsilon|^2 \, d\sigma < C.$$

Thus there exists a subsequence $s_\varepsilon \to s$ in $L^2(\Omega)$. Recall that $\theta_\varepsilon = e^{s_\varepsilon}$ and $\theta_\varepsilon \to \theta$ in $L^q(\Omega)$, see the last line of (5.1). Now we use Vitali's theorem to get

$$e^{s_\varepsilon} \to e^s \qquad \text{in} \;\; L^q(\Omega)$$

and

$$\theta = e^s \quad \text{with} \;\; s \in L^2(\Omega).$$

Hence $\theta > 0$ a.e. in Ω as $s > -\infty$ a.e. in Ω. $\square$

7. Effective viscous flux

To prove the strong convergence of the density we need an interesting quantity called effective viscous flux. To define it, we use the Helmholtz decomposition of the velocity

$$\boldsymbol{v}_\varepsilon = \nabla^\perp A_\varepsilon + \nabla\phi_\varepsilon, \tag{7.1}$$

where $\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$, the scalar function ϕ_ε is given by the system

$$\begin{aligned} \Delta\phi_\varepsilon &= \operatorname{div} \boldsymbol{v}_\varepsilon \qquad && \text{in} \;\; \Omega \\ \frac{\partial\phi_\varepsilon}{\partial\boldsymbol{n}} &= 0 && \text{at} \;\; \partial\Omega, \end{aligned} \tag{7.2}$$

and the field A_ε is given by

$$
\begin{aligned}
\Delta A_\varepsilon = \operatorname{rot} \boldsymbol{v}_\varepsilon = \omega_\varepsilon \qquad & \text{in } \ \Omega \\
\boldsymbol{n} \cdot \nabla^\perp A_\varepsilon = 0 \qquad & \text{at } \ \partial\Omega,
\end{aligned}
\tag{7.3}
$$

with $\operatorname{rot} \boldsymbol{v}_\varepsilon = \frac{\partial (v_\varepsilon)_2}{\partial x_1} - \frac{\partial (v_\varepsilon)_1}{\partial x_2}$ (as we consider the two-dimensional case; note that in 2D $\operatorname{rot} \boldsymbol{v}_\varepsilon$ is scalar function). The basic estimates (following from the standard elliptic estimates) for A_ε and ϕ_ε are

$$
\begin{aligned}
&\|\nabla \nabla^\perp A_\varepsilon\|_q \le C\|\omega_\varepsilon\|_q \qquad && \|\nabla^2 \nabla^\perp A_\varepsilon\|_q \le C\|\omega_\varepsilon\|_{1,q} \\
&\|\nabla^2 \phi_\varepsilon\|_q \le C\|\operatorname{div} \boldsymbol{v}_\varepsilon\|_q \qquad && \|\nabla^3 \phi_\varepsilon\|_q \le C\|\operatorname{div} \boldsymbol{v}_\varepsilon\|_{1,q}.
\end{aligned}
\tag{7.4}
$$

We have (in the weak sense)

$$
\begin{aligned}
-\mu \Delta \omega_\varepsilon = & -\frac{1}{2} \operatorname{rot} \operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \frac{1}{2} \operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon) \\
& + \operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{F}) \qquad \text{in } \ \Omega \\
\omega_\varepsilon = & \left(2\chi - \frac{f}{\mu}\right) \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\tau} \qquad \text{at } \ \partial\Omega,
\end{aligned}
\tag{7.5}
$$

where χ is the curvature of $\partial\Omega$, cf. [5].

The form of system (7.5) enables to formulate the following two problems for $\omega_\varepsilon = \omega_\varepsilon^1 + \omega_\varepsilon^2$:

$$
\begin{aligned}
-\mu \Delta \omega_\varepsilon^1 = -\frac{1}{2} \operatorname{rot}\left(\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v})\boldsymbol{v}\right) \qquad & \text{in } \ \Omega \\
\omega_\varepsilon^1 = 0 \qquad \text{at } \ \partial\Omega, &
\end{aligned}
\tag{7.6}
$$

and

$$
\begin{aligned}
-\mu \Delta \omega_\varepsilon^2 = & -\operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon) + \operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{F}) \qquad \text{in } \ \Omega \\
\omega_\varepsilon^2 = & \left(2\chi - \frac{f}{\mu}\right) \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\tau} \qquad \text{at } \ \partial\Omega.
\end{aligned}
\tag{7.7}
$$

