On evolutionary Navier-Stokes-Fourier type systems in three spatial dimensions

Miroslav Bulíček, Roger Lewandowski, Josef Málek

Abstract. In this paper, we establish the large-data and long-time existence of a suitable weak solution to an initial and boundary value problem driven by a system of partial differential equations consisting of the Navier-Stokes equations with the viscosity ν polynomially increasing with a scalar quantity k that evolves according to an evolutionary convection diffusion equation with the right hand side $\nu(k)|\mathbf{D}(\boldsymbol{v})|^2$ that is merely L^1-integrable over space and time. We also formulate a conjecture concerning regularity of such a solution.

Keywords: large data existence, suitable weak solution, Navier-Stokes-Fourier equations, incompressible fluid, the viscosity increasing with a scalar quantity, regularity, turbulent kinetic energy model

Classification: 35Q30, 35Q35, 76F60

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set and $T \in (0, \infty)$. Our goal is to prove the existence of a triple $(\boldsymbol{v}, k, p) : (0,T) \times \Omega \to \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}$ which solves, in $(0,T) \times \Omega$, the following nonlinear system of five partial differential equations

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}(\nu(k)\mathbf{D}(\boldsymbol{v})) = -\nabla p, \tag{1.2}$$

$$k_{,t} + \operatorname{div}(k\boldsymbol{v}) - \operatorname{div}(\mu(k)\nabla k) + \varepsilon(k) = \nu(k)|\mathbf{D}(\boldsymbol{v})|^2. \tag{1.3}$$

We complete the system (1.1)–(1.3) by the following initial and boundary conditions:

$$\begin{aligned} \boldsymbol{v}(0,x) &= \boldsymbol{v}_0(x) \\ k(0,x) &= k_0(x) \quad \text{and} \quad k_0(x) \geq 0 \end{aligned} \qquad \text{a.e. in } \Omega, \tag{1.4}$$

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$$\begin{aligned} &\boldsymbol{v}\cdot\boldsymbol{n}=0 \\ &\lambda\boldsymbol{v}_{\boldsymbol{\tau}}+(1-\lambda)\left(\nu(k)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right)_{\boldsymbol{\tau}}=0 \end{aligned} \qquad \text{on } (0,T)\times\partial\Omega, \tag{1.5}$$

$$k=0 \qquad \text{on } (0,T)\times\partial\Omega_D, \tag{1.6}$$

$$\nabla k\cdot\boldsymbol{n}=0 \qquad \text{on } (0,T)\times\partial\Omega_N. \tag{1.7}$$

Here, $\mathbf{D}(\boldsymbol{v})$ denotes the symmetric part of the gradient of the vector field $\boldsymbol{v}$, i.e., $2\mathbf{D}(\boldsymbol{v})=\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T$, $\boldsymbol{n}=\boldsymbol{n}(x)$ is the outer normal to the boundary located at $x\in\partial\Omega$, $\boldsymbol{w}_{\boldsymbol{\tau}}:=\boldsymbol{w}-(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{n}$ denotes the projection of a vector $\boldsymbol{w}=\boldsymbol{w}(x)$ to the tangent plane of the boundary at x, $\partial\Omega_D$ and $\partial\Omega_N$ are smooth subset of $\partial\Omega$ satisfying $\overline{\partial\Omega_D\cup\partial\Omega_N}=\partial\Omega$ and $\partial\Omega_D\cap\partial\Omega_N=\emptyset$. The parameter $\lambda\in[0,1]$ homotopically connects a homogeneous Neumann type boundary condition for $\lambda=0$ with the homogeneous Dirichlet boundary condition for $\lambda=1$. If $0<\lambda<1$, then $(1.5)_2$ is called Navier's slip boundary conditions. In this paper we assume that λ is any number from $[0,1)$.

Concerning the functions $\mu,\nu,\varepsilon:\mathbb{R}_+\to\mathbb{R}_+$, we require that they are continuous and that for certain $\alpha,\beta,\gamma\in[0,\infty)$ and two positive constants C_1,C_2 the following inequalities hold for all $k\in\mathbb{R}_+$:

$$\begin{aligned} C_1(1+k)^\alpha\le\nu(k)\le C_2(1+k)^\alpha, \\ C_1(1+k)^\beta\le\mu(k)\le C_2(1+k)^\beta, \\ C_1k^{1+\gamma}\le\varepsilon(k)\le C_2k^{1+\gamma}. \end{aligned} \tag{1.8}$$

Within the framework of weak solutions the term on the right hand side of (1.3) is not easy to handle. Thus, it is more appropriate to "equivalently" reformulate the system (1.1)–(1.3) in the following way. Defining the scalar quantity E as

$$E:=\frac{1}{2}|\boldsymbol{v}|^2+k, \tag{1.9}$$

we deduce the equation for E by taking the scalar product of (1.2) and $\boldsymbol{v}$ and by adding the result to (1.3). Doing so, we arrive at the equation

$$E_{,t}+\operatorname{div}\left(\boldsymbol{v}(E+p)\right)-\operatorname{div}\left(\mu(k)\nabla k\right)-\operatorname{div}\left(\nu(k)\mathbf{D}(\boldsymbol{v})\boldsymbol{v}\right)+\varepsilon(k)=0. \tag{1.10}$$

Of course, assuming that the multiplication of (1.2) by $\boldsymbol{v}$ is meaningful (or in other words, assuming that $\boldsymbol{v}$ is a possible test function in the weak formulation of (1.2)) the identities (1.3) and (1.10) are equivalent. However, in three spatial dimensions we usually do not know that $\boldsymbol{v}$ is an admissible test function and we cannot conclude the equivalence of (1.3) and (1.10). The main mathematical reason why we prefer (1.10) to (1.3) is the fact that in (1.10) all nonlinear terms are in divergence form and belong to a better space than L^1 while in (1.3) the term on the right hand side belongs usually to L^1 only. Consequently, it is easier to identify weak limits of all nonlinear quantities in (1.10) than in (1.3). These facts seem to be first specified and exploited in [13]. On the other hand, considering (1.10) we see that we have to deal with p, that can be usually omitted in the system (1.1)–(1.3) by using divergence-free test functions in (1.2). Moreover, assuming that we have a weak solution to (1.1)–(1.2) and (1.10) that in addition satisfies

$$k_{,t} + \operatorname{div}(k\boldsymbol{v}) - \operatorname{div}\left(\mu(k)\nabla k\right) + \varepsilon(k) \geq \nu(k)|\mathbf{D}(\boldsymbol{v})|^2, \tag{1.11}$$

in a weak sense, then it is natural to call such a solution a suitable weak solution in sense of Caffarelli, Kohn, Nirenberg, see [10]. Indeed, subtracting (1.11) from (1.10), one deduces

$$|\boldsymbol{v}|^2_{,t} + \operatorname{div}\left(\boldsymbol{v}(|\boldsymbol{v}|^2 + 2p)\right) - \operatorname{div}\left(2\nu(k)\mathbf{D}(\boldsymbol{v})\boldsymbol{v}\right) \leq 0, \tag{1.12}$$

that is the form of local energy inequality as appeared in the definition of suitable weak solution to Navier-Stokes system, see [10].

In this study we establish the following result.

Theorem 1.1. *Assume that μ, ν and ε satisfy* (1.8) *with*

$$0 \leq \alpha < \frac{2\beta}{5} + \frac{2}{3}, \qquad 0 \leq \gamma < \beta + \frac{2}{3}. \tag{1.13}$$

Then for any $\Omega \in \mathcal{C}^{1,1}$, $T > 0$, $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$ and $k_0 \in L^1(\Omega)$, $k_0 \geq 0$ a.e. in Ω, there exists a suitable weak solution $(\boldsymbol{v}, p, k)$ to Problem (1.1)–(1.7)*, that in particular fulfils* (1.1)–(1.2) *and* (1.9)–(1.11) *in the sense of distributions.*

The precise definition of the solution and formulation of the result is given in Theorem 2.1 below, see Section 2.

The system (1.4)–(1.7) with ν, μ and ε of the form (1.8) is interesting from the point of view of mathematical analysis of PDEs, in particular, from the point of view of regularity theory. We shall address this point next.

To simplify discussion below, we assume that ν, μ and ε are of the form

$$\nu(k) := \nu_0 k^\alpha, \quad \mu(k) := \mu_0 k^\alpha \quad \text{and} \quad \varepsilon(k) = \varepsilon_0 k^{2-\alpha}, \tag{1.14}$$

where μ_0 and ν_0 are positive constants and $\varepsilon_0 \geq 0$.

We formulate the following conjecture.

Conjecture 1.1. *Let $\alpha \in \mathbb{R}$, ν, μ and ε be of the form* (1.14)*. Then there exist $\delta > 0$ and $C^* > 0$ such that for any triple $(\boldsymbol{v}, p, k)$ solving* (1.1)–(1.2) *and* (1.10)–(1.11) *in the sense of distribution the following implication holds:*
If

$$\int_{-1}^{0} \int_{B_1(0)} \nu(k)|\mathbf{D}(\boldsymbol{v})|^2 \, dx \, dt \leq \delta \tag{1.15}$$

then

$$|\boldsymbol{v}(t,x)| \leq C^* \quad \textit{in} \quad (-\frac{1}{2}, 0) \times B_{\frac{1}{2}}(0). \tag{1.16}$$

This conjecture certainly holds for $\alpha \equiv 0$ since then the system (1.1)–(1.2) reduces to Navier-Stokes equation for which Conjecture 1.15 was proved in [10], see also [31]. To our best knowledge, Conjecture 1.15 is open for general values of positive α's. In what follows, we will show how Conjecture 1.1 implies that, for certain α's, any suitable weak solution has bounded velocity.

