The Kadison problem on a class of commutative Banach algebras with closed cone

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Abstract. The main result of the paper characterizes continuous local derivations on a class of commutative Banach algebra A that all of its squares are positive and satisfying the following property: Every continuous bilinear map Φ from $A \times A$ into an arbitrary Banach space B such that $\Phi(a,b) = 0$ whenever $ab = 0$, satisfies the condition $\Phi(ab,c) = \Phi(a,bc)$ for all $a,b,c \in A$.

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1. Introduction

The authors [2] introduce the class of Banach algebras A with the following property:

(1) Every continuous bilinear map Φ from $A \times A$ into an arbitrary Banach space B such that $\Phi(a,b) = 0$ whenever $ab = 0$, satisfies the condition $\Phi(ab,c) = \Phi(a,bc)$ for all $a,b,c \in A$.

In [1], [2] the authors provided several strong reasons that should motivate the investigation in this algebraic frame.

Let A be an algebra. A linear mapping $d : A \to A$ with the property that $d(ab) = ad(b) + d(a)b$ for all $a,b \in A$ is called a *derivation* on A. A linear mapping $\delta : A \to A$ is called a *local derivation* on A if for every element $a \in A$ there exists a derivation $d_a : A \to A$ (depending on a) such that $\delta(a) = d_a(a)$. The concept of local derivations was introduced by Kadison [6] who studied local derivations on von Neumann algebras and polynomial algebras. Moreover, Kadison [6] set out a program of study for local maps suggesting that local derivations could prove useful in building derivations with particular properties. This paper intends to contribute to this program by studying the continuous local derivations on commutative Banach algebras satisfying Property (1) and all of its squares are positive. Specially, we will show that if A is a commutative Banach algebra satisfying Property (1) and all of its squares are positive and $d : A \to A$ is a continuous local derivation, then d is a generalized multiplier on A, i.e.,

$$d(xy)zt = d(x)yzt$$

for all $x, y, z, t \in A$. In the case where A is in addition semiprime, the situation improves. Indeed, any continuous local derivation on A is a multiplier, i.e.,

$$d(xy) = d(x)y$$

for all $x, y \in A$. Finally, we study the notion of local derivations on Banach *Dedekind σ-complete* almost f-algebras.

We point out that all proofs are purely order theoretical and algebraic in nature and do not involve any analytical means. We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notations and concepts that are not explained in this paper, one can refer to the standard monographs [3], [7], [8], [9].

2. Preliminaries

In order to avoid unnecessary repetitions we will suppose that all vector lattices and ℓ-algebras under consideration are **Archimedean**.

In the following lines, we recall definitions and some basic facts about vector lattices and lattice-ordered algebras.

A vector lattice A that is at the same time a Banach space with a norm which satisfies the monotonicity condition:

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$$

is called *Banach lattice*.

A vector lattice A is called *Dedekind σ-complete* if every non-empty and at most countable subset of A which is bounded from above has a supremum.

In the next lines, we recall definitions and some basic facts about f-algebras. A (real) algebra A which is simultaneously a vector lattice such that the partial ordering and the multiplication in A are compatible, so $a, b \in A^+$ implies $ab \in A^+$ is called *lattice-ordered algebra* (briefly *ℓ-algebra*). In an ℓ-algebra A we denote the collection of all nilpotent elements of A by $N(A)$. An ℓ-algebra A is referred to be *semiprime* if $N(A) = \{0\}$. An ℓ-algebra A is called an *f-algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. Every unital f-algebra (i.e., an f-algebra with a unit element) is semiprime. The ℓ-algebra A is called an *almost f-algebra* whenever it follows from $a \wedge b = 0$ that $ab = 0$. The f-algebras and the almost f-algebras are automatically commutative and have positive squares.

We end this section with some definitions about bilinear maps on vector lattices. Let A and B be vector lattices. A bilinear map Ψ from $A \times A$ into B is said to be *orthosymmetric* if for all $(a, b) \in A \times A$, $a \wedge b = 0$ implies $\Psi(a, b) = 0$, see [4].

3. Main results

Next, we will consider continuous local derivations on ℓ-algebras. We remark that in general a continuous local derivation on a Banach algebras A with the property (1) is not a derivation. This is illustrated in the following example.