From these equations we get

$$
\|\omega_\varepsilon^1\|_2 \le C(1 + C(k)\sqrt{\varepsilon}), \tag{7.8}
$$

$$
\|\omega_\varepsilon^2\|_{1,q} \le C(1 + k^{1+\gamma\frac{q-2}{q}+\delta}), \tag{7.9}
$$

$\delta > 0$, arbitrarily small. In the first relation we used the approximative continuity equation (3.2):

$$
\operatorname{rot}(\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon) = \operatorname{rot}(\boldsymbol{v}_\varepsilon(\varepsilon h + \varepsilon\Delta\rho_\varepsilon - \varepsilon\rho_\varepsilon))
$$

and when estimating the term with $\Delta\rho_\varepsilon$, we use the fact that $\sqrt{\varepsilon}\|\nabla\rho_\varepsilon\|_2 \le C$. In the second estimate we have

$$\|\omega_\varepsilon^2\|_{1,q} \le C(\|\boldsymbol{v}_\varepsilon\|_{1-\frac{1}{q},q,\partial\Omega} + \|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon\|_q + 1),$$

and we use estimates from Lemma 5.1.

At last we are ready to introduce the fundamental quantity which is in fact the potential part of the momentum equation: the effective viscous flux. Using the Helmholtz decomposition in the approximative momentum equation we have

$$\begin{aligned} \nabla(-(2\mu+\nu)\Delta\phi_\varepsilon + P(\rho_\varepsilon,\theta_\varepsilon)) &= \mu\Delta\nabla^\perp A_\varepsilon + K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{F} \\ - K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon &- \frac{1}{2}\varepsilon hK(\rho_\varepsilon)\boldsymbol{v}_\varepsilon + \frac{1}{2}\varepsilon\rho_\varepsilon\boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon\Delta\rho_\varepsilon\boldsymbol{v}_\varepsilon. \end{aligned} \tag{7.10}$$

We define

$$G_\varepsilon = -(2\mu+\nu)\Delta\phi_\varepsilon + P(\rho_\varepsilon,\theta_\varepsilon) = -(2\mu+\nu)\operatorname{div}\boldsymbol{v}_\varepsilon + P(\rho_\varepsilon,\theta_\varepsilon) \tag{7.11}$$

and its limit version

$$G = -(2\mu+\nu)\operatorname{div}\boldsymbol{v} + \overline{P(\rho,\theta)}. \tag{7.12}$$

In the following lemmae we prove fundamental properties of the effective viscous flux.

Lemma 7.1.

$$\|G\|_\infty \le C(1+k^{1+\eta}) \quad \textit{with} \quad \eta > 0 \quad \textit{arbitrarily small}, \qquad \|G\|_2 \le C, \tag{7.13}$$

C independent of k and ε.

Proof: We pass to the limit in (7.10), getting

$$\nabla((-2\mu+\nu)\Delta\phi + \overline{P(\rho,\theta)}) = \mu\Delta\nabla^\perp A + \overline{K(\rho)\rho}\boldsymbol{F} - \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v};$$

from this equation we estimate $\|\nabla G\|_q$ $(q>2)$

$$\|\nabla G\|_q \le C(\|\Delta\nabla^\perp A\|_q + \|\overline{K(\rho)\rho}\boldsymbol{F}\|_q + \|\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q).$$

We still remember the results from Lemma 5.1 and so

$$\|\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q \le C\|\boldsymbol{v}\|_{1,q}^2\|\overline{K(\rho)\rho}\|_\infty \le Ck^{1+\eta},$$

with η arbitrarily small for $q\to 2$ (the second term from the second inequality form (5.5) is very small for $q\to 2$). Next, we have in the weak sense

$$\begin{aligned} -\mu\Delta\omega &= -\operatorname{rot}(\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}) - \operatorname{rot}(\overline{K(\rho)\rho}\boldsymbol{F}) \qquad \text{in } \ \Omega \\ \omega &= \left(2\chi - \frac{f}{\mu}\right)\boldsymbol{v}\cdot\boldsymbol{\tau} \qquad \text{at } \ \partial\Omega, \end{aligned}$$

thus

$$\|\omega\|_{1,q} \le C(\|\boldsymbol{v}\|_{1,q} + \|\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q + \|\overline{K(\rho)\rho}\boldsymbol{F}\|_q) \le C(1+k^{1+\eta}).$$

