On weakly s-permutably embedded subgroups

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Abstract. Suppose G is a finite group and H is a subgroup of G. H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G; H is called weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$. We investigate the influence of weakly s-permutably embedded subgroups on the p-nilpotency and p-supersolvability of finite groups.

Keywords: weakly s-permutably embedded subgroups, p-nilpotent, n-maximal subgroup

Classification: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be s-permutable (or s-quasinormal) [1] in G if H permutes with every Sylow subgroup of G. From Ballester-Bolinches and Pedraza-Aguilera [2], we know H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. In recent years, it has been of interest to use supplementation properties of subgroups to characterize properties of a group. For example, Wang [3] introduced the concept of c-normal subgroup. A subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. In 2007, Skiba [5] introduced the concept of weakly s-permutable subgroup. A subgroup H of a group G is said to be weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap K \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H. As a generalization of above subgroups, Y. Li, S. Qiao and Y. Wang [7] introduced a new subgroup embedding property in a finite group called weakly s-permutably embedded subgroup. In the present paper we characterize p-nilpotency of finite groups with the assumption that some n-maximal subgroups are weakly s-permutably embedded.

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2. Preliminaries

Definition 2.1. A subgroup H of a group G is said to be weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$.

Remark. Obviously, s-permutably embedding property implies weakly s-permutably embedding property. The converse does not hold in general. For example, suppose $G = S_4$, the symmetric group of degree 4. Take $H = <(34)>$. Then H is weakly s-permutably embedded in G, but not s-permutably embedded in G.

Lemma 2.2 ([7, Lemma 2.5]). *Let H be a weakly s-permutably embedded subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly s-permutably embedded in L.*
(2) *If $N \lhd G$ and $N \leq H \leq G$, then H/N is weakly s-permutably embedded in G/N.*
(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutably embedded in G/N.*

Lemma 2.3. *Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geq 1$. If $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$, then G is p-nilpotent.*

Proof: Suppose that the statement is not true and let G be a counterexample of minimal order. Obviously, every subgroup of G satisfies the hypothesis of the Lemma. The minimal choice of G implies that G is a minimal non-p-nilpotent group. By [11, III, 5.2 and IV, 5.4], $G = P \rtimes Q$ is a subdirect product of two Sylow subgroups. It is easy to see that every proper quotient group of G satisfies the hypothesis. Thus $\Phi(P) = \Phi(G) = 1$ and so P is an elementary abelian p-group. Since $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$ and $|\mathrm{Aut}(P)|$ divides $(p-1)(p^2-1)\dots(p^n-1)$ for $|P| \leq p^n$, we have $N_G(P)/C_G(P) = 1$. This induces that G is p-nilpotent by [6, Theorem 10.1.8]. The contradiction completes the proof. □

Lemma 2.4 ([8, A, 1.2]). *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U \cap VW = (U \cap V)(U \cap W)$;
(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.5 ([9, Lemma 2.3]). *Suppose that H is s-permutable in G, P a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

Lemma 2.6 ([9, Lemma 2.4]). *Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Lemma 2.7 ([18, Lemma A]). *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.8 ([4, Lemma 2.8]). *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

3. Main results

Theorem 3.1. *Let G be a group and p a prime such that $(|G|,(p-1)(p^2-1)\dots(p^n-1))=1$ for some integer $n\geq 1$. If there exists a Sylow p-subgroup P of G such that every n-maximal subgroup (if exists) of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Proof: Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G is not a non-abelian simple group.

By Lemma 2.3, $p^n||P|$ and so there exists a non-identity n-maximal subgroup P_n of P. By the hypothesis, P_n is weakly s-permutably embedded in G. Then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G=P_nT$ and $P_n\cap T\leq (P_n)_{se}$. If G is simple, then $T=G$ and so $P_n=(P_n)_{se}$ is s-permutably embedded in G. Thus there is an s-permutable subgroup K of G such that P_n is a Sylow p-subgroup of K. Since G is simple, we have $K_G=1$. By Lemma 2.5, P_n is s-permutable in G. Therefore $N_G(P_n)\geq O^p(G)=G$ by Lemma 2.7. It follows that $P_n\lhd G$, a contradiction.

