Boundary value problems for semilinear evolution inclusions: Carathéodory selections approach

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Abstract. In this paper we prove two existence theorems for abstract boundary value problems controlled by semilinear evolution inclusions in which the nonlinear part is a lower Scorza-Dragoni multifunction. Then, by using these results, we obtain the existence of periodic mild solutions.

Keywords: semilinear differential inclusion, selection theorem, mild solution, lower Scorza Dragoni multifunction, mild periodic solution.

Classification: Primary 34A60, 34G20

1. Introduction

In the setting of a separable Banach space X we prove the existence of mild solutions for abstract boundary value problems controlled by the following semilinear evolution inclusion

$$x' \in A(t)x + F(t, x)$$

in which $\{A(t)\}_{t\in[0,b]}$ is a family of densely defined linear operators generating an evolution operator T and F is a multifunction. At first we examine problems in which the boundary condition $Lx = \omega$ is present, L is here a continuous linear operator and $\omega \in X$. Then we consider problems with the more general boundary condition $Lx = Mx$, where M is a compact operator.

In the recent past, analogous problems for ordinary differential equations have been treated by many authors (for a wide bibliography on this subject see for instance [6]). In the last few years the attention has been given to abstract boundary value problems controlled by semilinear differential inclusions. We refer, for example, to the papers of Anichini-Zecca [2], Zecca-Zezza [20], N.S. Papageorgiou [16], [17], [18] and to the monograph [9]. The possibility of wide practical applications explains the growing interest in the investigation of such abstract boundary value problems. Among the others, the recent papers [15] and [1] study "degenerate" problems.

In the first part of this note we present two existence results of mild solutions by considering lower Scorza-Dragoni multifunctions (see Theorem 3.1 and Theorem 3.2). As concerns the operators of the family $\{A(t)\}_{t\in[0,b]}$ we allow that they

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are unbounded and so our theorems can be applied to the study of distributed parameter control problems (see [14]) and in mathematical physics (free boundary and obstacle problems, see [5]). We show that our two results extend in a broad sense the theorems proved in [18] (see Remarks 3.1, 3.2 and 3.3).

In the last section we illustrate the applicability of our results for the study of problems with periodic boundary conditions and we obtain two existence theorems which extend in a broad sense the results proved in [18] (see Remark 4.1).

2. Preliminaries

Throughout this paper J denotes the interval $[0,b], b>0$, of the real line (endowed with the Lebesgue measure μ) and X is a Banach space with the norm $\|\cdot\|$. Let $\mathcal{P}(X)$ denote the collection of all nonempty subsets of X. For every $A\in\mathcal{P}(X)$, we denote by $\|A\|=\sup_{a\in A}\|a\|$. The following notations will also be used

$$\mathcal{P}_{f(c)}(X)=\{S\in\mathcal{P}(X): S \text{ is closed (and convex)}\},$$
$$\mathcal{P}_{(w)k(c)}(X)=\{S\in\mathcal{P}(X): S \text{ is (weakly-)compact (and convex)}\}.$$

For every $A,B\in\mathcal{P}_f(X)$, we define the Hausdorff metric by

$$H(A,B)=\max\{\sup_{b\in B}\rho(b,A),\sup_{a\in A}\rho(a,B)\}$$

where $\rho(b,A)=\inf\{\|a-b\|: a\in A\}$.

Let T be a topological space. A multifunction $F: T\to\mathcal{P}_f(X)$ is said to be *H-continuous* if it is continuous from T into the metric space $(\mathcal{P}_f(X),H)$. A multifunction $F: T\to\mathcal{P}(X)$ is said to be *lower semicontinuous* (*l.s.c.*) *at a point* $t_0\in T$ if, for every open set $A\subseteq X$ such that $F(t_0)\cap A\neq\emptyset$, there exists a neighbourhood U of t_0 with the property $F(t)\cap A\neq\emptyset$, for every $t\in U$.

