The weak compactness of almost Dunford-Pettis operators

Belmesnaoui Aqzzouz, Aziz Elbour, Othman Aboutafail

Abstract. We characterize Banach lattices on which every positive almost Dunford-Pettis operator is weakly compact.

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1. Introduction and notation

Recall from Wnuk [4] that an operator T from a Banach lattice E into a Banach space F is said to be almost Dunford-Pettis if $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E.

As Dunford-Pettis operators [2], there exists an almost Dunford-Pettis operator which is not weakly compact. In fact, the identity operator of the Banach lattice $L^1([0,1])$ is almost Dunford-Pettis (because $L^1([0,1])$ has the positive Schur property) but not weakly compact. Conversely, there exists a weakly compact operator which is not almost Dunford-Pettis. In fact, the identity operator of the Banach lattice l^2 is weakly compact but not almost Dunford-Pettis.

The aim of this paper is to present some necessary and sufficient conditions for almost Dunford-Pettis operators being weakly compact. More precisely, we will prove that every almost Dunford-Pettis operator T from a Banach lattice E into a Banach space Y is weakly compact if and only if the norm of the topological dual E' is order continuous or Y is reflexive (Theorem 2.1). As a consequence, we obtain a characterization of the order continuity of a dual norm (Corollary 2.3) and a characterization of a reflexive Banach space (Corollary 2.4). After that, we will show that the class of almost Dunford-Pettis operators satisfies the domination property (Proposition 2.5). Finally, we will prove that the second power of every positive almost Dunford-Pettis operator T from a Banach lattice E into itself, is weakly compact if and only if the norm of E' is order continuous (Theorem 2.6).

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for every $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for every net (x_α) such that $x_\alpha \downarrow 0$ in

E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

We will use the term operator $T : E \longrightarrow F$ between two Banach spaces to mean a bounded linear mapping. An operator $T : E \longrightarrow F$ between two Banach lattices is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that every positive linear mapping on a Banach lattice is continuous.

We refer to [1] for unexplained terminology of the Banach lattice theory and positive operators.

2. Main results

Our first result gives necessary and sufficient conditions under which every almost Dunford-Pettis operator is weakly compact.

Theorem 2.1. *Let E be a Banach lattice and let Y be a Banach space. Then the following assertions are equivalent:*

1. *Every almost Dunford-Pettis operator $T : E \longrightarrow Y$ is weakly compact.*
2. *Every Dunford-Pettis operator $T : E \longrightarrow Y$ is weakly compact.*
3. *One of the following statements is valid:*
   - (a) *The norm of E' is order continuous.*
   - (b) *Y is reflexive.*

Proof: (1)$\Rightarrow$(2) Obvious.

(2)$\Rightarrow$(3) Assume by way of contradiction that the norm of E' is not order continuous and Y is not reflexive. By Theorem 2.4.14 of [3] we may assume that l^1 is a closed sublattice of E, and it follows from Proposition 2.3.11 of [3] that there is a positive projection P from E onto l^1.

Also, since the closed unit ball of Y is not weakly compact, it follows from Eberlein-Šmulian theorem [1, Theorem 3.40] that there exists a sequence (x_n) in B_Y which does not have any weakly convergent subsequence. We consider the operator S defined by

$$S : l^1 \longrightarrow Y, \quad (\lambda_n) \longmapsto \sum_{n=1}^{\infty} \lambda_n x_n.$$

Note that in view of

$$\sum_{n=1}^{\infty} \|\lambda_n x_n\| \leq \left(\sum_{n=1}^{\infty} |\lambda_n| \right) < \infty,$$

the series defining S converges in norm for every $(\lambda_n) \in l^1$, and then S is well defined. We have to prove that $T = S \circ P : E \longrightarrow Y$ is Dunford-Pettis. Since l^1 has the Schur property (i.e. every sequence weakly converging to zero in l^1 is norm convergent to zero [1, Theorem 4.32]), its identity operator $\mathrm{Id}_{l^1} : l^1 \longrightarrow l^1$ is Dunford-Pettis. So, the composed operator $T = S \circ \mathrm{Id}_{l^1} \circ P$ is Dunford-Pettis.

But the operator T is not weakly compact. In fact, note that $x_n = S \circ P(e_n)$ for all $n \in \mathbb{N}$, where e_n is the sequence with the n'th entry equal to 1 and all others to zero. Since the sequence (x_n) does not have any weakly convergent subsequence, we conclude that T is not weakly compact, which contradicts (2).

(a) $\Rightarrow$(1) By Proposition 3.6.12 of [3] it suffices to show that T is M-weakly compact (i.e. $\|Tx_n\| \longrightarrow 0$ for every disjoint sequence (x_n) in the closed unit ball B_E). To see this, let (x_n) be a disjoint sequence in B_E. Since the norm of E' is order continuous, it follows from Theorem 2.4.14 of [3] that $x_n \longrightarrow 0$ for $\sigma(E, E')$. Now, as T is almost Dunford-Pettis, $\|Tx_n\| \longrightarrow 0$ and hence T is M-weakly compact.