(2) G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover $\Phi(G)=1$.

Let N be a minimal normal subgroup of G. Consider G/N. We will show G/N satisfies the hypothesis of the theorem. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. If $|PN/N|\leq p^n$, then G/N is p-nilpotent by Lemma 2.3. So we may suppose $|PN/N|\geq p^{n+1}$. Let M_n/N be an n-maximal subgroup of PN/N. Then $M_n=N(M_n\cap P)$. Let $P_n=M_n\cap P$. It follows that $P_n\cap N=M_n\cap P\cap N=P\cap N$ is a Sylow p-subgroup of N. Since

$$p^n=|PN/N:M_n/N|=|PN:(M_n\cap P)N|=|P:M_n\cap P|=|P:P_n|,$$

P_n is an n-maximal subgroup of P. By the hypothesis, P_n is weakly s-permutably embedded in G, thus there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G=P_nT$ and $P_n\cap T\leq (P_n)_{se}$. So $G/N=M/N\cdot TN/N=P_nN/N\cdot TN/N$. Since $(|N:P_n\cap N|,|N:T\cap N|)=1$, $(P_n\cap N)(T\cap N)=N=N\cap G=N\cap(P_nT)$. By Lemma 2.6, $(P_nN)\cap(TN)=(P_n\cap T)N$. It follows that $(P_nN/N)\cap(TN/N)=(P_nN\cap TN)/N=(P_n\cap T)N/N\leq (P_n)_{se}N/N$. Since $(P_n)_{se}N/N$ is s-permutably embedded in G/N by [2, Lemma 2.1], we have that M_n/N is weakly s-permutably embedded in G. Therefore G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G)=1$ are obvious.

(3) $O_{p'}(G)=1$.

If $O_{p'}(G)\neq 1$, then $N\leq O_{p'}(G)$ by step (2). Since

$$G/O_{p'}(G)\cong (G/N)/(O_{p'}(G)/N)$$

is p-nilpotent, we have G is p-nilpotent, a contradiction.

(4) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (2) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Since $P \cap M < P$, there is a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Take an n-maximal subgroup P_n of P such that $P_n \leq P_1$. By the hypothesis, there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G = P_nT$ and $P_n \cap T \leq (P_n)_{se}$. So there is an s-permutable subgroup K of G such that $(P_n)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq (P_n)_{se} \leq P_1$, and so $P = N(P \cap M) = NP_1 = P_1$, a contradiction. If $K_G = 1$, by Lemma 2.7, $(P_n)_{se}$ is s-permutable in G. From Lemma 2.7 we have $O^p(G) \leq N_G((P_n)_{se})$. Since $(P_n)_{se}$ is subnormal in G, $P_n \cap T \leq (P_n)_{se} \leq O_p(G) = N$ by [12, Corollary 1.10.17]. Thus, $(P_n)_{se} \leq P_1 \cap N$ and

$$(P_n)_{se} \leq ((P_n)_{se})^G = ((P_n)_{se})^{O^p(G)P} = ((P_n)_{se})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N.$$

It follows that $((P_n)_{se})^G = 1$ or $((P_n)_{se})^G = P_1 \cap N = N$. If $((P_n)_{se})^G = 1$, then $P_n \cap T = 1$ and so $|T|_p = p^n$. Hence T is p-nilpotent by Lemma 2.3. Since $T \lhd \lhd G$, we have G is p-nilpotent, a contradiction. If $((P_n)_{se})^G = P_1 \cap N = N$, then $N \leq P_1$ and so $P = P_1$, a contradiction.

(5) The final contradiction.