We also have (see (7.4))

$$\|\Delta\nabla^\perp A\|_q \le C\|\nabla\omega\|_q;$$

all this, together with control of mean value of G (in fact, the mean value of G is the mean value of $\overline{P(\rho,\theta)}$) and Sobolev imbeddings finish the proof of the first inequality. The second inequality we get immediately due to the fact that

$$\|G\|_2 \le C(\|\nabla\boldsymbol{v}\|_2 + \|P(\rho,\theta)\|_2). \qquad \square$$

Lemma 7.2. *Up to a subsequence, we have for $\varepsilon \to 0^+$*

$$G_\varepsilon \to G \text{ strongly in } L^2(\Omega). \tag{7.14}$$

Proof:

$$\begin{aligned}\nabla(G_\varepsilon - G) &= (K(\rho_\varepsilon)\rho_\varepsilon - \overline{K(\rho)\rho})\boldsymbol{F} - \frac{1}{2}K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon \\ &- \frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) + \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \mu\Delta\nabla^\perp(A_\varepsilon - A).\end{aligned} \tag{7.15}$$

For the first term we have

$$(K(\rho_\varepsilon)\rho_\varepsilon - \overline{K(\rho)\rho})\boldsymbol{F} \rightharpoonup 0 \qquad \text{in } L^q(\Omega)\ \forall\, q<\infty,$$

so the first term gives us strong convergence. The second "part" is

$$\begin{aligned}&-\frac{1}{2}K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) + \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} \\ &= -\frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon - K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon + \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v},\end{aligned} \tag{7.16}$$

and for the first term it holds

$$-\frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon = -\frac{1}{2}\varepsilon\Delta\rho_\varepsilon\boldsymbol{v}_\varepsilon + \frac{1}{2}\varepsilon\rho_\varepsilon\boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon hK(\rho)\boldsymbol{v}_\varepsilon,$$

in which the first term converges to zero strongly in $W^{-1,2}$ (this determines the space of convergence) and the other two terms converge to zero weakly in $L^q(\Omega)$, $q<\infty$. The other two terms in (7.16) converge to zero weakly in L^q due to the fact that $\operatorname{div}(\overline{K(\rho)\rho}\boldsymbol{v}) = 0$.

Finally we have to look at the last term of (7.15). First we show that

$$\nabla(\omega_\varepsilon - \omega) = B^1_\varepsilon + B^2_\varepsilon,$$

where $B^1_\varepsilon \rightharpoonup 0$ in $L^2(\Omega)$ and $B^2_\varepsilon \to 0$ in $W^{-1,2}(\Omega)$. We have

$$\Delta(\omega_\varepsilon - \omega) = -\frac{1}{2}\operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon) - \frac{1}{2}\operatorname{rot}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon)$$
$$+ \operatorname{rot}(\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \operatorname{rot}((K(\rho_\varepsilon)\rho_\varepsilon - \overline{K(\rho)\rho})\boldsymbol{F}) \qquad \text{in } \ L^2(\Omega)$$
$$\omega_\varepsilon - \omega = \left(2\chi - \frac{f}{\mu}\right)(\boldsymbol{v}_\varepsilon - \boldsymbol{v})\cdot\boldsymbol{\tau} \qquad \text{at } \ \partial\Omega.$$

Now, as above, we can estimate the difference $\omega_\varepsilon - \omega$ by $B^1_\varepsilon + B^2_\varepsilon$, where $B^1_\varepsilon \to 0$ strongly in $W^{-1,2}(\Omega)$ and $B^2_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega)$. Recalling

$$\|\Delta\nabla^\perp(A_\varepsilon - A)\|_{-1,2} \le \|\nabla(\omega_\varepsilon - \omega)\|_{-1,2},$$

we conclude that

$$\Delta\nabla^\perp(A_\varepsilon - A) = B^1_\varepsilon + B^2_\varepsilon,$$

where $B^1_\varepsilon \to 0$ in $W^{-1,2}(\Omega)$ and $B^2_\varepsilon \rightharpoonup 0$ in $L^2(\Omega)$. Therefore $G_\varepsilon - G \to$ const. in $L^2(\Omega)$. But we notice that