(b) $\Rightarrow$(1) In this situation, every operator from E into Y is weakly compact. □

If in Theorem 2.1, the Banach space Y is a Banach lattice, we obtain

Theorem 2.2. *Let E and F be two Banach lattices. Then the following assertions are equivalent:*

(1) *Every positive almost Dunford-Pettis operator $T : E \longrightarrow F$ is weakly compact.*
(2) *Every positive Dunford-Pettis operator $T : E \longrightarrow F$ is weakly compact.*
(3) *One of the following statements is valid:*
   (a) *The norm of E' is order continuous.*
   (b) *F is reflexive.*

Proof: It suffices to prove that (2) $\Rightarrow$(3). But this is the same as the implication (2) $\Rightarrow$(3) in the Theorem 2.1 by observing that the Banach lattice F is reflexive if and only if the positive part of its unit ball is weakly compact. □

As a consequence of Theorem 2.2, we obtain the following characterization of the order continuity of the dual norm:

Corollary 2.3. *Let E be a Banach lattice. Then the following statements are equivalent:*

(1) *Every almost Dunford-Pettis operator T from E into E is weakly compact.*
(2) *Every positive almost Dunford-Pettis operator T from E into E is weakly compact.*
(3) *The norm of E' is order continuous.*

Another consequence of Theorem 2.1 is given by the following interesting result on reflexive Banach spaces:

Corollary 2.4. *For a Banach space Y, the following statements are equivalent:*

(1) *If E is an infinite dimensional AL-space, then every operator $T : E \longrightarrow Y$ is weakly compact.*
(2) *Every operator $T : L^1[0, 1] \longrightarrow Y$ is weakly compact.*
(3) *Every operator $T : l^1 \longrightarrow Y$ is weakly compact.*
(4) *Y is reflexive.*

Proof: (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious (because $L^1[0,1]$ and l^1 are AL-spaces).

(2) $\Rightarrow$ (4) If every operator $T : L^1[0,1] \longrightarrow Y$ is weakly compact, then it follows from Theorem 2.1 that the norm of $(L^1[0,1])'$ is order continuous or Y is reflexive. But the norm of $(L^1[0,1])' = L^\infty[0,1]$ is not order continuous. So Y is reflexive.

(3) $\Rightarrow$ (4) Repeat the argument of the implication (2) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1) Obvious. $\square$

Finally, note that there exists a Banach lattice E and there exists an almost Dunford-Pettis operator T from E into E such that its second power operator T^2 is not weakly compact. In fact, the operator $T = \mathrm{Id}_{l^1}$ is almost Dunford-Pettis but its second power $T^2 = \mathrm{Id}_{l^1}$ is not weakly compact.

To give necessary and sufficient conditions for which each positive almost Dunford-Pettis operator T from E into E admits a second power operator T^2 which is weakly compact, we need to establish a result on the domination for the class of almost Dunford-Pettis operators.

Proposition 2.5. *Let E and F be two Banach lattices and let $S, T : E \longrightarrow F$ be two operators such that $0 \leq S \leq T$ and T is almost Dunford-Pettis. Then S is also almost Dunford-Pettis.*

Proof: Let (x_n) be a disjoint sequence of E such that $x_n \longrightarrow 0$ for $\sigma(E, E')$; it follows from [4, Remark 1] that $|x_n| \longrightarrow 0$ for $\sigma(E, E')$. Since T is almost Dunford-Pettis, thus $\|T(|x_n|)\| \longrightarrow 0$. Using the inequalities $|S(x_n)| \leq S(|x_n|) \leq T(|x_n|)$, we see that $\|S(x_n)\| \leq \|T(|x_n|)\|$ for all n, from which we get $\|S(x_n)\| \longrightarrow 0$, and hence the operator S is almost Dunford-Pettis. $\square$

Now, we give our second main result.

Theorem 2.6. *Let E be a Banach lattice. Then the following conditions are equivalent:*

1. *For all positive operators S and T from E into E such that $0 \leq S \leq T$ and T is almost Dunford-Pettis, the operator S is weakly compact.*
2. *Every positive almost Dunford-Pettis operator T from E into E is weakly compact.*
3. *For every positive almost Dunford-Pettis operator T from E into E, the second power operator T^2 is weakly compact.*
4. *The norm of E' is order continuous.*

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) If the norm in E' is not order continuous, then E contains a positively complemented copy of l^1, and clearly a positive projection $P : E \longrightarrow l^1 \longrightarrow E$ is a Dunford-Pettis operator from E into E whose square $P^2 = P$ is not weakly compact because P fixes a copy of l^1.

(4) $\Rightarrow$ (1) Let S and T be two operators from E into E such that $0 \leq S \leq T$ and T is almost Dunford-Pettis. By Proposition 2.5, the operator S is almost Dunford-Pettis. Finally, the result follows from Theorem 2.1. $\square$

References

[1] Aliprantis C.D., Burkinshaw O., *Positive Operators*, reprint of the 1985 original, Springer, Dordrecht, 2006.

[2] Aqzzouz B., Nouira R., Zraoula L., *Les opérateurs de Dunford-Pettis positifs qui sont faiblement compacts*, Proc. Amer. Math. Soc. **134** (2006), 1161–1165.

[3] Meyer-Nieberg P., *Banach Lattices*, Universitext, Springer, Berlin, 1991.

[4] Wnuk W., *Banach lattices with the weak Dunford-Pettis property*, Atti Sem. Mat. Fis. Univ. Modena **42** (1994), no. 1, 227–236.

Belmesnaoui Aqzzouz:
Université Mohammed V-Souissi, Faculté des Sciences Economiques, Juridiques et Sociales, Département d'Economie, B.P. 5295, SalaAljadida, Morocco

E-mail: baqzzouz@hotmail.com

Aziz Elbour, Othman Aboutafail:
Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra, Morocco

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