Indeed, assume that a triple $(\boldsymbol{v}, k, p)$ solve (1.1)–(1.2) and (1.10)–(1.11) on some neighborhood of $(0,0)$ that contains for some $\ell_0 > 0$ a set $(-\ell_0^A, 0) \times B_{\ell_0}(0)$ with some $A > 0$ specified below. Then we rescale the triple in the following way. For any $\ell \leq \ell_0$ we define for some $B > 0$

$$\begin{aligned}\boldsymbol{v}_\ell(t,x) &:= \ell^B \boldsymbol{v}(\ell^A t, \ell x),\\ p_\ell(t,x) &:= \ell^{2B} p(\ell^A t, \ell x),\\ k_\ell(t,x) &:= \ell^{2B} k(\ell^A t, \ell x).\end{aligned}$$

It is easy to show that if we choose A, B such that

$$A := \frac{2-2\alpha}{1-2\alpha}, \qquad B := \frac{1}{1-2\alpha}$$

and assume that $\alpha \neq \frac{1}{2}$ then the triple $(\boldsymbol{v}_\ell, p_\ell, k_\ell)$ solves (1.1)–(1.2) and (1.10)–(1.11) in the sense of distribution in $(-1,0)\times B_1(0)$. Next, we apply Conjecture 1.1 on the rescaled velocity $\boldsymbol{v}_\ell$. Hence, using the standard substitution theorem we see that we need to show that

$$\begin{aligned}\delta &\geq \int_{-1}^0 \int_{B_1(0)} \nu(k_\ell)|\mathbf{D}(\boldsymbol{v}_\ell)|^2 \, dx\, dt\\ &= \int_{-1}^1 \int_{B_1(0)} \ell^{2B\alpha+2B+2}(k(\ell^A t, \ell x))^\alpha |\mathbf{D}(\boldsymbol{v}(\ell^A t, \ell x))|^2 \, dx\, dt\\ &= \int_{-\ell^A}^0 \int_{B_\ell(0)} \ell^{2B\alpha+2B+2-A-3}(k(t,x))^\alpha |\mathbf{D}(\boldsymbol{v}(t,x))|^2 \, dx\, dt\\ &= \ell^{\frac{6\alpha-1}{1-2\alpha}} \int_{-\ell^A}^0 \int_{B_\ell(0)} k^\alpha |\mathbf{D}(\boldsymbol{v})|^2 \, dx\, dt.\end{aligned} \tag{1.17}$$

Interestingly, we see that for $\frac{1}{6} \leq \alpha < \frac{1}{2}$ we can choose ℓ so small that the premise of Conjecture 1.1 is fulfilled. As its consequence, we conclude that $\boldsymbol{v}_\ell$ is bounded in $(-1/2, 0) \times B_{1/2}(0)$ and $\boldsymbol{v}$ is bounded in $(-(\ell/2)^\alpha, 0) \times B_{\ell/2}(0)$. Even more, it follows from (1.17), Conjecture 1.1 and the standard covering argument procedure that, for $\alpha < \frac{1}{6}$, the Hausdorff dimension of the set $\mathcal{S}$ of possible singularities of $\boldsymbol{v}$ (here, the point of singularity is defined such (t,x) that $\boldsymbol{v}$ is not bounded in any neighborhood of (t,x)) is bounded by

$$d(\mathcal{S}) < \frac{1-6\alpha}{1-2\alpha}, \tag{1.18}$$

which is consistent with the standard estimate of possible singular set for the Navier-Stokes equations.

To summarize, the system (1.4)–(1.7) with ν, μ and ε of the form (1.14) is an interesting system from the point of view of regularity theory. Before however one starts to study regularity property of any solution one needs to establish its existence, and this is the subject of this paper. While the statement of Theorem 1.1 for $\alpha = 0$ was investigated in [8], the case $\alpha > 0$ is analyzed in this study. Note that for $\varepsilon \equiv 0$ and $\beta = \alpha$, Theorem 1.1 guarantees the existence of solution for $0 \le \alpha < \frac{10}{9}$.

There are two main reasons motivating us to analyze the problem (1.1)–(1.8). The first one comes from the large-data analysis of turbulent models. The second reason is connected with the question of large-data qualitative mathematical properties of flows of incompressible heat-conducting Newtonian fluids. We shall discuss the both issues in what follows.

(1) Kolmogorov model. The problem in consideration (1.1)–(1.3) is closely related to the so-called turbulent kinetic energy model; then $\boldsymbol{v}$ represents the statistical mean (averaged) velocity of the fluid, p is associated to the statistical mean normal stress - the averaged pressure, ν stands for the viscosity, μ is the eddy diffusion and k denotes the turbulent kinetic energy defined as $\frac{1}{2}\sum_{j=1}^{3}\overline{|v_j'|^2}$, whereas $\boldsymbol{v}'$ is the velocity of fluctuations and $\overline{z}$ stands for the averaging of the quantity z. The term on the right hand side of (1.3) represents the energy that the large scales transmit onto the small scales, and the last term of the left hand side of (1.3) measures the energy rate returned by the small scales to the large scales. Usually, the quantities ν, μ and ε are depending on the mixing length scale ℓ that is a positive given function or it is driven by another evolutionary equation.

In fact, one of the first models of this type was proposed by Kolmogorov in [15], see also the paper No. 48 in [16] or Appendix in [30]. Based on local properties of turbulence and incorporating, as Kolmogorov clearly states, (unspecified) crude approximations, he formulates a closed system of equations of the form

$$\operatorname{div}\overline{\boldsymbol{v}} = 0 \tag{1.19}$$

$$\overline{\boldsymbol{v}}_{,t} + \operatorname{div}(\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}) = -\nabla\left(\frac{\overline{p}}{\varrho} + b\right) + A\operatorname{div}\left(2\,\frac{b}{\omega}\,\mathbf{D}(\overline{\boldsymbol{v}})\right), \tag{1.20}$$

$$\omega_{,t} + \operatorname{div}(\omega\overline{\boldsymbol{v}}) = -\tfrac{7}{11}\omega^2 + A'\operatorname{div}\left(\frac{b}{\omega}\nabla\omega\right), \tag{1.21}$$

$$b_{,t} + \operatorname{div}(b\overline{\boldsymbol{v}}) = -b\omega + \tfrac{4}{3}A\,\frac{b}{\omega}\,|\mathbf{D}(\overline{\boldsymbol{v}})|^2 + A''\operatorname{div}\left(\frac{b}{\omega}\,\nabla b\right), \tag{1.22}$$

where the velocity of the fluid is the sum of the averaged velocity $\overline{\boldsymbol{v}}$ and the velocity of fluctuations $\boldsymbol{v}'$, $\overline{p}$ is the averaged pressure, $b := \frac{1}{3}\sum_{j=1}^{3}\overline{|v_j'|^2}$ is one third of the sum of averaged square of the components of the velocity of fluctuations, and ω is related to the length scale ℓ through the relation $\omega := C\sqrt{b}/\ell$, C, A,

A' and A'' are constants. Equations (1.20)–(1.22) coincide exactly[1] with equations (1)–(3) in [15], [16], [30], we merely completed the system by the constraint of incompressibility (1.19). Thus, we obtain a closed system of six equations for $(\overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{p}, \ell, b)$.

Next, assuming that ℓ is a given known function, equation (1.21) is redundant. Thus, setting $\boldsymbol{v} := \overline{\boldsymbol{v}}$, $p := \frac{\overline{p}}{\varrho} + b$ and noticing that $b\omega = \frac{C}{\ell} b\sqrt{b}$ and $\frac{b}{\omega} = \frac{\ell}{C}\sqrt{b}$, the system (1.19)–(1.22) simplifies to

$$\operatorname{div} \boldsymbol{v} = 0 \tag{1.23}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \operatorname{div}\left(\frac{2\ell A}{C}\sqrt{b}\,\mathbf{D}(\boldsymbol{v})\right), \tag{1.24}$$

$$b_{,t} + \operatorname{div}(b\boldsymbol{v}) = -\frac{C}{\ell} b\sqrt{b} + \frac{4\ell A}{3C}\sqrt{b}\,|\mathbf{D}(\boldsymbol{v})|^2 + \operatorname{div}\left(\frac{\ell A''}{C}\sqrt{b}\,\nabla b\right). \tag{1.25}$$

Setting $k := \frac{3}{2}b$, $\nu(k) := \frac{2\ell A}{C}\sqrt{b} = \frac{2\sqrt{2}\ell A}{\sqrt{3}C}\sqrt{k}$, $\mu(k) = \frac{\ell\sqrt{2}A''}{\sqrt{3}C}\sqrt{k}$ and $\varepsilon(k) = \frac{\sqrt{2}C}{\sqrt{3}\ell}\sqrt{k}k$ we arrive at the system of the form (1.1)–(1.3) that is subject of investigation in this paper. Note that the quantity E introduced in (1.9) (that plays an important role in our analysis) is the sum of the kinetic energy associated to the averaged velocity and the turbulent kinetic energy $k = \frac{1}{2}\sum_{j=1}^{3} \overline{|v_j'|^2}$.

Although the model (1.1)–(1.3) describes complicated turbulent behavior in a simplified manner (see for example discussion in [30]), it is quite popular and efficient in various applications. It is used for instance in oceanography ([5], [32], [20]), in marine engineering ([22], [28]), etc., and surprisingly gives very accurate numerical results in comparison with experimental data. In certain applications, this model thus "prevents" the computational analysts from dealing with the $(k - \varepsilon)$ model (see the original work due to Launder and Spalding [17], and also [26] for more details) that is from the computational point of view very costly.

The derivation of models such as (1.1)–(1.3) is mainly based on dimensional analysis and physical assumptions on the turbulence (see [26] and [20]) that lead to the following forms for ν and μ

$$\nu(k) = \nu_0 + \nu_1\sqrt{k} \quad \text{and} \quad \mu(k) = \mu_0 + \mu_1\sqrt{k}\,, \tag{1.26}$$

where $\nu_0 \geq 0$, $\nu_1 \geq 0$, $\mu_0 > 0$ and $\mu_1 > 0$ are constants. Note that the case (1.26) with ν_0, ν_1, μ_0, μ_1 positive is covered by Theorem 1.1. There are also works towards the mathematical justification of the k-equation (1.3) from the Navier-Stokes equations ([11], [25], [12]), but a transparent and consistent derivation of these models is, to our best knowledge, missing. The limitations and applicability of the model in consideration are one of the topics studied in our forthcoming paper.

[1] In fact, we follow the translation given by Spalding in [30]. There seems to be a misprint concerning the definition of $\overline{\epsilon}$ in [15].

From the point of view of analysis of turbulent kinetic energy models the result presented in this paper can be considered as a natural continuation of Theorem 4 in [21] since it solves the problem formulated in [21] that has been left open. Also, Theorem 4 in [21] that concerns the case when both ν and μ are bounded function of k proves that in three spatial dimensions the limit equation for k deduced from approximated solutions satisfies a variational inequality. This paper gives two essential novel contributions to the analysis of (1.1)–(1.3). First, the unknown k is shown to fulfill the equation for E (see (1.10) above) rather than equation (1.3), and second, it investigates three-dimensional flows with ν and k that are unbounded functions of k. The price we pay for dealing with (1.10) rather than with (1.3) is that we need to introduce globally integrable pressure and this is the reason why we are not able to extend, at the current state, the theory to Dirichlet boundary condition for the velocity (the case $\lambda = 1$ in $(1.5)_2$).