$$\int_\Omega (G_\varepsilon - G)\,dx = \int_\Omega \Delta(\phi_\varepsilon - \phi)\,dx + \int_\Omega (P(\rho_\varepsilon,\theta_\varepsilon) - \overline{P(\rho,\theta)})\,dx \to 0,$$

as

$$\int_{\partial\Omega} \frac{\partial\phi}{\partial\boldsymbol{n}}dS = \int_{\partial\Omega}\frac{\partial\phi_\varepsilon}{\partial\boldsymbol{n}}dS = 0,$$

hence the constant is zero. □

8. Limit passage

Theorem 8.1. *There exists sufficiently large $k_0 > 0$ such that for $k > k_0$ we have*

$$\frac{k-2}{k}(k-2)^\gamma \ge 1 + \|G\|_\infty \tag{8.1}$$

and for a subsequence $\varepsilon \to 0+$ it holds

$$\lim_{\varepsilon\to 0+} |\{x\in\Omega : \rho_\varepsilon(x) > k-2\}| = 0. \tag{8.2}$$

In particular, $\overline{K(\rho)\rho} = \rho$ a.e. in Ω.

Proof: We define a smooth function $M : \mathbb{R}^+_0 \to [0,1]$ such that

$$M(t) = \begin{cases} 1 & \text{for } \ t \le k-2 \\ \in (0,1) & \text{for } \ k-2 < t < k-1 \\ 0 & \text{for } \ k-1 \le t \end{cases}$$

and $M'(t) < 0$ for $t \in (k-2, k-1)$. Now, we multiply the approximative continuity equation (3.2) by M^l ($l \in \mathbb{N}$) and integrate over Ω. As

$$\varepsilon \int_\Omega M^l(\rho_\varepsilon)\Delta\rho_\varepsilon \, dx = -\varepsilon l \int_\Omega M^{l-1}(\rho_\varepsilon)M'(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2 \, dx \geq 0,$$

we get

$$\int_\Omega \left(\int_0^{\rho_\varepsilon(x)} tlM^{l-1}(t)M'(t) \, dt \right) \operatorname{div} \boldsymbol{v}_\varepsilon \geq R_\varepsilon$$

with $R_\varepsilon \to 0$ as $\varepsilon \to 0$. Now, we recall definitions of G and M to get

$$\begin{aligned} &- (k-2) \int_\Omega \Big(\int_0^{\rho_\varepsilon(x)} lM^{l-1}(t)M'(t) \, dt \Big) P(\rho_\varepsilon, \theta_\varepsilon) \, dx \\ &\leq k \Big| \int_\Omega \Big(\int_0^{\rho_\varepsilon(x)} -lM^{l-1}(t)M'(t) \, dt \Big) G_\varepsilon \, dx \Big| + R_\varepsilon. \end{aligned}$$

Thus the properties of M lead to the following inequality

$$\frac{k-2}{k} \int_{\{\rho_\varepsilon > k-2\}} (1 - M^l(\rho_\varepsilon))P(\rho_\varepsilon, \theta_\varepsilon) \, dx \leq \int_{\{\rho_\varepsilon > k-2\}} (1 - M^l(\rho_\varepsilon))|G_\varepsilon| \, dx + |R_\varepsilon|.$$

From the explicit form of approximative pressure function (3.5) we see

$$\begin{aligned} &\frac{k-2}{k}(k-2)^\gamma |\{\rho_\varepsilon > k-2\}| - \frac{k-2}{k} \|P(\rho_\varepsilon, \theta_\varepsilon)\|_2 \|M^l(\rho_\varepsilon)\|_2 \\ &\leq \|G\|_\infty |\{\rho_\varepsilon > k-2\}| + \|G - G_\varepsilon\|_1 + |R_\varepsilon|. \end{aligned}$$

Using inequality (7.13) (see Lemma 7.2) we are able to choose k_0 so large that for all $k > k_0$ we have (8.1), $\|G\|_\infty \leq Ck^{1+\eta}$ and $\gamma > 2$.