Example 1. Take $A = \mathbb{R}^2$ with the usual operations and order. For all $(a, b), (a', b') \in A$, we define the following multiplication

$$(a, b) * (a', b') = (0, aa').$$

A simple verification shows that A is a Banach algebras A with the property (1) under the multiplication $*$. Observe that any derivation $d : A \to A$ is uniquely defined by two real numbers a, b so that

$$d(x, y) = d_{a,b}(x, y) = (ax, bx + 2ay).$$

The identity of A, denoted by Id_A, is a continuous local derivation. Indeed, if $x, y, b \in \mathbb{R}$, then $\mathrm{Id}_A(0, y) = d_{1/2,b}(0, y)$ and $\mathrm{Id}_A(x, y) = d_{1,-y/x}(x, y)$ if $x \neq 0$. But Id_A is not a derivation on A.

The next propositions are an essential ingredient for our main results.

Proposition 1. *Let A be a commutative Banach algebra satisfying Property* (1) *and with all squares positive, and let $d : A \to A$ be a local derivation. Then*

$$d(x)yz = xd(y)z = 0$$

for all $z \in A$ and as soon as $xy = 0$.

Proof: Let $d : A \to A$ be a local derivation and let $x, y \in A$ such that $xy = 0$. It follows that there exists a derivation δ_x on A such that $d(x)y = \delta_x(x)y$. Since

$$0 = \delta_x(xy) = \delta_x(x)y + x\delta_x(y)$$

it follows that

$$0 = (\delta_x(x)y + x\delta_x(y))^2 = (\delta_x(x)y)^2 + (x\delta_x(y))^2.$$

Consequently,

$$(\delta_x(x)y)^2 = (x\delta_x(y))^2 = 0.$$

By using the same argument as in [5, Theorem 3.9(i)], we have

$$(\delta_x(x)y - z)^2 \geq 0$$

for all $z \in A$, $n \in \mathbb{Z}$. Therefore

$$2n\delta_x(x)yz \leq z^2$$

for all $z \in A$, $n \in \mathbb{Z}$. As A is assumed to be Archimedean, we deduce that

$$\delta_x(x)yz = d(x)yz = 0$$

for all $z \in A$ and as soon as $x \wedge y = 0$. Using the same argument, we prove that

$$xd(y)z = 0$$

for all $z \in A$ and as soon as $x \wedge y = 0$, which gives the desired result. □

Theorem 1. *Let A be a commutative Banach algebra satisfying Property* (1) *and with all squares positive, and let $d : A \to A$ be a continuous local derivation. Then*

$$d(xt)yz = d(x)yzt$$

for all $x, y, z, t \in A$.

Proof: Let $z \in A$, then according to the previous proposition, the following bilinear map $\Psi_z : A \times A \to A$ defined by $\Psi_z(x, y) = d(x)yz$, for all $x, y \in A$, satisfies the property $\Psi_z(x, y) = 0$ as soon as $xy = 0$. Since A is a commutative Banach algebra satisfying Property (1) and all its squares are positive, it follows that

$$\begin{aligned} d(xt)yz &= \Psi_z(xt, y) \\ &= \Psi_z(x, yt) \\ &= d(x)yzt \end{aligned}$$

for all $x, y, z, t \in A$, as required. □

Corollary 1. *Let A be a semiprime commutative Banach algebra satisfying Property* (1) *and with all squares positive, and let $d : A \to A$ be a continuous local derivation. Then d is a multiplier on A, i.e.,*

$$d(xy) = d(x)y$$

for all $x, y \in A$.

Proof: Let $x, y, z \in A$. As it is seen in the proof of the previous theorem,

$$(d(xy) - d(x)y)^3 = 0.$$

Consequently, $d(xy)z - d(x)yz \in N(A)$. Since A is semiprime, it follows that

$$d(xy) - d(x)y = 0$$

for all $x, y \in A$, as required. □

Corollary 2. *Let A be a unitary commutative Banach algebra satisfying Property* (1) *and with all squares positive, and let $d : A \to A$ be a continuous local derivation. Then d is a multiplier on A, i.e.,*

$$d(xy) = d(x)y$$

for all $x, y \in A$.

Before continuing with the next result, we recall the following notion.

Let A be a vector lattice and let $0 \leq a \in A$. An element $0 \leq e \in A$ is called a *component* of a if $e \wedge (a - e) = 0$.

Our purpose now is to describe continuous derivations for the class of Banach Dedekind σ-complete almost f-algebras. The next proposition is an essential ingredient for the next result.

Proposition 2. *Let A be a Dedekind σ-complete vector lattice, let B be a Banach space and let Φ be a continuous bilinear map from $A \times A$ into B such that $\Phi(a, b) = 0$ whenever $a \wedge b = 0$. Then Φ is symmetric.*

Proof: Let $a, b \in A$ and let $e = |a| + |b|$. It is well known that the order ideal I_e generated by e in A can be seen as an f-algebra with e as unit (where its f-algebra multiplication is denoted by juxtaposition), see [8, Remark 19.5].