We finish this part by recalling several related results and approaches. The system (1.1)–(1.3) was first studied in [19] and [21]. Assuming that the eddy viscosity is a bounded function of k, the author establishes the existence of weak (distributional) solutions in the steady-state case and in the evolutionary 2D case if both k and $\boldsymbol{v}$ satisfy homogeneous Dirichlet boundary conditions. These results have been generalized in many ways and for other boundary conditions, as for instance to flows of two interacting fluids such as the Ocean and the Atmosphere ([3], [4], [1]). There are very few uniqueness results that are mainly obtained under smallness assumptions on the total variation of the eddy viscosity or the source term, and they concern steady-state flows ([2], [6]). In order to analyze models with unbounded eddy viscosities (that are important, see (1.26)) several different tools were developed, mostly for some simplified models (such as steady-state models, models without convective terms, and even without the pressure). We refer the interested reader to Lewandowski and Murat [20, Chapter 5] for details concerning renormalized solutions, or to [14] (energy solutions in special function spaces) or to [18] (energy solutions with periodic boundary conditions).

(2) Navier-Stokes-Fourier system. Associating k with the internal energy (or temperature) and setting $\varepsilon \equiv 0$, the system (1.1)–(1.3) describes unsteady flows of incompressible heat-conducting fluids in which the Cauchy stress $\mathbf{T}$ and the heat flux $\boldsymbol{q}$ are given by the constitutive equations of the form

$$\mathbf{T} := -p\mathbf{I} + \nu(k)\mathbf{D}(\boldsymbol{v}) \quad \text{and} \quad \boldsymbol{q} := \mu(k)\nabla k. \tag{1.27}$$

The system of equations (1.1)–(1.3) together with (1.27) is called the incompressible Navier-Stokes-Fourier system, where ν denotes the kinematical viscosity of the fluid and μ is the heat conductivity. In most liquids, that are well approximated as incompressible materials, the internal energy is proportional to the temperature and the viscosity *decreases* with increasing temperature. This is just opposite scenario than that described by the assumptions (1.8). Although the Navier-Stokes-Fourier system with the viscosity satisfying (1.8) is not reflecting experimental observations it would be definitely of interest to know that there are unsteady flows of a class of Newtonian fluids that exist for large data and the velocity is bounded.

The large data existence result presented here can be viewed as the extension of the approach (that is based on the appropriate form of the balance of energy) originally developed in [13] and [8] where the Navier-Stokes-Fourier system with the *bounded* viscosity and the heat conductivity is treated; the spatially-periodic problem is analyzed in [13] while flows in bounded domains satisfying the Navier's slip boundary conditions are studied in [8]. Naumann [27] studied the model with the temperature dependent viscosity and the heat conductivity, he however uses equation (1.3) instead of (1.10); due to difficulties to identify the limit the dissipative term at the right-hand side of (1.3) his concept of solution is weaker than that introduced in [13], [8] and used in this paper as well. For the sake of completeness, we remark that Lions [23, Section 3.4] studies the case where the viscosity and the heat conductivity are positive constants (temperature independent) and provides two approaches (different from that presented here) how the problem can be investigated in order to establish long-time and large-data existence results.

The paper is organized as follows. After introducing relevant function spaces, we establish, in Section 2, the main result that includes the precise definition of suitable weak solutions to (1.1)–(1.3). Then, in Section 3, we introduce two-level approximations depending on parameters n and m and prove the main result. Since the existence of solutions to the (m,n)-approximation, for a fixed n and m, is given in [8, Appendix] we focus on the analysis of the limit behavior of the solutions $(\boldsymbol{v}^{m,n}, p^{m,n}, k^{m,n})$ first as $n \to \infty$ and then as $m \to \infty$.

2. Main result

In order to state the main result with all details we need to clarify the notation of relevant function spaces. For the velocity field, we define

$$
\begin{aligned}
W^{1,p}_{\boldsymbol{n}} &:= \left\{ \boldsymbol{v} \in W^{1,p}(\Omega)^3 : \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \right\}, \\
W^{1,p}_{\boldsymbol{n},\mathrm{div}} &:= \left\{ \boldsymbol{v} \in W^{1,p}_{\boldsymbol{n}} : \ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \right\}, \\
W^{-1,p'}_{\boldsymbol{n}} &:= \left(W^{1,p}_{\boldsymbol{n}} \right)^*, \quad W^{-1,p'}_{\boldsymbol{n},\mathrm{div}} := \left(W^{1,p}_{\boldsymbol{n},\mathrm{div}} \right)^*, \\
L^2_{\boldsymbol{n},\mathrm{div}} &:= \overline{W^{1,2}_{\boldsymbol{n},\mathrm{div}}}^{\|\,\|_2}.
\end{aligned}
$$

We also introduce the natural space for k; for some fixed $\beta \in \mathbb{R}_+$ we set

$$
\begin{aligned}
\mathcal{E}_\beta := \Big\{ & k \in L^\infty(0,T; L^1(\Omega)) : \ k \geq 0 \text{ a.e.}, \\
& ((1+k)^s - 1) \in L^2(0,T; W^{1,2}_D(\Omega)) \text{ for all } s < \frac{\beta+1}{2} \Big\},
\end{aligned}
$$

where $W^{1,2}_D(\Omega) := \{ k \in W^{1,2}(\Omega);\ k = 0 \text{ on } \partial\Omega_D \}$.

Note that by using standard interpolation technique the following continuous embedding holds (we show it in the proof of the main theorem) for $\beta \in [0,1]$

$$\mathcal{E}_\beta \hookrightarrow L^r(0,T;L^r(\Omega)^3) \cap L^q(0,T;W_D^{1,q}(\Omega)^3) \ \text{ for all } r < \frac{3\beta+5}{3} \text{ and } q < \frac{3\beta+5}{4}\,.$$

If $\beta > 1$ then $q = 2$ in the above embedding.

Moreover, in what follows we use the abbreviation $(a,b)_A := \int_A ab$ whenever $ab \in L^1(A)$. In case that $A = \Omega$ we also omit writing the subscript Ω. The same notation is used for vector- and tensor-valued functions as well.

We formulate the main result of this paper.

Theorem 2.1. *Let $\Omega \in \mathcal{C}^{1,1}$, $T > 0$, $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$ and $k_0 \in L^1(\Omega)$, $k_0 \geq 0$ a.e. in Ω, be given arbitrarily. Assume that ν, μ and ε satisfy (1.8) with α, β and γ fulfilling*

$$0 \leq \alpha < \frac{2\beta}{5} + \frac{2}{3}\,, \qquad 0 \leq \gamma < \beta + \frac{2}{3}\,. \tag{2.1}$$

Then there exist a triple $(\boldsymbol{v}, p, k)$ and E given as

$$E = \frac{1}{2}|\boldsymbol{v}|^2 + k,$$

satisfying

$$\boldsymbol{v} \in \mathcal{C}_{\mathrm{weak}}(0,T;L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}), \tag{2.2}$$

$$\boldsymbol{v}_{,t} \in L^{q'}(0,T;W_{\boldsymbol{n}}^{-1,q'}) \text{ for all } q < \min\left\{\frac{5}{3}, 2 - \frac{2\alpha}{\alpha+\beta+\frac{5}{3}}\right\}, \tag{2.3}$$

$$k \in \mathcal{E}_\beta, \tag{2.4}$$

$$k_{,t} \in \mathcal{M}(0,T;W^{-1,1+\delta}) \text{ for certain } \delta > 0 \text{ small}, \tag{2.5}$$

$$p \in L^q(0,T;L^q(\Omega)) \text{ for all } q < \min\left\{\frac{5}{3}, 2 - \frac{2\alpha}{\alpha+\beta+\frac{5}{3}}\right\}, \tag{2.6}$$

$$\sqrt{\nu(k)}\mathbf{D}(\boldsymbol{v}) \in L^2(0,T;L^2(\Omega)^{3\times 3}), \tag{2.7}$$

$$E_{,t} \in L^{1+\delta}(0,T;W_D^{-1,1+\delta}(\Omega)) \text{ for certain } \delta > 0 \text{ small}, \tag{2.8}$$

and fulfilling

$$\begin{aligned}\int_0^T \langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - (\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{w}) + \frac{\lambda}{1-\lambda}(\boldsymbol{v},\boldsymbol{w})_{\partial\Omega} + (\nu(k)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{w}))\, dt \\ = \int_0^T (p, \operatorname{div}\boldsymbol{w})\, dt \qquad \text{for all } \boldsymbol{w} \in L^\infty(0,T;W^{1,\infty}_{\boldsymbol{n}}),\end{aligned} \tag{2.9}$$

$$\begin{aligned}\int_0^T \langle E_{,t}, w\rangle - (\boldsymbol{v}(E+p), \nabla w) + (\mu(k)\nabla k, \nabla w) + (\varepsilon(k), w)\, dt \\ = -\int_0^T (\nu(k)\mathbf{D}(\boldsymbol{v})\boldsymbol{v}, \nabla w)\, dt \qquad \textit{for all } w \in L^\infty(0,T; W_D^{1,\infty}(\Omega)),\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\int_0^T \langle k_{,t}, w\rangle - (k\boldsymbol{v}, \nabla w) + (\mu(k)\nabla k, \nabla w) + (\varepsilon(k), w)\, dt \\ \geq \int_0^T (\nu(k)|\mathbf{D}(\boldsymbol{v})|^2, w)\, dt \qquad \textit{for all } w \in \mathcal{C}(0,T; W_D^{1,\infty}(\Omega)).\end{aligned} \tag{2.11}$$

Moreover, the initial conditions are attained in the following sense

$$\lim_{t\to 0+} \left(\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 + \|k(t) - k_0\|_1\right) = 0. \tag{2.12}$$

It is worth of noticing that Theorem 2.1 covers the interesting case $\alpha = \beta = \gamma$ for $0 \leq \alpha < 10/9$. In particular, the case (1.26) is included.