Therefore,

$$\begin{aligned} &|\{x \in \Omega : \rho_\varepsilon(x) > k-2\}| \\ &\leq C \left(\|M^l(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > k-2\})} + \|G - G_\varepsilon\|_{L^1(\Omega)} + |R_\varepsilon| \right). \end{aligned} \tag{8.3}$$

Now, for fixed $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$

$$C(\|G - G_\varepsilon\|_1 + |R_\varepsilon|) \leq \frac{\delta}{2}. \tag{8.4}$$

We fix ε and then consider the sequence $\{M^l(\rho_\varepsilon)1_{\{\rho_\varepsilon > k-2\}}\}_{l \in \mathbb{N}}$. We see that it monotonely pointwise converges to zero. Thus by the Lebesgue theorem we are able to find $l = l(\varepsilon, \delta)$ such that

$$C\|M^l(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > k-3\})} \leq \frac{\delta}{2}. \tag{8.5}$$

From (8.3), (8.4) and (8.5) we obtain

$$\lim_{\varepsilon\to 0} |\{x \in \Omega; \rho_\varepsilon(x) > k-2\}| \le \delta. \tag{8.6}$$

As $\delta > 0$ is arbitrarily small, Theorem 8.1 is proved. □

We have (for the proof see [6]):

Lemma 8.1. *It holds*

$$\int_\Omega \overline{P(\rho,\theta)\rho}\, dx \le \int_\Omega G\rho\, dx, \tag{8.7}$$

and

$$\int_\Omega \overline{P(\rho,\theta)}\rho\, dx = \int_\Omega G\rho\, dx. \tag{8.8}$$

Now, using (8.7) and (8.8), together with the elementary properties of weak limits ($\rho\overline{P(\rho,\theta)} \le \overline{P(\rho,\theta)\rho}$ a.e. in Ω) we get

$$\rho\overline{P(\rho,\theta)} \le \overline{P(\rho,\theta)\rho} \quad \text{a.e. in } \Omega, \qquad \text{i.e. } \overline{\rho^{\gamma+1}} + \overline{\rho^2}\theta = \overline{\rho^\gamma}\rho + \rho^2\theta \quad \text{a.e. in } \Omega.$$

The same elementary properties tell us that $\overline{\rho^{\gamma+1}} \ge \overline{\rho^\gamma}$ and $\overline{\rho^2}\theta \ge \rho^2\theta$, so

$$\overline{\rho^2}\theta = \rho^2\theta \qquad \text{a.e. in } \Omega.$$

The limit temperature is a.e. positive, thus $\overline{\rho^2} = \rho^2$ a.e. and

$$\lim_{\varepsilon\to 0} \|\rho_\varepsilon - \rho\|_2^2 = \overline{\rho^2} - \rho^2 = 0.$$

Next, let us check the strong convergence of the velocity gradient. Due to Theorem 8.1 and Lemma 6.1 we have

$$P(\rho_\varepsilon,\theta_\varepsilon) \to p(\rho,\theta) \qquad \text{strongly in } L^2(\Omega). \tag{8.9}$$

From (8.9) and (7.14) we deduce that (recall (7.11) and (7.12))

$$\operatorname{div} \boldsymbol{v}_\varepsilon \to \operatorname{div} \boldsymbol{v} \qquad \text{strongly in } L^2(\Omega). \tag{8.10}$$

Additionally, from the properties of the vorticity we already know that

$$\operatorname{rot} \boldsymbol{v}_\varepsilon \to \operatorname{rot} \boldsymbol{v} \qquad \text{strongly in } L^2(\Omega). \tag{8.11}$$

All this together with (7.4) and Korn's inequality implies

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \qquad \text{strongly in } W^{1,2}(\Omega). \tag{8.12}$$

Due to the bound of $\boldsymbol{v}_\varepsilon$ in $W^{1,q}$ we have

$$\mathbf{S}(\boldsymbol{v}_\varepsilon):\nabla\boldsymbol{v}_\varepsilon \to \mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v} \qquad \text{strongly in } L^q(\Omega), \tag{8.13}$$

$1 \le q < \infty$. Now, it is time for a little summary. We know that

$$\begin{aligned} \rho_\varepsilon &\to \rho && \text{in } L^q(\Omega) \text{ for } q<\infty \\ \boldsymbol{v}_\varepsilon &\to \boldsymbol{v} && \text{in } W^{1,q}(\Omega) \text{ for } q<\infty \\ \theta_\varepsilon &\to \theta && \text{in } L^q(\Omega) \text{ for } q<\infty \\ \theta_\varepsilon &\rightharpoonup \theta && \text{in } W^{1,2}(\Omega) \text{ if } m\ge 2 \\ \theta_\varepsilon &\rightharpoonup \theta && \text{in } W^{1,1+\delta}(\Omega) \text{ for } \delta<1 \text{ if } m<2. \end{aligned} \tag{8.14}$$