Let (Ω,Σ) be a measurable space. A multifunction $F:\Omega\to\mathcal{P}(X)$ is said to be *measurable* if the set $F^-(U)=\{t\in\Omega: F(t)\cap U\neq\emptyset\}$ is measurable, for every open set $U\subseteq X$, while F is said to be *graph measurable* if $\operatorname{Gr}F=\{(\omega,x)\in\Omega\times X: x\in F(\omega)\}\in\Sigma\times\mathcal{B}(X)$, with $\mathcal{B}(X)$ the Borel σ-field of X. Moreover F is said to be *scalarly measurable* if for every x^* belonging to the dual topological space X^*, the function $\omega\mapsto\sigma(x^*,F(\omega))=\sup\{\langle x^*,x\rangle: x\in F(\omega)\}$ is measurable, where $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair (X,X^*).

By S_F^p, $1\le p\le\infty$, we denote the set of all selections of a multifunction $F: J\to\mathcal{P}(X)$ that belong to the Lebesgue-Bochner space $L^p(J,X)$ (see [8, Definition A.3.89]), i.e. $S_F^p=\{f\in L^p(J,X): f(t)\in F(t)\text{ a.e. on } J\}$.

A multifunction $F: J\times X\to\mathcal{P}_f(X)$ is said to satisfy the *Scorza-Dragoni property* if, for every $\varepsilon>0$, there exists a closed subset J_ε of J, $\mu(J\backslash J_\varepsilon)<\varepsilon$, such that $F_{|J_\varepsilon\times X}$ is H-continuous.

A multifunction $F: J\times X\to\mathcal{P}(X)$ is said to verify the *lower Scorza-Dragoni property* if, for every $\varepsilon>0$, there exists a closed subset J_ε of J, $\mu(J\backslash J_\varepsilon)<\varepsilon$, such that $F_{|J_\varepsilon\times X}$ is lower semicontinuous.

Finally we recall that a function $f : J \times X \to X$ is said to be a *Carathéodory selection* of a multifunction $F : J \times X \to \mathcal{P}(X)$ if, for every $x \in X$, $f(\cdot, x)$ is measurable; for every $t \in J$, $f(t, \cdot)$ is continuous; $f(t, x) \in F(t, x)$, a.e. $t \in J$, for every $x \in X$.

Moreover we recall the following basic definitions.

Let $\Delta = \{(t, s) \in J \times J : 0 \le s \le t \le b\}$ be fixed. A two parameter family $\{T(t, s)\}_{(t,s)\in\Delta}$, $T(t, s) : X \to X$ bounded linear operators, is called an *evolution system* if

(i) $T(t, t) = I$, $t \in J$,
(ii) $T(t, r)T(r, s) = T(t, s)$, $0 \le s \le r \le t \le b$,
(iii) $(t, s) \mapsto T(t, s)$ is strongly continuous on Δ, i.e., for every $x \in X$, the map $(t, s) \mapsto T(t, s)x$ is continuous on Δ (see e.g. [11]),

and we denote with $T : \Delta \to \mathcal{L}(X)$ the respective *evolution operator* (see e.g. [19]), where $\mathcal{L}(X)$ is the space of all bounded linear operators from X to X. Taking (iii) into account, we note that there exists a constant $\mathcal{M} > 0$ such that

$$\sup_{(t,s)\in\Delta} \|T(t, s)\|_{\mathcal{L}} \le \mathcal{M}, \tag{2.1}$$

where $\|T(t, s)\|_{\mathcal{L}} = \sup_{\|x\|\le 1} \|T(t, s)x\| \le \mathcal{M}$.

Let $\{A(t)\}_{t\in J}$ be a family of linear operators, $A(t) : \mathcal{D}(A) \subseteq X \to X$, $\mathcal{D}(A)$ not depending on t and being a dense subset of X. In order to obtain our existence results it is not necessary to precise the way in which the family $\{A(t)\}_{t\in J}$ generates an evolution operator. Usually is said that $\{A(t)\}_{t\in J}$ *generates an evolution operator* $T : \Delta \to \mathcal{L}(X)$ if there exists an evolution system $\{T(t, s)\}_{(t,s)\in\Delta}$ such that on the region $\mathcal{D}(A)$ each operator $T(t, s)$ is strongly differentiable relative to t and s and

$$\frac{\partial T(t, s)}{\partial t} = A(t)T(t, s) \qquad \text{and} \qquad \frac{\partial T(t, s)}{\partial s} = -T(t, s)A(s)$$

(see e.g. [11], [12], [19]).