We also remark that all terms in (2.9)–(2.11) are meaningful; the most critical term is the last term in (2.10) and the L^1-integrability of this term leads to the restriction $(2.1)_1$. Indeed, noticing that $\nu(k)\mathbf{D}(\boldsymbol{v})\boldsymbol{v} = \sqrt{\nu(k)}\mathbf{D}(\boldsymbol{v})\boldsymbol{v}\sqrt{\nu(k)}$ and $\sqrt{\nu(k)}\mathbf{D}(\boldsymbol{v}) \in L^2(0,T; L^2(\Omega)^{3\times 3})$, $\boldsymbol{v} \in L^{10/3}(0,T; L^{10/3}(\Omega)^3)$ and $\sqrt{\nu(k)} \in L^{\frac{3\beta+5}{3}-s}(0,T; L^{\frac{3\beta+5}{3}-s}(\Omega))$ we observe, by applying the Hölder inequality that

$$\nu(k)\mathbf{D}(\boldsymbol{v})\boldsymbol{v} \in L^1(0,T; L^1(\Omega)) \quad \Longleftrightarrow \quad 0 \leq \alpha < \frac{2\beta}{5} + \frac{2}{3},$$

which is the first condition in (2.1). The second condition $(2.1)_2$ is required in order to know that $\varepsilon(k)$ belongs to a better space than $L^1(0,T; L^1(\Omega))$, which is needed to establish the compactness of the terms involving $\varepsilon(k)$.

3. Proof of Theorem 2.1

First we introduce a notation of various truncated functions. For any $m \in \mathbb{R}_+$, we define the function T_m through

$$T_m(y) := \begin{cases} y & \text{if } |y| \leq m, \\ m \operatorname{sgn}(y) & \text{if } |y| > m, \end{cases} \tag{3.1}$$

and we use the symbol Θ_m to denote the primitive function to T_m, i.e.,

$$\Theta_m(y) := \int_0^y T_m(\tau)\, d\tau. \tag{3.2}$$

For β introduced in $(1.8)_2$ and for arbitrary $s \geq 0$, we also introduce the function Φ_s by the formula

$$\Phi_s(y) := \int_0^y (1+\tau)^{\frac{\beta-s-1}{2}}\, d\tau = \frac{2}{\beta-s+1}\left[(1+y)^{\frac{\beta-s+1}{2}} - 1\right]. \tag{3.3}$$

Finally, we consider a smooth non-increasing function G such that $G(y) = 1$ when $y \in [0,1]$ and $G(y) = 0$ for $y \geq 2$, and define G_m as

$$G_m(y) := G\left(\frac{y}{m}\right). \tag{3.4}$$

The primitive function to G_m is then defined through

$$\Gamma_m(y) := \int_0^y G_m(\tau)\, d\tau. \tag{3.5}$$

The first part of the proof takes inspiration in the method developed in [8]. We start with a "semi"-Galerkin approximation. Let $\{\boldsymbol{w}_k\}_{k=1}^{\infty}$ be a basis of $W^{1,2}_{\boldsymbol{n},\mathrm{div}} \cap W^{2,4}(\Omega)^d$, which exists due to the separability of this space. We look for $(\boldsymbol{v}^{n,m}, k^{n,m})$, where

$$\boldsymbol{v}^{n,m} := \sum_{i=1}^{n} c_i^{n,m}(t)\boldsymbol{w}_i(x), \qquad \text{and} \qquad k^{n,m} \geq 0 \quad a.e.$$

fulfill the equations

$$\begin{aligned}(\boldsymbol{v}^{n,m}_{,t}, \boldsymbol{w}_i) - \left(G_m(|\boldsymbol{v}^{n,m}|^2)\boldsymbol{v}^{n,m} \otimes \boldsymbol{v}^{n,m}, \nabla \boldsymbol{w}_i\right) + \frac{\lambda}{1-\lambda}(\boldsymbol{v}^{n,m}, \boldsymbol{w}_i)_{\partial\Omega}& \\ + (\nu(T_m(k^{n,m}))\mathbf{D}(\boldsymbol{v}^{n,m}), \mathbf{D}(\boldsymbol{w}_i)) = 0 \qquad \text{for all } i = 1, \dots, n,&\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\int_0^T \langle k^{n,m}_{,t}, w\rangle - (\boldsymbol{v}^{n,m} k^{n,m}, \nabla w) + (\mu(k^{n,m})\nabla k^{n,m}, \nabla w) + (\varepsilon(k^{n,m}), w)\, dt \\ &= \int_0^T (\nu(T_m(k^{n,m}))|\mathbf{D}(\boldsymbol{v}^{n,m})|^2, w)\, dt \quad \text{ for all } w \in L^2(0,T; W^{1,2}_D(\Omega)),\end{aligned} \tag{3.7}$$

as well as the initial conditions of the form

$$\begin{aligned}&\boldsymbol{v}^{n,m}(0,x) := \boldsymbol{v}^n_0(x) := \sum_{i=1}^{n} c_i^0 \boldsymbol{w}_i \text{ with } c_i^0 := (\boldsymbol{v}_0, \boldsymbol{w}_i), \\ &\lim_{t\to 0} \|k^{n,m}(t) - k^n_0\|_2^2 = 0 \text{ with } k^n_0 := k_0 * \eta_{\frac{1}{n}},\end{aligned} \tag{3.8}$$

where $\eta_{\frac{1}{n}}$ is the standard regularizing kernel of radii $\frac{1}{n}$ and k_0 is extended by 0 outside of Ω. Note that $\boldsymbol{v}^n_0 \to \boldsymbol{v}_0$ strongly in $L^2(\Omega)$ and that $k^n_0 \to k_0$ strongly in $L^1(\Omega)$.

The existence of the solution to (3.6)–(3.8) is established in [8, Appendix] and here we merely state the result concerning large-data and long-time existence proved therein.

Theorem 3.1. *Let arbitrary $n, m \in \mathbb{N}$ be fixed. Assume that all assumptions of Theorem 2.1 hold. Then there exist $(\boldsymbol{c}^{n,m}, k^{n,m})$ solving (3.6)–(3.8) such that*

$$\boldsymbol{c}^{n,m} \in W^{1,2}(0,T)^n, \tag{3.9}$$

$$k^{n,m} \in L^\infty(0,T; L^\infty(\Omega)) \cap L^2(0,T; W_D^{1,2}(\Omega)), \tag{3.10}$$

$$k^{n,m}_{,t} \in L^2(0,T; W_0^{-1,2}(\Omega)). \tag{3.11}$$

3.1 Limit $n \to \infty$. Since $m \in \mathbb{N}$ is fixed in this subsection, we write $(\boldsymbol{v}^n, k^n)$ instead of $(\boldsymbol{v}^{n,m}, k^{n,m})$, where $(\boldsymbol{v}^{n,m}, k^{n,m})$ denotes a solution to (3.6)–(3.8). Our goal is to study the convergence in equations (3.6)–(3.7) if $n \to \infty$. We will follow the procedure developed in [8] that we have to modify in order to treat unbounded coefficients ν and μ. This is why we investigate this limiting process here rigorously and in detail.

3.1.1 Uniform estimates on $\boldsymbol{v}^n$. Multiplying the i-th equation in (3.6) by c_i^n and then summing over $i = 1, \dots n$ we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^n\|_2^2 - \frac{1}{2}(G_m(|\boldsymbol{v}^n|^2)\boldsymbol{v}^n, \nabla|\boldsymbol{v}^n|^2) + \frac{\lambda}{1-\lambda}\|\boldsymbol{v}^n\|_{\partial\Omega,2}^2 \\ &\qquad + \int_\Omega \nu(T_m(k^n))|\mathbf{D}(\boldsymbol{v}^n)|^2\, dx = 0. \end{aligned} \tag{3.12}$$

Next, using the fact that $\boldsymbol{v}^n \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and $\operatorname{div} \boldsymbol{v}^n = 0$ in Ω we deduce that

$$\frac{1}{2}(G_m(|\boldsymbol{v}^n|^2)\boldsymbol{v}^n, \nabla|\boldsymbol{v}^n|^2) = \frac{1}{2}(\boldsymbol{v}^n, \nabla\Gamma_m(|\boldsymbol{v}^n|^2)) = -\frac{1}{2}(\operatorname{div} \boldsymbol{v}^n, \Gamma_m(|\boldsymbol{v}^n|^2)) = 0.$$

Thus, we conclude from (3.12) that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_2^2 + 2\int_0^T \int_\Omega \nu(T_m(k^n))|\mathbf{D}(\boldsymbol{v}^n)|^2\, dx\, dt \le \|\boldsymbol{v}_0^n\|_2^2 \le C(\boldsymbol{v}_0) < \infty. \tag{3.13}$$

It then follows from $(1.8)_1$ and the Korn inequality that

$$\int_0^T \|\boldsymbol{v}^n(t)\|_{1,2}^2\, dt \le C(C_1^{-1}, \boldsymbol{v}_0) < \infty. \tag{3.14}$$

Moreover, using the standard interpolation inequality, (3.13)–(3.14) implies that

$$\int_0^T \|\boldsymbol{v}^n\|_{\frac{10}{3}}^{\frac{10}{3}}\, dt \le C. \tag{3.15}$$

Note finally that it follows from (3.6) and (3.13)–(3.14) that

$$\int_0^T \|\boldsymbol{v}^n_{,t}\|^2_{W^{-1,2}_{\boldsymbol{n},\mathrm{div}}} \le C(m). \tag{3.16}$$

3.1.2 Estimates on k^n uniform w.r.t. both m and n. Setting $w := T_1(k^n)$ in (3.7) (note that $T_1(k^n)$ is a possible test function) we obtain the identity

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Theta_1(k^n)\,dx - (\boldsymbol{v}^n,\nabla\Theta_1(k^n)) + (\mu(k^n)\nabla k^n, T_1'(k^n)\nabla k^n)&\\ +(\varepsilon(k^n),T_1(k^n)) = (\nu(T_m(k^n))|\mathbf{D}(\boldsymbol{v}^n)|^2, T_1(k^n)).&\end{aligned} \tag{3.17}$$

Since $\operatorname{div}\boldsymbol{v}^n = 0$ in Ω and $\boldsymbol{v}^n\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, the second term on the left hand side vanishes. Moreover, using (1.8), we see that the third term on the left hand side is nonnegative. Thus, integrating (3.17) over time, using $(1.8)_3$ to estimate the last term on the left hand side from below and using (3.13) to bound the right hand side of (3.17), we conclude that

$$\sup_{t\in(0,T)}\|\Theta_1(k^n(t))\|_1 + C\int_0^T \|k^n\|^{\gamma+1}_{\gamma+1}\,dt \le C + \|\Theta_1(k^n_0)\|_1. \tag{3.18}$$