If $N \cap P \leq \Phi(P)$, then N is p-nilpotent by J. Tate's theorem ([11, IV, 4.7]). Hence, by $N_{p'}$ char $N \lhd G$, $N_{p'} \leq O_{p'}(G) = 1$. It follows that N is a p-group. Then $N \leq O_p(G) = 1$, a contradiction. Consequently, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. We take an n-maximal subgroup P_n of P such that $P_n \leq P_1$. By the hypothesis, P_n is weakly s-permutably embedded in G. Then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_n)_{se}$ of G contained in P_n such that $G = P_nT$ and $P_n \cap T \leq (P_n)_{se}$. So there is an s-permutable subgroup K of G such that $(P_n)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_n)_{se} \cap N$ is a Sylow p-subgroup of N. We know that $(P_n)_{se} \cap N \leq P_n \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p-subgroup of N, so $(P_n)_{se} \cap N = P_n \cap N = P \cap N$. Consequently, $P = (N \cap P)P_1 = (P_n \cap N)P_1 = P_1$, a contradiction. Therefore $K_G = 1$. By Lemma 2.5, $(P_n)_{se}$ is s-permutable in G and so $(P_n)_{se} \lhd \lhd G$. Hence $P_n \cap T \leq (P_n)_{se} \leq O_p(G) = 1$. Since $|T|_p = p^n$, T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a normal Hall p'-subgroups of G, a contradiction. □

Theorem 3.2. *Let p be a prime and $\mathcal{F}$ a saturated formation containing all p-nilpotent groups. Suppose that G is a group with $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$ for some integer $n \geq 1$. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup*

E such that $G/E \in \mathfrak{F}$ and E has a Sylow p-subgroup such that every n-maximal subgroup (if exists) of P is weakly s-permutably embedded in G.

Proof: The necessity is obvious. We need only to prove the sufficiency. Suppose that the assertion is not true and let G be a counterexample of minimal order. By Lemma 2.1, every n-maximal subgroup of P is weakly s-permutably embedded in E. Hence by Theorem 3.1, E is p-nilpotent. Obviously $E \neq G$. Let T be a normal Hall p'-subgroup of E. Now we divide the proof into the following steps:

(1) $T = 1$, and so $P = E \lhd G$.

Assume that $T \neq 1$. Because T is a normal Hall p'-subgroup of E and $E \lhd G$, $T \lhd G$. We claim that G/T (with respect to E/T) satisfies the hypothesis. In fact, $(G/T)/(E/T) \cong G/E \in \mathfrak{F}$ and E/T is a p-group. Suppose that M_n/T is an n-maximal subgroup of PT/T and $P_n = M_n \cap P$. Then P_n is an n-maximal subgroup of P and $M_n = P_nT$. By the hypothesis, P_n is weakly s-permutably embedded in G. By Lemma 2.1, $M_n/T = P_nT/T$ is weakly s-permutably embedded in G/T. The minimal choice of G implies that $G/T \in \mathfrak{F}$. It is easy to see that $G \in \mathfrak{F}$ from [8, Proposition IV. 3.11], a contradiction. Hence $T = 1$ and $P = E \trianglelefteq G$.

(2) Suppose that Q is a Sylow q-subgroup of G, where q is a prime divisor of $|G|$ and $q \neq p$. Then $PQ = P \times Q$.

By (1), $P = E \trianglelefteq G$. So PQ is a subgroup of G. By Lemma 2.1, every n-maximal subgroup of P is weakly s-permutably embedded in PQ. Hence by Theorem 3.1, we have that PQ is p-nilpotent. It follows that $Q \trianglelefteq PQ$ and so $PQ = P \times Q$.

(3) Final contradiction.