We return to the approximative energy equation (3.4) and explain the limit passage in a more detailed way. We have

$$\begin{aligned} &\int_\Omega (1+\theta_\varepsilon^m)\frac{\varepsilon+\theta_\varepsilon}{\theta_\varepsilon}\nabla\theta_\varepsilon\cdot\nabla\phi\,dx + \int_{\partial\Omega} L(\theta_\varepsilon)(\theta_\varepsilon-\theta_0)\phi\,d\sigma + \int_{\partial\Omega}\varepsilon\ln\theta_\varepsilon\phi\,d\sigma \\ &-\int_\Omega \Big(\Big(\int_0^{\rho_\varepsilon(x)} K(t)\,dt\Big)\boldsymbol{v}_\varepsilon\cdot\nabla(\theta_\varepsilon\phi) + K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla(\theta_\varepsilon\phi)\Big)dx \\ &+\int_\Omega \Big(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\theta_\varepsilon\phi + \operatorname{div}(\theta_\varepsilon\boldsymbol{v}_\varepsilon\phi)\int_0^{\rho_\varepsilon(x)} K(t)\,dt\Big)dx \\ &= \int_\Omega \mathbf{S}(\boldsymbol{v}_\varepsilon):\nabla\boldsymbol{v}_\varepsilon\phi\,dx. \end{aligned} \tag{8.15}$$

From (8.14) we see that

$$(1+\theta_\varepsilon^m)\frac{\varepsilon+\theta_\varepsilon}{\theta_\varepsilon}\nabla\theta_\varepsilon \rightharpoonup (1+\theta^m)\nabla\theta \qquad \text{in } L^q(\Omega) \text{ for } q<2$$

and passing to the limit with the last four terms in the LHS of (8.15) we get (using the strong convergence of the density)

$$\begin{aligned} &\int_\Omega (-\rho\boldsymbol{v}\cdot\nabla(\theta\phi) - \rho\boldsymbol{v}\cdot\nabla(\theta\phi) + \rho\phi\boldsymbol{v}\cdot\nabla\theta + \operatorname{div}(\theta\phi\boldsymbol{v})\rho)\,dx \\ &= \int_\Omega (-\rho\theta\boldsymbol{v}\cdot\nabla\phi + \rho\theta\operatorname{div}\boldsymbol{v}\phi)\,dx. \end{aligned} \tag{8.16}$$

To pass with the boundary term we have to use the interpolation inequality (recall that $\theta_n \to \theta$ in any $L^q(\Omega)$ and $l+1=m$):

$$\int_{\partial\Omega}|\theta|^{l+1} \le c\|\nabla\theta\|_q\|\theta\|^l_{\frac{lq}{q-1}},$$

which implies

$$\|\theta_n - \theta_m\|_{l+1,\partial\Omega}^{l+1} \le c\|\nabla(\theta_n - \theta_m)\|_q^{\frac{1}{l+1}} \|(\theta_n - \theta_m)\|_{\frac{lq}{q-1}}^{l+\frac{l}{l+1}},$$

where the first term is bounded and the second one converges to zero; hence the boundary term also converges to zero.

The very last thing to do is to prove that the limit temperature $\theta \in W^{1,p}(\Omega)$. Defining

$$\Phi(\theta) = \int_0^\theta (1 + t^m)\, dt$$

we rewrite the limit equation using the Kirchhoff transform as in Section 4 in terms of $\Phi(\theta)$. Testing it by $\Phi(\theta)$, we get as in Section 4 that $\theta \in L^\infty(\Omega)$. One more iteration in the energy equation and we see that $\theta \in W^{1,p}(\Omega)$ for $p < \infty$. This finishes the proof of Theorem 2.1.

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Charles University, Mathematical Institute, Sokolovská 83,
186 75 Prague 8, Czech Republic

E-mail: petra.pech@post.cz
pokorny@karlin.mff.cuni.cz

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