Finally, using the simple estimate for the growth of Θ_1 we get that

$$\sup_{t\in(0,T)}\|k^n(t)\|_1 + C\int_0^T \|k^n\|^{\gamma+1}_{\gamma+1}\,dt \le C + \|k_0\|_1 < \infty. \tag{3.19}$$

Next, recalling that $k^n \ge 0$ a.e. in Ω we consider $w = (1+k^n)^{-s} - 1$ with $s > 0$ small and observe that such w is an admissible test function in (3.7), in particular $\|w\|_\infty \le 2$ and $w \in L^2(0,T;W^{1,2}_D(\Omega))$ for each $n\in\mathbb{N}$. Inserting such w into (3.7), using the fact that $\operatorname{div}\boldsymbol{v}^n = 0$ and the estimates established in (3.13) and (3.19), we get

$$\int_0^T\int_\Omega \mu(k^n)(1+k^n)^{-s-1}|\nabla k^n|^2\,dx\,dt \le C(s^{-1}). \tag{3.20}$$

Consequently, using the assumption $(1.8)_2$ and recalling the definition of Φ_s, see (3.3), we conclude that (using the fact that Φ_s has zero trace on Ω_D)

$$\begin{aligned}\int_0^T \|\Phi_s(k^n)\|^2_{1,2}\,dt &\le C\int_0^T \|\nabla\Phi_s(k^n)\|^2_2\,dt\\ &\le C\int_0^T\int_\Omega \mu(k^n)(1+k^n)^{-s-1}|\nabla k^n|^2\,dx\,dt \le C(s^{-1}).\end{aligned} \tag{3.21}$$

Using the first inequality in

$$c^{-1}((1+x)^{\frac{\beta-s+1}{2}} - 1) \le \Phi_s(x) \le c(1+x)^{\frac{\beta-s+1}{2}}, \qquad (x\ge 0) \tag{3.22}$$

the embedding $W_D^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and $(3.21)_1$ we observe that

(3.23)
$$\int_0^T \|k^n\|_{3(\beta-s+1)}^{\beta-s+1}\, dt \le C(1+\int_0^T \|\Phi_s(k^n)\|_{1,2}^2\, dt) \le C(s^{-1}) \text{ for all } s>0 \text{ small.}$$

Then, referring to the standard interpolation inequality

$$\|u\|_{\beta-s+\frac{5}{3}} \le \|u\|_1^{1-a}\|u\|_{3(\beta-s+1)}^{a} \qquad \text{with } a := \frac{\beta-s+1}{\beta-s+\frac{5}{3}}, \tag{3.24}$$

applied onto k^n we conclude from (3.19) and (3.23) that

(3.25)
$$\int_0^T \|k^n\|_{\beta-s+\frac{5}{3}}^{\beta-s+\frac{5}{3}}\, dt \le \int_0^T \|k^n\|_1^{\frac{2}{3}}\|k^n\|_{3(\beta-s+1)}^{\beta-s+1}\, dt \overset{(3.20)}{\underset{(3.23)}{\le}} C(s^{-1}) \text{ for all } s>0 \text{ small.}$$

Notice that the estimate (3.25) is better than the second estimate in (3.19) since we assume that $\gamma < \beta + \frac{2}{3}$, see $(2.1)_2$. Moreover, using the Hölder inequality and the estimates (3.15) and (3.25), it is easy to deduce that (note that the specific value of a small parameter s differs from s in (3.25))

$$\int_0^T \|\boldsymbol{v}^n k^n\|_{\frac{10}{9}\frac{3\beta+5}{\beta+5}-s}^{\frac{10}{9}\frac{3\beta+5}{\beta+5}-s}\, dt \le C(s^{-1}) \qquad \text{for all } s>0 \text{ small.} \tag{3.26}$$

Concerning the estimate on the gradient of k^n, we consider first the case $\beta \in [0,1]$ and we set $q := \frac{3\beta-3s+5}{4}$. Combining the estimates stated in (3.20) and (3.25), we conclude that

$$\begin{aligned}
\int_0^T \|\nabla k^n\|_q^q &\le C\int_0^T\int_\Omega \left(\mu(k^n)(1+k^n)^{-s-1}|\nabla k^n|^2\right)^{\frac{q}{2}}(1+k^n)^{\frac{q(s+1-\beta)}{2}}\, dx\, dt\\
&\le C\left(\int_0^T\int_\Omega \mu(k^n)(1+k^n)^{-s-1}|\nabla k^n|^2\, dx\, dt\right)^{\frac{q}{2}}\left(\int_0^T \|1+k^n\|_{\beta+\frac{5}{3}-s}^{\beta+\frac{5}{3}-s}\, dt\right)^{\frac{2-q}{2}}\\
&\le C(s^{-1}).
\end{aligned}$$

If $\beta > 1$ we can always find $s>0$ small enough so that $\beta - s - 1 > 0$. Consequently[2],

$$\begin{aligned}
&\int_0^T \|\nabla k^n\|_{\frac{3\beta+5-s}{4}}^{\frac{3\beta+5-s}{4}} \le C(s^{-1}) \qquad \text{for all } s>0 \text{ small} \qquad \text{for } \beta\in[0,1],\\
&\int_0^T \|\nabla k^n\|_2^2 \le C \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \beta>1.
\end{aligned} \tag{3.27}$$

[2]Note that the estimates (3.27) and (3.25) are better than those derived in [14] and [21].

Similarly, the estimates (3.21)–(3.25) together with $(1.8)_2$ imply that

$$\int_0^T \|\mu(k^n)\nabla k^n\|_{\frac{3\beta+5}{3\beta+4}-s}^{\frac{3\beta+5}{3\beta+4}-s} \le C(s^{-1}) \qquad \text{for all } s>0 \text{ small.} \tag{3.28}$$

Finally, using the above established estimates it is not difficult to observe (see [7] for details) that

$$\int_0^T \|k^n_{,t}\|_{-1,r-s}\,dt \le C(s^{-1}) \qquad \text{for all } s>0 \text{ small} \tag{3.29}$$

with r given by

$$r := \min\left\{\frac{3\beta+5}{3\beta+4}, \frac{10}{9}\frac{3\beta+5}{\beta+5}\right\}. \tag{3.30}$$

3.1.3 Limit $n\to\infty$. Letting $n\to\infty$ and using (3.13), (3.15), (3.16), (3.25) and (3.27), and using the convention that a selected sequence is denoted again as the original one, we can find a subsequence such that[3]

$$\boldsymbol{v}^n \rightharpoonup^* \boldsymbol{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}}), \tag{3.31}$$

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}})\cap L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}(\Omega)^3), \tag{3.32}$$

$$\boldsymbol{v}^n_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^2(0,T;W^{-1,2}_{\boldsymbol{n},\mathrm{div}}), \tag{3.33}$$

$$k^n \rightharpoonup k \qquad \text{weakly in } L^q(0,T;W^{1,q}_D(\Omega)) \text{ for all } q<\min\left\{\frac{3\beta+5}{4},2\right\}, \tag{3.34}$$

$$k^n \rightharpoonup k \qquad \text{weakly in } L^\omega(0,T;L^\omega(\Omega)) \text{ for all } 1\le\omega<\frac{3\beta+5}{3}, \tag{3.35}$$

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{8}{3}}(0,T;L^{\frac{8}{3}}(\partial\Omega)^3). \tag{3.36}$$

In addition, using the generalized version of the Aubin-Lions compactness lemma (see [29]) together with (3.33) and (3.29) leads to the conclusions that

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{strongly in } L^q(0,T;L^q(\Omega)^3) \text{ for all } q<\frac{10}{3}, \tag{3.37}$$

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{strongly in } L^q(0,T;L^q(\partial\Omega)^3) \text{ for all } q<\frac{8}{3}, \tag{3.38}$$

$$k^n \to k \qquad \text{strongly in } L^q(0,T;L^q(\Omega)) \text{ for all } q<\frac{3\beta+5}{3}, \tag{3.39}$$

and consequently we show that (at least for a suitable subsequence)

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{a.e. in } (0,T)\times\Omega, \tag{3.40}$$

$$k^n \to k \qquad \text{a.e. in } (0,T)\times\Omega, \tag{3.41}$$

[3] For the proof of (3.36) and (3.38) see [7].

(3.42)
$$\Phi_s(k^n) \rightharpoonup \Phi_s(k) \quad \text{weakly in } L^2(0,T;W_D^{1,2}(\Omega)) \text{ for all } s>0 \text{ small.}$$

Moreover, using the Fatou lemma, (3.19) and (3.41) we can conclude that

$$\sup_{t\in(0,T)} \|k(t)\|_1 \le C. \tag{3.43}$$

Concerning limits in the nonlinear terms in (3.6) and (3.7) we first easily observe (recall that $\nu(T_m(k^n))$ is a bounded a.e. convergent sequence as $n\to\infty$) that

$$\sqrt{\nu(T_m(k^n))}\mathbf{D}(\boldsymbol{v}^n) \rightharpoonup \sqrt{\nu(T_m(k))}\mathbf{D}(\boldsymbol{v}) \quad \text{weakly in } L^2(0,T;L^2(\Omega)^{3\times3}), \tag{3.44}$$
$$\nu(T_m(k^n))\mathbf{D}(\boldsymbol{v}^n) \rightharpoonup \nu(T_m(k))\mathbf{D}(\boldsymbol{v}) \quad \text{weakly in } L^2(0,T;L^2(\Omega)^{3\times3}). \tag{3.45}$$

Next, having the assumption on γ, see $(1.8)_3$, one can also obtain by using (3.34), (3.39) and the Vitali theorem that

$$\varepsilon(k^n)\to\varepsilon(k) \qquad \text{strongly in } L^q(0,T;L^q(\Omega)) \text{ for all } q<\frac{3\beta+5}{3(\gamma+1)}. \tag{3.46}$$

Also, it is a consequence of (3.28) that there is some $\boldsymbol{q}$ such that

$$\mu(k^n)\nabla k^n \rightharpoonup \boldsymbol{q} \qquad \text{weakly in } L^q(0,T;L^q(\Omega)^3) \text{ for all } q<\frac{3\beta+5}{3\beta+4}. \tag{3.47}$$

In order to identify $\boldsymbol{q}$, we first remark that it is enough to show that

$$\lim_{n\to\infty}\int_0^T (\mu(k^n)\nabla k^n,\boldsymbol{\varphi})\,dt = \int_0^T (\mu(k)\nabla k,\boldsymbol{\varphi})\,dt \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}((0,T)\times\Omega).$$

However, using the assumption $(1.8)_2$ concerning μ and the convergence results (3.39) and (3.42) we observe that

$$\int_0^T (\mu(k^n)\nabla k^n,\boldsymbol{\varphi})\,dt = \int_0^T (\underbrace{\mu(k^n)(1+k^n)^{-\frac{\beta-s-1}{2}}}_{\text{strongly in } L^2}\underbrace{\nabla\Phi_s(k^n)}_{\text{weakly in } L^2},\boldsymbol{\varphi})\,dt$$
$$\overset{n\to\infty}{\to}\int_0^T (\mu(k)(1+k)^{-\frac{\beta-s-1}{2}}\nabla\Phi_s(k),\boldsymbol{\varphi})\,dt = \int_0^T (\mu(k)\nabla k,\boldsymbol{\varphi})\,dt.$$

Consequently, $\boldsymbol{q}=\mu(k)\nabla k$.