3. Main results

First of all we prove the following selection theorem.

Proposition 3.1. *Let X be a Banach space. If a multifunction $F : J \times X \to \mathcal{P}_{fc}(X)$ has the lower Scorza-Dragoni property, then there exists a Carathéodory selection $f : J \times X \to X$ of F.*

Proof: From the lower Scorza-Dragoni property, for every $j \in \mathbb{N}$, we can say that there exists a closed set $K_j \subset J$, $\mu(J \backslash K_j) < 2^{-j}$, such that $F_j = F_{|K_j \times X}$ is lower semicontinuous. Since F_j satisfies all assumptions of Michael's selection Theorem (see [13, Theorem 3.2″]) we can claim that there exists a continuous function $f_j : K_j \times X \to X$ such that $f_j(t,x) \in F(t,x)$, for all $(t,x) \in K_j \times X$. Now we consider the function $f : J \times X \to X$ defined by

$$f(t,x) = \begin{cases} f_j(t,x), & t \in K_j \setminus \bigcup_{i<j} K_i,\ j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

We want to show that f is a Carathéodory selection of F.

By using the continuity of f_j it is easy to check that, for each $t \in J$, the function $f(t,\cdot)$ is continuous on X. Now, fixing $x \in X$, in order to prove the measurability of $f(\cdot,x)$ we consider an open set $A \subset X$. We have

$$f^-(A,x) = \{t \in J : f(t,x) \in A\} = H_1 \cup H_2,$$

where

$$H_1 = \{t \in \bigcup_{j \in \mathbb{N}} K_j : f(t,x) \in A\}$$

and

$$H_2 = \{t \in J \setminus \bigcup_{j \in \mathbb{N}} K_j : f(t,x) \in A\}.$$

Let us note that the set H_1 can be rewritten as follows

$$H_1 = \bigcup_{j \in \mathbb{N}} A_j, \text{ where } A_j = f_j^-(A,x) \setminus \bigcup_{i<j} K_i.$$

Fixing $j \in \mathbb{N}$, by continuity of $f_j(\cdot,x)$ on K_j we have that there exists an open set $I_j \subset J$ such that $f_j^-(A,x) = I_j \cap K_j$. Therefore the set $f_j^-(A,x)$ is measurable. We can deduce that H_1 is measurable. On the other hand, as regards the set H_2 there are two possibilities: if $0 \in A$ then $H_2 = J \setminus \bigcup_{j \in \mathbb{N}} K_j$, while if $0 \notin A$ then $H_2 = \emptyset$. So, H_2 is also measurable.

Finally we observe that if $(t,x) \in (\bigcup_{j \in \mathbb{N}} K_j) \times X$ then there exists $\widetilde{j} = \min\{j \in \mathbb{N} : t \in K_j\}$ such that $f(t,x) = f_{\widetilde{j}}(t,x) \in F(t,x)$. Being

$$\mu(J \setminus \bigcup_{j \in \mathbb{N}} K_j) \leq \mu(J \setminus K_j) < 2^{-j}, \quad j \in \mathbb{N},$$

we can deduce that the set $J \setminus \bigcup_{j \in \mathbb{N}} K_j$ has measure zero. Therefore we can conclude that f is a Carathéodory selection of F. □

Now, we are able to prove the existence of mild solutions for the following boundary value problem governed by a semilinear differential inclusion

$$(\mathcal{P}) \qquad \begin{cases} x' \in A(t)x + F(t,x) \\ Lx = \omega \end{cases}$$

where $\omega \in X$, $L : C(J,X) \to X$ is an operator and the family $\{A(t)\}_{t\in J}$ satisfies the assumption

$H(A)$: $\{A(t)\}_{t\in J}$ is a family of linear operators $A(t) : \mathcal{D}(A) \subseteq X \to X$, with $\mathcal{D}(A)$ not depending on t and dense in X, generating an evolution system $\{T(t,s)\}_{(t,s)\in\Delta}$ such that the operator $T(t,s)$ is compact, for $t - s > 0$.