Let H be an arbitrary non-identity normal subgroup of G contained in P and G_p a Sylow p-subgroup of G. By (2), we have $HQ = H \times Q$ for any Sylow q-subgroup of G with $q \neq p$. This induces that $O^p(G) \leq C_G(H)$ and $[H, G] = [H, G_pO^p(G)] = [H, G_p] \trianglelefteq G$. We claim that $[H, G_p] < H$. Indeed, if $[H, G_p] = H$, then for any non-negative integer t, $H = [H, G_p, \ldots, G_p] \leq G_p^{t+1}$, where the number of G_p in $[H, G_p, \ldots, G_p]$ is t, which contradicts [8, Theorem A.10.3]. Thus $[H, G_p] < H$ and consequently there exists a normal subgroup K of G such that H/K is a chief factor of G and $[H, K] \leq K$. This implies that $H/K \leq Z(G/K)$. Then we obtain that $G \in \mathfrak{F}$ since $G/P \in \mathfrak{F}$. The final contradiction completes the proof. □

Corollary 3.3. *Let p be the smallest prime dividing the order of a group G. Assume that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Corollary 3.4 ([16, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Assume that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c^*-normal in G, then G is p-nilpotent.*

Corollary 3.5 ([9, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is s-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 3.6 ([7, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Corollary 3.7 ([19, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-permutable in G, then G is p-nilpotent.*

Corollary 3.8 ([20, Theorem 3.2]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-permutable in G, then G is p-nilpotent.*

Corollary 3.9 ([10, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is s-permutably embedded in G, then G is p-nilpotent.*

Corollary 3.10 ([13, Theorem 3.4]). *Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 3.11 ([17, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c-normal or s-permutably embedded in G, then G is p-nilpotent.*

Theorem 3.12. *Let p be a prime, G a p-solvable group and H a normal subgroup of G such that G/H is p-supersolvable. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is weakly s-permutably embedded in G, then G is p-supersolvable.*

Proof: Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) G has a unique minimal normal subgroup N contained in H such that G/N is p-supersolvable.

Let N be a minimal normal subgroup of G contained in H. Since P is the Sylow p-subgroup of H, PN/N is the Sylow p-subgroup of H/N. Let M/N be a maximal subgroup of PN/N; then $M = (M \cap P)N$. Let $P_1 = M \cap P$. Obviously, P_1 is the maximal subgroup of P. Since G is p-solvable, N is elementary abelian p-group or p'-group. If N is p'-group, then $M/N = P_1N/N$. If N is p-group, then $M/N = P_1/N$. By hypothesis, P_1 is weakly s-permutably embedded in G and so M/N is weakly s-permutably embedded in G/N by Lemma 2.1. Since $(G/N)/(H/N) \cong G/H$ is p-supersolvable, G/N satisfies all the hypotheses of our theorem. It follows that G/N is p-supersolvable by the minimality of G. Clearly,

N is the unique minimal normal subgroup of G contained in H as the class of p-supersolvable group is a saturated formation.

(2) $O_{p'}(G) = 1$.

If $T = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/T$. Clearly, $\overline{G}/\overline{H} \cong G/HT$ is p-supersolvable by the p-supersolvability of G/H, where $\overline{H} = HT/T$. Let $\overline{P_1} = P_1T/T$ be a maximal subgroup of PT/T. We may assume that P_1 is a maximal subgroup of P. Since P_1 is weakly s-permutably embedded in G, the subgroup P_1T/T is weakly s-permutably embedded in G/T by Lemma 2.1. The minimality of G yields that $\overline{G}$ is p-supersolvable, and so G is also p-supersolvable, a contradiction.

(3) The final contradiction.