All above established convergence results are not sufficient to take the limit in the nonlinear term at the right hand side of (3.7). However, since m is fixed and $\boldsymbol{v}=\boldsymbol{v}^m$ is an admissible test function in (3.6) we can use energy equality method here. First, we notice that it follows from (3.31)–(3.33), (3.37) and (3.45) that

$$
\begin{aligned}
&\int_0^T \langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - \big(G_m(|\boldsymbol{v}|^2)\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{w}\big)\ dt + \int_0^T (\nu(T_m(k))\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{w}))\ dt \\
&+ \frac{\lambda}{1-\lambda}\int_0^T (\boldsymbol{v},\boldsymbol{w})_{\partial\Omega}\ dt = 0 \qquad \text{for all } \boldsymbol{w}\in L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}).
\end{aligned} \tag{3.48}
$$

Moreover, using (3.31)–(3.33) and (3.44) it is standard to deduce (see for example [24]) that

$$
\boldsymbol{v}\in \mathcal{C}([0,T];L^2_{\boldsymbol{n},\mathrm{div}}) \quad \text{and} \quad \boldsymbol{v}(0)=\boldsymbol{v}_0\,.
$$

Next, we shall show that we can replace the weak convergence in (3.45) by the strong one. For this purpose, we first integrate (3.12) w.r.t. time $t\in(0,T)$ and obtain

$$
\begin{aligned}
\int_0^T \|\sqrt{\nu(T_m(k^n))}\mathbf{D}(\boldsymbol{v}^n)\|_2^2\ dt &= -\frac{1}{2}\|\boldsymbol{v}^n(T)\|_2^2 + \frac{1}{2}\|\boldsymbol{v}_0^n\|_2^2 - \int_0^T \frac{\lambda}{1-\lambda}\|\boldsymbol{v}^n\|_{2,\partial\Omega}^2\ dt \\
&= -\frac{1}{2}\|\boldsymbol{v}^n(T)-\boldsymbol{v}(T)\|_2^2 + \frac{1}{2}\|\boldsymbol{v}_0^n-\boldsymbol{v}_0\|_2^2 - \int_0^T \langle \boldsymbol{v}_{,t}, \boldsymbol{v}^n-\boldsymbol{v}\rangle + \langle \boldsymbol{v}^n_{,t}, \boldsymbol{v}\rangle\ dt \\
&\quad - \int_0^T \frac{\lambda}{1-\lambda}\|\boldsymbol{v}^n\|_{2,\partial\Omega}^2\ dt.
\end{aligned}
$$

Therefore, letting $n\to\infty$ we deduce from (3.32), (3.33), (3.38) and (3.8) that

$$
\begin{aligned}
\limsup_{n\to\infty}\int_0^T \|\sqrt{\nu(T_m(k^n))}\mathbf{D}(\boldsymbol{v}^n)\|_2^2\ dt \le &-\int_0^T \langle \boldsymbol{v}_{,t},\boldsymbol{v}\rangle\ dt \\
&- \int_0^T \frac{\lambda}{1-\lambda}\|\boldsymbol{v}\|_{2,\partial\Omega}^2\ dt.
\end{aligned} \tag{3.49}
$$

Next, setting $\boldsymbol{w}:=\boldsymbol{v}$ in (3.48) and using (3.49) we obtain

$$
\limsup_{n\to\infty}\int_0^T \|\sqrt{\nu(T_m(k^n))}\mathbf{D}(\boldsymbol{v}^n)\|_2^2\ dt \le \int_0^T \|\sqrt{\nu(T_m(k))}\mathbf{D}(\boldsymbol{v})\|_2^2\ dt. \tag{3.50}
$$

Consequently, as (3.44) implies that

$$
\int_0^T \|\sqrt{\nu(T_m(k))}\mathbf{D}(\boldsymbol{v})\|_2^2\ dt \le \liminf_{n\to\infty}\int_0^T \|\sqrt{\nu(T_m(k^n))}\mathbf{D}(\boldsymbol{v}^n)\|_2^2\ dt \tag{3.51}
$$

we finally conclude that

$$
\sqrt{\nu(T_m(k^n))}\mathbf{D}(\boldsymbol{v}^n) \to \sqrt{\nu(T_m(k))}\mathbf{D}(\boldsymbol{v}) \quad \text{strongly in } L^2(0,T;L^2(\Omega)^{3\times 3}), \tag{3.52}
$$

or saying differently

$$
\nu(T_m(k^n))|\mathbf{D}(\boldsymbol{v}^n)|^2 \to \nu(T_m(k))|\mathbf{D}(\boldsymbol{v})|^2 \quad \text{strongly in } L^1(0,T;L^1(\Omega)). \tag{3.53}
$$

Finally, using (3.7), (3.29) and (3.53) we observe that

$$k^n_{,t} \rightharpoonup k_{,t} \qquad \text{weakly in } L^1(0,T;W_D^{-1,r-s}(\Omega)) \quad \text{for all } s>0 \text{ small,} \tag{3.54}$$

with r given by (3.30). At this point, it is easy to take the limit in (3.7) and arrive at

$$\begin{aligned} &\int_0^T \langle k_{,t}, w\rangle - (\boldsymbol{v}k, \nabla w) + (\mu(k)\nabla k, \nabla w) + (\varepsilon(k), w)\, dt \\ &\quad = \int_0^T (\nu(T_m(k))|\mathbf{D}(\boldsymbol{v})|^2, w)\, dt \qquad \text{for all } w \in L^\infty(0,T;W_D^{1,\infty}(\Omega)). \end{aligned} \tag{3.55}$$

3.1.4 Attainment of initial data k_0. We first integrate (3.17) w.r.t. time over $(0,t)$ and obtain (note that the second term vanishes and the third and fourth terms are nonnegative)

$$\|\Theta_1(k^n(t))\|_1 \le \int_0^t \nu(T_m(k^n))|\mathbf{D}(\boldsymbol{v}^n)|^2\, dx\, d\tau + \|\Theta_1(k_0^n)\|_1.$$

Next, we let $n\to\infty$. Using the nonnegativity of Θ_1, the point-wise convergence of k^n, see (3.41), and the Fatou lemma we are able to take limit in the term at the left hand side with corresponding inequality sign. On the other hand, using (3.53) we are able to identify limit of the first term on the right hand side and therefore we obtain for almost all time $t \in (0,T)$

$$\|\Theta_1(k(t))\|_1 \le \int_0^t \nu(T_m(k))|\mathbf{D}(\boldsymbol{v})|^2\, dx\, d\tau + \|\Theta_1(k_0)\|_1, \tag{3.56}$$

which implies that

$$\limsup_{t\to 0+} \|\Theta_1(k(t))\|_1 \le \|\Theta_1(k_0)\|_1. \tag{3.57}$$

Next, setting in (3.55) $w := T_1(k^n)(\Theta_1(k^n))^{-\frac{1}{2}}\varphi\chi_{[0,t]}$ where $\varphi\in\mathcal{D}(\Omega)$, $\varphi\ge 0$, we obtain (note that w is an admissible test function)

$$\begin{aligned} &2(\sqrt{\Theta_1(k^n(t))}, \varphi) - 2\int_0^t (\boldsymbol{v}^n\sqrt{\Theta_1(k^n)}, \nabla\varphi)\, d\tau \\ &+ \int_0^t\int_\Omega \mu(k^n)\left(T_1'(k^n)(\Theta_1(k^n))^{-\frac{1}{2}} - \frac{1}{2}(T_1(k^n))^2(\Theta_1(k^n))^{-\frac{3}{2}}\right)|\nabla k^n|^2\varphi\, dx\, d\tau \\ &+ \int_0^t (\mu(k^n)T_1(k^n)(\Theta_1(k^n))^{-\frac{1}{2}}\nabla k^n, \nabla\varphi)\, d\tau \\ &+ \int_0^t (\varepsilon(k^n), T_1(k^n)(\Theta_1(k^n))^{-\frac{1}{2}}\varphi)\, d\tau \\ &= \int_0^t (\nu(T_m(k^n))|\mathbf{D}(\boldsymbol{v}^n)|^2, T_1(k^n)(\Theta_1(k^n))^{-\frac{1}{2}}\,\varphi)\, d\tau + 2(\sqrt{\Theta_1(k_0^n)}, \varphi). \end{aligned}$$

Observing that the integrand in the third integral is non-positive and the first integral on the right hand side is nonnegative, we can neglect both of them by replacing the equality sign by the inequality[4]. Then we let $n \to \infty$. Applying all convergence results established above, it is standard to conclude that for almost all times $t \in (0,T)$

$$\begin{aligned}&(\sqrt{\Theta_1(k(t))}, \varphi) - \int_0^t (\boldsymbol{v}\sqrt{\Theta_1(k)}, \nabla\varphi)\, d\tau + \frac{1}{2}\int_0^t (\mu(k)T_1(k)(\Theta_1(k))^{-\frac{1}{2}}\nabla k, \nabla\varphi)\, d\tau \\ &+ \frac{1}{2}\int_0^t (\varepsilon(k), T_1(k)(\Theta_1(k))^{-\frac{1}{2}}\varphi)\, d\tau \geq 2(\sqrt{\Theta_1(k_0)}, \varphi).\end{aligned}$$

Finally, letting $t \to 0_+$ we observe that

$$\liminf_{t\to 0+}(\sqrt{\Theta_1(k(t))}, \varphi) \geq (\sqrt{\Theta_1(k_0)}, \varphi) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega),\ \varphi \geq 0.$$