We recall that a function $x \in C(J,X)$ is said to be a *mild solution* for $(\mathcal{P})$ if

$$\begin{aligned} x(t) &= T(t,0)x(0) + \int_0^t T(t,s)f(s)\,ds,\ t \in J, \text{ where } \ f \in S^1_{F(\cdot,x(\cdot))}; \\ Lx(t) &= \omega,\ t \in J. \end{aligned}$$

At first, we prove the existence of mild solutions for $(\mathcal{P})$ when F has the *lower Scorza-Dragoni property.*

Theorem 3.1. *Let X be a separable Banach space whose dual X^* is separable. We assume that $H(A)$ is satisfied and we suppose the following hypotheses on the data*

$H(F)$: *$F : J \times X \to \mathcal{P}_{fc}(X)$ is a multifunction such that*
- (1) *$(t,x) \to F(t,x)$ has the lower Scorza-Dragoni property;*
- (2) *there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$, where $\varphi_k \in L^p_+(J)$, $1 \le p \le \infty$, $\sup_{\|x\|\le k} \|F(t,x)\| \le \varphi_k(t)$, a.e. on J, and such that $\underline{\lim}_{k\to\infty} \frac{1}{k}\int_0^b \varphi_k(t)\,dt = \beta < \infty$.*

$H(L)$: *$L : C(J,X) \to X$ is a linear, continuous operator satisfying:*
- H_0: *the operator $\widehat{L} \in \mathcal{L}(X)$ defined by $\widehat{L}(x) = L(T(\cdot,0)x)$, $x \in X$, is a bijection;*
- H_1: *$(\mathcal{M}\|\widehat{L}^{-1}\|_{\mathcal{L}}\|L\|_{\mathcal{L}} + 1)\ \mathcal{M}\beta < 1$, where $\mathcal{M}$ and β are as in (2.1) and $H(F)(2)$, respectively.*

Then $(\mathcal{P})$ admits at least one mild solution.

Proof: From Proposition 3.1 the multifunction $F : J \times X \to \mathcal{P}_{fc}(X)$ has a Carathéodory selection $f : J \times X \to X$. Let us consider the multifunction $G : J \times X \to \mathcal{P}_{kc}(X)$ defined as

$$G(t,x) = \{f(t,x)\}, \quad (t,x) \in J \times X. \tag{3.1}$$

Now, for each $x \in X$, because $f(\cdot,x)$ is measurable we can say that the multifunction $G(\cdot,x)$ is measurable. Moreover, for each $t \in J$, $f(t,\cdot)$ is continuous and so it is easy to check that $G(t,\cdot)$ is H-continuous. Then G is a Carathéodory multifunction with compact and convex values in the separable Banach space X. Hence, from Proposition 7.16 of [8], we get that G satisfies the Scorza-Dragoni property.

Moreover, taking into account that f is a selection of F, by using $H(F)(2)$ we have that the mentioned sequence $(\varphi_k)_k$ is such that $\sup_{\|x\|\le k} \|G(t,x)\| \le \varphi_k(t)$, a.e. $t \in J$, $k \in \mathbb{N}$. Now, since X^* has the Radon-Nikodym property (see [8, Proposition A.3.97]), from Theorem 1 of [17] we have that there exists at least one mild solution $\widetilde{x} \in C(J, X)$ for the following problem:

$$(\mathcal{P})_{\text{ext}} \qquad \begin{cases} x' \in A(t)x + \text{ext}\ G(t,x) \\ Lx = \omega \end{cases}$$

where ext $G(t,x)$ denotes the set of extreme points of $G(t,x)$. Since ext $G(t,x) = \{f(t,x)\} \subset F(t,x)$, we can say that $\widetilde{x}$ is also a mild solution for $(\mathcal{P})$. □

Remark 3.1. Let us remark that Theorem 3.1 extends in a broad sense Theorems 3.1 and 3.2 proved in [18]. First of all we observe that $(\mathcal{P})$ (in which $\omega \in X$ is fixed) can be written as the following problem

$$(\widetilde{\mathcal{P}}) \qquad \begin{cases} x' \in A(t)x + F(t,x) \\ Lx = Mx \end{cases}$$

where $M : C(J,X) \to X$ is the operator defined by $Mx = \omega$, $x \in C(J,X)$.