Since G is p-solvable, N is an elementary abelian p-group by step (2). If N is contained in all maximal subgroups of G, then $N \leq \Phi(G)$ and so G is p-supersolvable, a contradiction. Hence there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Applying Lemma 2.8, we have $O_p(H) \cap M \lhd G$. Therefore $O_p(H) \cap M = 1$ and $N = O_p(H)$. Let G_p be a Sylow p-subgroup of G containing P. Then $G_p = P(G_p \cap M)$ and $G_p \cap M < G_p$. Take a maximal subgroup G_1 of G containing $G_p \cap M$ and set $P_1 = G_1 \cap P$. Then $G_p \cap M = G_1 \cap M$ and $G_1 = P_1(G_p \cap M)$. Moreover, P_1 is maximal in P. By the hypothesis, P_1 is weakly s-permutably embedded in G. Then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. So there is an s-permutable subgroup K of G such that $(P_1)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then we can take a minimal normal subgroup N_1 of G such that $N_1 \leq K_G$. Since G is p-solvable, from (2), N_1 must be a p-subgroup, so that $N_1 \leq (P_1)_{se} \leq P \leq H$ and indeed $N_1 = N$ by step (1). Furthermore, $G_p = N(G_p \cap M) \leq P_1(G_p \cap M) = G_1$, a contradiction. Therefore $K_G = 1$ and, by Lemma 2.5, $(P_1)_{se}$ is s-permutable in G. By [12, Corollary 1.10.17], $P_1 \cap T \leq (P_1)_{se} \leq N$. Since $|G : T|$ is a number of p-power and $T \lhd \lhd G$, $O^p(G) \leq T$. We know $G/O^p(G)$ is p-subgroup, so $G/O^p(G)$ is p-supersolvable and $G/(N \cap O^p(G)) \lesssim G/N \times G/O^p(G)$ is p-supersolvable. Then $N \cap O^p(G) \neq 1$. Since N is the minimal subgroup, $N \cap O^p(G) = N$ and $N \leq O^p(G)$. It follows that $N \leq T$. Thus we have $P_1 \cap T = P_1 \cap N = (P_1)_{se}$ is s-permutable in G. Since $G_1 = P_1(G_p \cap M)$ and $P_1 = (P_1 \cap N)(P \cap M)$, we have $G_1 = (P_1 \cap N)(G_p \cap M)$. Now let Q be a Sylow q-subgroup of M with $q \neq p$. Then Q is also a Sylow q-subgroup of G, and hence $(P_1 \cap N)Q = Q(P_1 \cap N)$. Since $G_p \cap M$ is a Sylow p-subgroup of M, the set $(P_1 \cap N)M$ forms a group. The maximality of M implies that either $(P_1 \cap N)M = G$ or $(P_1 \cap N)M = M$. If the former holds, then $G_p = G_1(G_p \cap M) = G_1$, a contradiction. Thus we must have $(P_1 \cap N)M = M$, that is, $P_1 \cap N \leq M$. It follows that $P_1 \cap N = 1$. Since $P_1 \cap N$ is a maximal subgroup of N, we have N is a cyclic of order p. Thus G is p-supersolvable, a final contradiction. □

Corollary 3.13 ([16, Theorem 3.5]). *Let p be a prime, G a p-solvable group and H a normal subgroup of G such that G/H is p-supersolvable. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c^*-normal in G, then G is p-supersolvable.*

Corollary 3.14 ([14, Theorem 3.1]). *Let p be a prime, G a p-solvable group and H a normal subgroup of G such that G/H is p-supersolvable. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c-normal in G, then G is p-supersolvable.*

Corollary 3.15 ([15, Theorem 3.10]). *Let p be a prime, G a p-solvable group and H a normal subgroup of G such that G/H is p-supersolvable. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is s-permutably embedded in G, then G is p-supersolvable.*

Corollary 3.16 ([20, Theorem 3.3]). *Let p be a prime and G a p-solvable group. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is s-permutable in G, then G is p-supersolvable.*

Corollary 3.17. *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is weakly s-permutably embedded in G, then G is supersolvable.*

Proof: Let p is the smallest prime divisor of $|G|$. The supersolvability of G/H implies that G/H is p-nilpotent. By Corollary 3.3, G is p-nilpotent. Furthermore G is solvable. Applying Theorem 3.12, it is easy to see that G is supersolvable. □

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