Let $L = \{k \in I_e; k = \sum_{i=1}^{n} \alpha_i e_i, \alpha_i \in \mathbb{R}, e_i \text{ is a component of } e,\ n \in \mathbb{N}^*\}$. By using the Freudenthal spectral Theorem [7, Theorem 40.2], we deduce that, for each $0 \leq x \in I_e$, there exists a sequence k_n of the form $\sum_{i=1}^{n} \alpha_i e_i, \alpha_i \in \mathbb{R}_+$, where e_i is a component of e, such that

$$0 \leq x - k_n \leq \frac{1}{n} e.$$

Hence $k_n \to x\,(r.u)$. Therefore L is a dense vector subspace of I_e.

We claim that the restriction of Φ to $L \times L$, denoted also by Φ, is symmetric. To this end, let $k, k' \in L$. It follows that $k = \sum_{i=1}^{i=n} \alpha_i e_i$ and $k' = \sum_{j=1}^{j=m} \beta_j f_j$, where e_i, f_j are components of e. Then

$$\begin{aligned}\Phi(k, k') &= \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \alpha_i \beta_j \Phi(e_i, f_j) \\ &= \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \alpha_i \beta_j \Phi(e_i(f_j + (e - f_j)), f_j(e_i + (e - e_i))).\end{aligned}$$

Then

$$\Phi(e_i, f_j) = \Phi(e_i f_j, f_j) + \Phi(e_i(e - f_j), f_j)$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Since

$$e_i(e - f_j) \wedge f_j = 0,$$

we have

$$e_i(e - f_j)f_j = 0,$$
$$\Phi(e_i, f_j) = \Phi(e_i f_j, f_j).$$

Using the same argument, we deduce that

$$\Phi(e_i, f_j) = \Phi(e_i f_j, e_i f_j) = \Phi(f_j e_i, e_i f_j) = \Phi(f_j, e_i).$$

Therefore

$$\begin{aligned}\Phi(k, k') &= \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \alpha_i\beta_j\Phi(e_i, f_j)\\ &= \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \alpha_i\beta_j\Phi(f_j, e_i)\\ &= \Phi(k', k).\end{aligned}$$

Hence the restriction of Φ to $L \times L$ is symmetric. Now since $f, g \in I_e$ and since L is dense in I_e, there exists $f_n, g_n \in L$, for all $n \in \mathbb{N}$, such that $f_n \to f\,(r.u)$ and $g_n \to g\,(r.u)$. By the continuity of Φ, we have

$$\Phi(f_n, g_n) \to \Phi(f, g) \quad \text{and} \quad \Phi(g_n, f_n) \to \Phi(g, f).$$

Since $\Phi(f_n, g_n) = \Phi(g_n, f_n)$, it follows that $\Phi(f, g) = \Phi(g, f)$, which gives the desired result. □

We have gathered now all prerequisites for the proof of our the following result.

Theorem 2. *Any Banach Dedekind σ-complete almost f-algebra A satisfies the property* (1).

PROOF: Let Φ be a continuous bilinear map from $A \times A$ into a Banach space B such that $\Phi(a, b) = 0$ whenever $ab = 0$. We shall prove that $\Phi(ab, c) = \Phi(a, bc)$ for all $a, b, c \in A$. To this end let $a, b, c \in A$. By the previous result $\Phi(ab, c) = \Phi(c, ab)$. Let us define the Φ_b be a continuous bilinear map from $A \times A$ into B defined by $\Phi_b(x, y) = \Phi(x, yb)$. The bilinear Φ_b satisfies the hypothesis of the previous result, thus Φ_b is symmetric. It follows that

$$\begin{aligned}\Phi(c, ab) &= \Phi_b(c, a)\\ &= \Phi_b(a, c)\\ &= \Phi(a, bc)\end{aligned}$$

for all $a, b, c \in A$ and we are done. □

The next corollaries turns to be immediate consequences of the previous theorems.

Corollary 3. *Let A be a Banach Dedekind σ-complete almost f-algebra and let $d : A \to A$ be a continuous local derivation. Then*

$$d(xt)yz = d(x)yzt$$

for all $x, y, z, t \in A$.

Corollary 4. *Let A be a semiprime Banach Dedekind σ-complete almost f-algebra and let $d : A \to A$ be a continuous local derivation. Then d is a multiplier on A, i.e.,*

$$d(xy) = d(x)y$$

for all $x, y \in A$.

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