The following multifunction $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{fc}(\mathbb{R})$

$$F(t,x) = \begin{cases} \{0\}, & x \in \mathbb{N}_0 \\ [0,n], & x \in \,]n-1, n[\, \cup \,]-n, 1-n[\,,\ n \in \mathbb{N} \end{cases}$$

satisfies all hypotheses of Theorem 3.1 but not the property $H(F)(3)$ of Theorem 3.1 of [18] or $H(F)_1(3)$ of Theorem 3.2 of [18]. Indeed, any sequence $(\varphi_k)_{k\in\mathbb{N}}, \varphi_k \in L^1_+([0,1])$, with the property

$$\sup_{\|x\|\le k} \|F(t,x)\| = \max_{n\le k}\{\|[0,n]\|\} = k \le \varphi_k(t), \text{ a.e. on } [0,1]$$

it is such that $\underline{\lim}_{k\to\infty} \frac{1}{k}\int_0^1 \varphi_k(t)\,dt \neq 0$.

Now we obtain an existence result of mild solutions for the more general problem $(\widetilde{\mathcal{P}})$.

Theorem 3.2. *Let X be a separable Banach space. We assume that $H(A)$, $H(L)$ and H_0 are satisfied and we suppose the following hypotheses on the data*

$H(F)_1$: *$F : J \times X \to \mathcal{P}_{fc}(X)$ is a multifunction such that*
- (1) *$(t,x) \to F(t,x)$ has the lower Scorza-Dragoni property;*
- (2) *there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$, where $\varphi_k \in L^1_+(J)$, $\sup_{\|x\|\le k}\|F(t,x)\| \le \varphi_k(t)$, a.e. on J, and such that $\underline{\lim}_{k\to\infty} \frac{1}{k}\int_0^b \varphi_k(t)\,dt = 0$.*

$H(M)$: $M : C(J,X) \to X$ is a compact operator such that
$\lim_{\|u\|\to\infty} \frac{\|M(u)\|}{\|u\|} = 0$.

Then $(\widetilde{\mathcal{P}})$ admits at least one mild solution, i.e. a function $x \in C(J,X)$ verifying $x(t) = T(t,0)x(0) + \int_0^t T(t,s)f(s)\,ds$, $t \in J$, where $f \in S^1_{F(\cdot,x(\cdot))}$, and such that $Lx(t) = Mx(t)$, $t \in J$.

Proof: First we observe that, by proceeding as in Theorem 3.1, we can consider the multifunction $G : J \times X \to \mathcal{P}_{kc}(X)$ defined as in (3.1). From Proposition 1.6 of [8] we can deduce that f is measurable and so we have that G is scalarly measurable (see [8, Proposition 2.39, p. 166]). Moreover, for every $t \in J$, the multifunction $G(t,\cdot) : X \to \mathcal{P}_{kc}(X)$ is H-continuous, so we can say that it is upper semicontinuous from X into X_w, where X_w denotes the Banach space X endowed with the weak topology.

Next, by using the fact that f is a selection of F, the condition $H(F)_1(2)$ implies that the mentioned sequence $(\varphi_k)_{k\in\mathbb{N}}$ is such that $\sup_{\|x\|\le k} \|G(t,x)\| \le \varphi_k(t)$, a.e. $t \in J$, $k \in \mathbb{N}$.

Now we are in position to apply Theorem 3.1 of [18] and so we can say that there exists at least one mild solution $\widetilde{x} \in C(J,X)$ for problem

$$(\mathcal{P})_G \qquad \begin{cases} x' \in A(t)x + G(t,x) \\ Lx = Mx \end{cases}$$

Recalling the definition of G, we can conclude that $\widetilde{x}$ is also a mild solution for $(\widetilde{\mathcal{P}})$. □

Remark 3.2. We note that our Theorem 3.2 extends in a broad sense Theorem 3.1 of [18]. Indeed there exist multifunctions verifying the hypotheses of Theorem 3.2, but not all conditions required in Theorem 3.1 of [18]. For example, we can consider the multifunction $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{kc}(\mathbb{R})$ defined as follows

$$F(t,x) = \begin{cases} \{0\}, & x \in \mathbb{N}_0 \\ [0, \frac{1}{k}], & x \in \,]k-1, k[\, \cup \,]-k, 1-k[\,,\ k \in \mathbb{N}. \end{cases}$$