Thus, using the density argument, (3.43) and the fact that $\Theta_1(k)$ has at most linear growth in k, we finally deduce that

(3.58)
$$\liminf_{t\to 0+}(\sqrt{\Theta_1(k(t))}, \varphi) \geq (\sqrt{\Theta_1(k_0)}, \varphi) \quad \text{for all } \varphi \in L^2(\Omega),\ \varphi \geq 0 \text{ a.e. in } \Omega.$$

Consequently, it is then easy to observe that

$$\begin{aligned}&\lim_{t\to 0+} \|\sqrt{\Theta_1(k(t))} - \sqrt{\Theta_1(k_0)}\|_2^2 \\ &\quad = \lim_{t\to 0+}\Big(\|\Theta_1(k(t))\|_1 + \|\Theta_1(k_0)\|_1 - 2(\sqrt{\Theta_1(k(t))}, \sqrt{\Theta_1(k_0)})\Big) \\ &\quad \overset{(3.57),(3.58)}{\leq} \|\Theta_1(k_0)\|_1 + \|\Theta_1(k_0)\|_1 - 2(\sqrt{\Theta_1(k_0)}, \sqrt{\Theta_1(k_0)}) = 0,\end{aligned}$$

which finally leads to

$$\lim_{t\to 0+}\|k(t) - k_0\|_1 = 0. \tag{3.59}$$

3.2 Limit $m \to \infty$. In the previous subsection, we established the existence of $(\boldsymbol{v}^m, k^m)$ fulfilling, for every $m \in \mathbb{N}$ fixed, the weak formulations (3.48) and (3.55). Before summarizing the estimates for $(\boldsymbol{v}^m, k^m)$ that are uniform with respect to m, we take the advantage of considered slip boundary conditions ($0 \leq \lambda < 1$ in (1.5)) and introduce the integrable pressure.

For any $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{n}}$ we observe that the Helmholtz decomposition $\boldsymbol{w} = \boldsymbol{w}_{\text{div}} + \nabla\varphi$ with φ having zero mean over Ω and solving $-\Delta\varphi = \operatorname{div}\boldsymbol{w}$ in Ω and homogeneous Neumann problem on $\partial\Omega$ is compatible with (1.5) for $0 \leq \lambda < 1$. Indeed, noticing

[4] At this level of approximation, we even do not need this simplification because we are able to identify the limit of corresponding quantities. However, it will not be the case in the final passage to the limit and we will be forced to use such procedure.

that

$$\int_0^T \langle \boldsymbol{v}^m_{,t}, \boldsymbol{w}\rangle \, dt = \int_0^T \langle \boldsymbol{v}^m_{,t}, \boldsymbol{w}_{\mathrm{div}}\rangle \, dt, \tag{3.60}$$

we can extend the definition domain for $\boldsymbol{v}^m_{,t}$ and observe that $\boldsymbol{v}^m_{,t} \in L^2(0,T; W^{-1,2}_{\boldsymbol{n}})$.

Let us introduce p^m as the solution of the following problem

$$\begin{aligned}(p^m, \triangle\varphi) = {}& (\nu(T_m(k^m))\mathbf{D}(\boldsymbol{v}^m), \nabla^{(2)}\varphi) + \frac{\lambda}{1-\lambda}(\boldsymbol{v}^m, \nabla\varphi)_{\partial\Omega} \\ & - (G_m(|\boldsymbol{v}|^2)\boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla^2\varphi) \text{ for all } \varphi \in W^{2,2}(\Omega),\ \nabla\varphi \in W^{1,2}_{\boldsymbol{n}}.\end{aligned} \tag{3.61}$$

Taking $\boldsymbol{w} \in L^2(0,T,W^{1,2}_{\boldsymbol{n}})$ arbitrarily, applying the Helmholtz decomposition on such $\boldsymbol{w}$, taking the sum of (3.48) with the test function $\boldsymbol{w}_{\mathrm{div}}$ and (3.61) and using (3.60) we obtain the following identity

$$\begin{aligned}&\int_0^T \langle \boldsymbol{v}^m_{,t}, \boldsymbol{w}\rangle - \big(G_m(|\boldsymbol{v}^m|^2)\boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla \boldsymbol{w}\big) + (\nu(T_m(k^m))\mathbf{D}(\boldsymbol{v}^m), \mathbf{D}(\boldsymbol{w}))\, dt \\ &+ \frac{\lambda}{1-\lambda}\int_0^T (\boldsymbol{v}^m, \boldsymbol{w})_{\partial\Omega}\, dt = \int_0^T (p^m, \operatorname{div}\boldsymbol{w})\, dt \text{ for all } \boldsymbol{w} \in L^2(0,T;W^{1,2}_{\boldsymbol{n}}).\end{aligned} \tag{3.62}$$

It is easy to check from (3.62) that such normalized p^m is uniquely determined by a given solution $(\boldsymbol{v}^n, k^n)$.

We also recall that the m-approximation satisfies (3.55) that we repeat for brevity. It reads as

$$\begin{aligned}&\int_0^T \langle k^m_{,t}, w\rangle - (\boldsymbol{v}^m k^m, \nabla w) + (\mu(k^m)\nabla k^m, \nabla w) + (\varepsilon(k^m), w)\, dt \\ &= \int_0^T (\nu(T_m(k^m))|\mathbf{D}(\boldsymbol{v}^m)|^2, w)\, dt \quad \text{ for all } w \in L^\infty(0,T;W^{1,\infty}_D(\Omega)).\end{aligned} \tag{3.63}$$

Next, we recall the uniform bound on $(\boldsymbol{v}^m, p^m)$ and derive the uniform bound on the pressure p^m that will be needed in what follows. First, referring to lower semicontinuity of the norms and the Fatou lemma we get from (3.13) and (3.19)

$$\begin{aligned}\sup_{t\in(0,T)} \big(\|\boldsymbol{v}^m(t)\|_2^2 + \|k^m(t)\|_1\big) + \int_0^T\int_\Omega \nu(T_m(k^m))|\mathbf{D}(\boldsymbol{v}^m)|^2\, dx\, dt \\ + \int_0^T \|k^m\|_{\gamma+1}^{\gamma+1}\, dt \le C.\end{aligned} \tag{3.64}$$

Moreover, using (3.64) and the standard embedding of Sobolev functions to the space of traces together with the standard interpolation inequalities one can deduce, see [9, Lemma 1.12] for details, that

$$\int_0^T \int_{\partial\Omega} |\boldsymbol{v}^m|^{\frac{8}{3}}\, dS\, dt + \int_0^T \|\boldsymbol{v}^m\|_{\frac{10}{3}}^{\frac{10}{3}}\, dt \le C. \tag{3.65}$$

In addition, referring again to the lower semicontinuity of the norms we obtain from (3.21) and (3.25)–(3.28)

$$\begin{aligned}\int_0^T &\|\Phi_s(k^m)\|_{1,2}^2 + \|\boldsymbol{v}^m k^m\|_{\frac{10}{9}\frac{3\beta+5}{\beta+5}-s}^{\frac{10}{9}\frac{3\beta+5}{\beta+5}-s} + \|k^m\|_{\beta+\frac{5}{3}-s}^{\beta+\frac{5}{3}-s} + \|\nabla k^m\|_{\min(2,\frac{3\beta+5}{4})-s}^{\min(2,\frac{3\beta+5}{4})-s}\, dt \\ &+ \int_0^T \|\mu(k^m)\nabla k^m\|_{\frac{3\beta+5}{3\beta+4}-s}^{\frac{3\beta+5}{3\beta+4}-s}\, dt \le C(s^{-1}) \qquad \text{for all } s>0 \text{ small.}\end{aligned} \tag{3.66}$$

Next, observing that

$$\nu(T_m(k^m))\mathbf{D}(\boldsymbol{v}^m) = \sqrt{\nu(T_m(k^m))}\mathbf{D}(\boldsymbol{v}^m)\sqrt{\nu(T_m(k^m))}\,,$$

and recalling that according to (3.64) $\sqrt{\nu(T_m(k^m))}\mathbf{D}(\boldsymbol{v}^m)$ is uniformly bounded in $L^2(0,T;L^2(\Omega)^{3\times 3})$ and according to (3.66) $\sqrt{\nu(T_m(k^m))}$, which grows as $(1+k^m)^{\alpha/2}$, is bounded uniformly in $L^{\frac{2}{\alpha}(\beta+\frac{5}{3}-s)}(0,T;L^{\frac{2}{\alpha}(\beta+\frac{5}{3}-s)}(\Omega))$, we conclude that

$$\begin{aligned}&\int_0^T \|\nu(T_m(k^m))\mathbf{D}(\boldsymbol{v}^m)\|_{q_0-s}^{q_0-s}\, dt \le C(s^{-1}) \quad \text{for all } s>0 \text{ small,}\\ &\qquad\qquad\qquad \text{with } q_0 := \frac{2(3\beta+5)}{3\alpha+3\beta+5}\,.\end{aligned} \tag{3.67}$$

Similarly, incorporating also the second estimate in (3.65), we observe that

$$\begin{aligned}&\int_0^T \|\nu(T_m(k^m))\mathbf{D}(\boldsymbol{v}^m)\boldsymbol{v}^m\|_{w_0-s}^{w_0-s}\, dt \le C(s^{-1}) \quad \text{for all } s>0 \text{ small,}\\ &\qquad\qquad\qquad \text{with } w_0 := \frac{10(3\beta+5)}{15\alpha+24\beta+40}\,.\end{aligned} \tag{3.68}$$

Note that the assumption $(2.1)_1$ guarantees that $w_0>1$.