Remark 3.3. Let us note that if we restrict Theorem 3.2 of [18] to the class of multifunctions $F : J \times X \to \mathcal{P}_{fc}(X)$ we can say that our Theorem 3.2 improves Theorem 3.2 of [18]. Indeed by Theorem 3.5 of [7] we have that $\mathcal{B}(J \times X) \times \mathcal{B}(X)$-graph measurability coincides with measurability so, by using Theorem 2.1 of [3], the hypotheses on F required in Theorem 3.2 of [18] imply the assumptions of Theorem 3.2. On the other hand, there exist multifunctions satisfying the hypotheses of our Theorem 3.2 but not all hypotheses of Theorem 3.2 of [18], as the following example proves:

Example 3.1. Put $C \subset [0,1]$ the Cantor set such that $\mu(C) = 0$, where μ is the Lebesgue measure in $\mathbb{R}$, let $f : [0,1] \to [0,2]$ be a function so defined

$$f(x) = f_C(x) + x, \quad x \in [0,1]$$

where f_C is the Vitali-Cantor function. It is easy to see that f is an homomorphism and that $\mu(f(C) \setminus \{0\}) = 1$. Fixed H a non Lebesgue measurable subset of $f(C) \setminus \{0\}$, we consider $M = f^{-1}(H)$. Now we are in position to define the multifunction $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{fc}(\mathbb{R})$ where

$$F(t,x) = \begin{cases} [0,2], & t \in M, \\ [0,1], & (t,x) = (0,0), \\ \{0\}, & \text{otherwise.} \end{cases}$$

First of all we observe that, for each $\varepsilon > 0$, by using the regularity of Lebesgue measure there exists a compact subset $C_\varepsilon \subset [0,1] \setminus C$ such that $\mu([0,1] \setminus C_\varepsilon) < \frac{\varepsilon}{2}$. Now, the closed set $J_\varepsilon = C_\varepsilon \cap [\frac{\varepsilon}{2}, 1]$ is such that $\mu([0,1] \setminus J_\varepsilon) < \varepsilon$ and $F_{|J_\varepsilon \times \mathbb{R}}$ is s.c.i.. Moreover it is easy to check that F has also the other properties of our Theorem 3.2.

On the other hand, F does not verify all the assumptions of Theorem 3.2 of [18]. In fact, taking into account that M is not a Borel measurable set, the multifunction F is not $\mathcal{B}([0,1] \times [0,1]) \times \mathcal{B}(\mathbb{R})$-graph measurable.

Remark 3.4. Obviously, taking into account that in Theorem 3.1 we also require the separability of the dual space X^*, there exist multifunctions that satisfy the properties of Theorem 3.2 but not all the hypotheses of Theorem 3.1 (for example if Ω is a separable measurable space, $X = L^1(\Omega)$ is a separable Banach space whose dual $X^* = L^\infty(\Omega)$ is not separable (cf. [4, p. 98])). Moreover there exist multifunctions that verify the hypotheses of Theorem 3.1 but not the hypotheses of Theorem 3.2. To justify this assertion we can consider the multifunction of Remark 3.1.

4. Periodic solutions

In this section we use the results of Section 3 to establish the existence of periodic mild solutions for the following boundary value problem

$$(\mathcal{PP}) \qquad \begin{cases} x' \in A(t)x + F(t,x) \\ x(0) = x(b) \end{cases}$$

We need the following stronger hypothesis on $\{A(t)\}_{t \in J}$

$H(A)_p$: $\{A(t)\}_{t \in J}$ is a family of linear operators, $A(t) : \mathcal{D}(A) \subseteq X \to X$, with $\mathcal{D}(A)$ not depending on t and dense in X, generating an evolution system $\{T(t,s)\}_{(t,s) \in \Delta}$ such that the operator $T(t,s)$ is compact, for $t - s > 0$, and

$$T(b,0)x = x \;\Leftrightarrow\; x = 0. \tag{4.1}$$

From condition (4.1) we deduce that $\mathrm{Ker}(T(b,0) - I) = \{0\}$ (where I is the identity operator). By invoking Fredholm's alternative Theorem (see [8, Theorem A.3.125]), (4.1) implies that $R(T(b,0) - I) = X$. Therefore the operator $T(b,0) - I$ is a bijection. Being $T(b,0) - I \in \mathcal{L}(X)$, by Banach's Theorem (see [10, Theorem 2, p. 229]) we can deduce $(T(b,0) - I)^{-1} \in \mathcal{L}(X)$ and so there exists a constant $\widetilde{\mathcal{K}} > 0$ such that

$$\|(T(b,0) - I)^{-1}\|_{\mathcal{L}} \leq \widetilde{\mathcal{K}}. \tag{4.2}$$

Now, we can prove

Theorem 4.1. *Let X be a separable Banach space whose dual X^* is separable. We suppose that $H(A)_p$ is satisfied, the multifunction F verifies $H(F)$ of Theorem 3.1 and*

H_1': $(2\mathcal{M}\widetilde{\mathcal{K}}+1)\mathcal{M}\beta < 1$, where $\mathcal{M}, \widetilde{\mathcal{K}}$ and β are as in (2.1), (4.2) and $H(F)(2)$, respectively.

Then $(\mathcal{PP})$ admits at least one mild periodic solution, i.e. a function $x \in C(J,X)$ verifying $x(t) = T(t,0)x(0) + \int_0^t T(t,s)f(s)\,ds$, $t \in J$, where $f \in S^1_{F(\cdot,x(\cdot))}$, and such that $x(0) = x(b)$.

Proof: Let us consider the operator $L : C(J,X) \to X$ defined by $Lx = x(b) - x(0)$. It is easy to check that L is a continuous and linear operator. Moreover, let $\widehat{L} : X \to X$ be the operator $\widehat{L}x = L(T(\cdot,0)x)$, $x \in X$. By recalling the definition of L and $H(A)_p$ we clearly have

$$\widehat{L} = T(b,0) - I \in \mathcal{L}(X) \tag{4.3}$$

and there exists $\widehat{L}^{-1} \in \mathcal{L}(X)$, so L satisfies H_0 of Theorem 3.1. On the other hand, thanks to the definition of L we can write

$$\|L\|_{\mathcal{L}} = \sup_{\|x\|_C \leq 1} \|x(b) - x(0)\| \leq 2$$

and, by using H_1' and (4.2), we have

$$(\mathcal{M}\|\widehat{L}^{-1}\|_{\mathcal{L}}\ \|L\|_{\mathcal{L}} + 1)\ \mathcal{M}\beta < 1.$$

Then, H_1 of Theorem 3.1 is also verified. By using operator L, fixed $\omega = 0$, the problem $(\mathcal{PP})$ can be rewritten as $(\mathcal{P})$. So, by applying Theorem 3.1 we can conclude that $(\mathcal{PP})$ has at least one mild periodic solution. □

Now we prove the existence of mild periodic solutions in the case that X^* is not necessarily separable.

Theorem 4.2. *Let X be a separable Banach space. If $H(A)_p$ is satisfied and the multifunction F verifies $H(F)_1$ of Theorem 3.2, then $(\mathcal{PP})$ admits at least one mild periodic solution.*

Proof: As in the previous theorem we can say that the operator $L : C(J,X) \to X$ defined by $Lx = x(b) - x(0)$ satisfies $H(L)$ and H_0. Now, let us consider $M : C(J,X) \to X$ defined as $Mx = 0$, $x \in C(J,X)$. The operator M is clearly compact and such that $\lim_{\|u\|\to\infty} \frac{\|M(u)\|}{\|u\|} = 0$. Therefore M verifies hypothesis $H(M)$ of Theorem 3.2.

Now, by using the operators L and M, the problem $(\mathcal{PP})$ can be rewritten as $(\widetilde{\mathcal{P}})$. So by applying Theorem 3.2 we can conclude the existence of at least one mild periodic solution for $(\mathcal{PP})$. $\square$

Remark 4.1. Thanks to the examples presented in Remarks 3.1, 3.2 and 3.3 we can say that our periodic results extend in a broad sense the analogous theorems proved in [18].

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(*Received* July 18, 2010, *revised* July 18, 2010)