At this point, we can deduce from (3.61) the estimates for $\{p^m\}$ that will be uniform with respect to m. We consider φ with zero mean over Ω solving the homogeneous Neumann problem $-\triangle\varphi = |p^m|^{q-2}p^m - \frac{1}{|\Omega|}\int_\Omega |p^m|^{q-2}p^m\, dx$ and inserting it into (3.61). Using the estimates on $\{\boldsymbol{v}^m\}$ and the Hölder inequality we obtain

$$\int_0^T \|p^m\|_{z_0-s}^{z_0-s}\, dt \le C(s^{-1}) \quad \text{for all } s>0 \text{ small}, \tag{3.69}$$
$$\text{with } z_0 := \min\left(\frac{5}{3}, \frac{2(3\beta+5)}{3\alpha+3\beta+5}\right).$$

Finally, using equation (3.62) and the above estimates we conclude that

$$\int_0^T \|\boldsymbol{v}^m_{,t}\|^{z_0-s}_{W^{-1,z_0-s}_{\boldsymbol{n}}}\, dt \le C(s^{-1}) \quad \text{for all } s>0 \text{ small}. \tag{3.70}$$

Similarly as in the previous subsection, using (3.55), (3.64) and (3.66) we deduce that

$$\int_0^T \|k^m_{,t}\|_{W^{-1,r-s}_D}\, dt \le C(s^{-1}) \text{ for all } s>0 \text{ small and } r \text{ defined in (3.30)}. \tag{3.71}$$

Having all uniform estimates (3.64), (3.65), (3.66), (3.69), (3.70) and (3.71), and using the generalized version of the Aubin-Lions compactness lemma we find subsequences that we again label in the same way as the original sequences such that (we use the convention that $s>0$ is small but arbitrary)

$$\boldsymbol{v}^m \rightharpoonup^* \boldsymbol{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}}), \tag{3.72}$$
$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}) \cap L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}(\Omega)^3), \tag{3.73}$$
$$\boldsymbol{v}^m_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^{z_0-s}(0,T;W^{-1,z_0-s}_{\boldsymbol{n}}) \text{ for } z_0 \text{ from (3.69)}, \tag{3.74}$$
$$p^m \rightharpoonup p \qquad \text{weakly in } L^{z_0-s}(0,T;L^{z_0-s}(\Omega)) \text{ for } z_0 \text{ from (3.69)}, \tag{3.75}$$
$$k^m \rightharpoonup k \qquad \text{weakly in } L^q(0,T;W^{1,q}_D(\Omega)) \text{ for all } q < \min(2, \frac{3\beta+5}{4}), \tag{3.76}$$
$$k^m_{,t} \rightharpoonup^* k_{,t} \qquad \text{weakly}^* \text{ in } \mathcal{M}(0,T;W^{-1,r-s}_D(\Omega)) \text{ for } r \text{ from (3.30)}, \tag{3.77}$$
$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{8}{3}}(0,T;L^{\frac{8}{3}}(\partial\Omega)^3), \tag{3.78}$$
$$\boldsymbol{v}^m \to \boldsymbol{v} \qquad \text{strongly in } L^q(0,T;L^q(\Omega)^3) \text{ for all } q < \frac{10}{3}, \tag{3.79}$$
$$\boldsymbol{v}^m \to \boldsymbol{v} \qquad \text{strongly in } L^q(0,T;L^q(\partial\Omega)^3) \text{ for all } q < \frac{8}{3}, \tag{3.80}$$
$$k^m \to k \qquad \text{strongly in } L^q(0,T;L^q(\Omega)) \text{ for all } q < \frac{3\beta+5}{3}, \tag{3.81}$$
$$\boldsymbol{v}^m \to \boldsymbol{v} \qquad \text{a.e. in } \Omega\times(0,T), \tag{3.82}$$
$$k^m \to k \qquad \text{a.e. in } \Omega\times(0,T), \tag{3.83}$$
$$\Phi_s(k^m) \rightharpoonup \Phi_s(k) \quad \text{weakly in } L^2(0,T;W^{1,2}_D(\Omega)). \tag{3.84}$$

Moreover, using the same procedure as in the previous subsection we can conclude that

$$\sup_{t\in(0,T)} \|k(t)\|_1 \leq C. \tag{3.85}$$

Similarly, as in the previous subsection, see (3.47), we can verify that

$$\mu(k^m)\nabla k^m \rightharpoonup \mu(k)\nabla k \quad \text{weakly in } L^q(0,T;L^q(\Omega)^3) \text{ for all } q < \frac{3\beta+5}{3\beta+4}\,. \tag{3.86}$$

Moreover, it follows from (3.64) that there is an $\mathbf{S} \in L^2(0,T;L^2(\Omega)^{3\times 3})$ such that

$$\sqrt{\nu(T_m(k^m))}\mathbf{D}(\boldsymbol{v}^m) \rightharpoonup \mathbf{S} \qquad \text{weakly in } L^2(0,T;L^2(\Omega)^{3\times 3}). \tag{3.87}$$

To identify $\mathbf{S}$ we first observe that (3.83), the growth assumption $(1.8)_1$, (3.66) and Vitali's theorem imply that

$$\nu(T_m(k^m)) \to \nu(k) \quad \text{strongly in } L^q(0,T;L^q(\Omega)) \text{ for all } q < \frac{3\beta+5}{3\alpha}\,. \tag{3.88}$$

Since the assumption (2.1) guarantees that $\frac{3\beta+5}{3\alpha} > 2$, it follows from (3.73) and (3.88) that

$$\mathbf{S} = \sqrt{\nu(k)}\mathbf{D}(\boldsymbol{v}) \text{ a.e. in } (0,T)\times\Omega. \tag{3.89}$$

Similarly, using (3.67), we can deduce that

$$\nu(T_m(k^m))\mathbf{D}(\boldsymbol{v}^m) \rightharpoonup \mathbf{S}_2 \quad \text{weakly in } L^q(0,T;L^q(\Omega)^{3\times 3}) \text{ for all } q < q_0\,. \tag{3.90}$$

To identify $\mathbf{S}_2$ it is then enough to combine (3.87), (3.89) and (3.88) to obtain that

$$\mathbf{S}_2 = \nu(k)\mathbf{D}(\boldsymbol{v}) \text{ a.e. in } (0,T)\times\Omega.$$

At this point, we can complete the proof of Theorem 2.1. First note that (3.72)–(3.88) implies that the triple $(\boldsymbol{v},k,p)$ satisfies (2.2)–(2.7). Next, the above established convergences (3.72)–(3.90) suffice to prove (2.9) by letting $m\to\infty$ in (3.62). Similarly, letting $m\to\infty$ in (3.63) we deduce (2.11), using the weak lower semicontinuity of the last term in (3.63).

Then, setting in (3.62) $\boldsymbol{w} := \boldsymbol{v}^m w$ with arbitrary $w \in L^\infty(0,T;W_D^{1,\infty}(\Omega))$ and adding the result to (3.63) we arrive at

(3.91)
$$\int_0^T \langle E^m_{,t}, w\rangle - (\boldsymbol{v}^m(p^m+k^m),\nabla w) - (G_m(|\boldsymbol{v}^m|^2)\boldsymbol{v}^m\otimes\boldsymbol{v}^m, \nabla(\boldsymbol{v}^m w))\,dt$$
$$+\int_0^T (\nu(T_m(k^m))\mathbf{D}(\boldsymbol{v}^m)\boldsymbol{v}^m,\nabla w) + (\mu(k^m)\nabla k^m,\nabla w) + (\varepsilon(k^m),w)\,dt = 0,$$

where we set

$$E^m := \frac{1}{2}|\boldsymbol{v}^m|^2 + k^m.$$

Noticing that the third term in (3.91) can be simplified by using integration by parts and also the fact that $\operatorname{div} \boldsymbol{v}^m = 0$ in Ω, we get

$$\begin{aligned}(G_m(|\boldsymbol{v}^m|^2)\boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla(\boldsymbol{v}^m w))& \\ &= \frac{1}{2}(w\boldsymbol{v}^m, \nabla\Gamma_m(|\boldsymbol{v}^m|^2)) + (G_m(|\boldsymbol{v}^m|^2)|\boldsymbol{v}^m|^2\boldsymbol{v}^m, \nabla w)) \\ &= ((G_m(|\boldsymbol{v}^m|^2)|\boldsymbol{v}^m|^2 - \frac{1}{2}\Gamma_m(|\boldsymbol{v}^m|^2))\boldsymbol{v}^m, \nabla w).\end{aligned}$$

From (3.91) we can obtain the estimate on the time derivative of E^m and by selecting a subsequence observe that

$$E^m_{,t} \rightharpoonup E_{,t} \quad \text{weakly in } L^q(0,T; W_D^{-1,q}(\Omega)), \text{ where } E := \frac{1}{2}|\boldsymbol{v}|^2 + k, \tag{3.92}$$

for all $1 < q < \min\left\{\frac{10}{9}, w_0, \frac{3\beta+5}{3\beta+4}\right\}$; w_0 is introduced in (3.68).

Finally, setting $m \to \infty$ in (3.91) it is standard to obtain (2.10).

3.2.1 Attainment of initial condition. We aim to prove (2.12). The first part, i.e., the attainment of the initial velocity $\boldsymbol{v}_0$ is standard and we refer the reader to [24]. To establish the second part we use the similar procedure as in the previous subsection with only one essential change. First part follows the procedure from the preceding subsection and we deduce that

$$\liminf_{t\to 0+}(\sqrt{\Theta_1(k(t))}, \varphi) \geq (\sqrt{\Theta_1(k_0)}, \varphi) \text{ for all } \varphi \in L^2(\Omega),\ \varphi \geq 0 \text{ a.e. in } \Omega. \tag{3.93}$$

To finish the proof of (2.12) it is then enough to obtain

$$\limsup_{t\to 0+} \|\Theta_1(k(t))\|_1 \leq \|\Theta_1(k_0)\|_1 \tag{3.94}$$

and the same arguments as in preceding subsection then leads to (2.12). To prove (3.94) we have to proceed differently. Rewriting (3.56) again as

$$\|\Theta_1(k^m(t))\|_1 \leq \int_0^t \int_\Omega \nu(T_m(k^m))|\mathbf{D}(\boldsymbol{v}^m)|^2 \, dx \, dt + \|\Theta_1(k_0)\|_1, \tag{3.95}$$

we can replace the first term on the right hand side by using $\boldsymbol{w} := \boldsymbol{v}^n \chi_{[0,t]}$ as a test function in (3.62). Hence, after neglecting the boundary integral, because of correct sign, we get

$$\|\Theta_1(k^m(t))\|_1 \leq -\|\boldsymbol{v}^m(t)\|_2^2 + \|\boldsymbol{v}_0\|_2^2 + \|\Theta_1(k_0)\|_1. \tag{3.96}$$

Therefore, passing to the limit w.r.t. m we get after using the Fatou lemma and weak lower semicontinuity of norm that

$$\|\Theta_1(k(t))\|_1 \leq -\|\boldsymbol{v}(t)\|_2^2 + \|\boldsymbol{v}_0\|_2^2 + \|\Theta_1(k_0)\|_1. \tag{3.97}$$

Consequently, using also the first part of (2.12) then leads to (3.94). Thus, the proof is complete.

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