Isolated points and redundancy

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Abstract. We describe the isolated points of an arbitrary topological space (X,τ). If the τ-specialization pre-order on X has enough maximal elements, then a point $x \in X$ is an isolated point in (X,τ) if and only if x is both an isolated point in the subspaces of τ-kerneled points of X and in the τ-closure of $\{x\}$ (a special case of this result is proved in Mehrvarz A.A., Samei K., *On commutative Gelfand rings*, J. Sci. Islam. Repub. Iran **10** (1999), no. 3, 193–196). This result is applied to an arbitrary subspace of the prime spectrum $\mathrm{Spec}(R)$ of a (commutative with nonzero identity) ring R, and in particular, to the space $\mathrm{Spec}(R)$ and the maximal and minimal spectrum of R. Dually, a prime ideal P of R is an isolated point in its Zariski-kernel if and only if P is a minimal prime ideal. Finally, some aspects about the redundancy of (maximal) prime ideals in the (Jacobson) prime radical of a ring are considered, and we characterize when $\mathrm{Spec}(R)$ is a scattered space.

Keywords: maximal (minimal) spectrum of a ring, scattered space, isolated point, prime radical, Jacobson radical

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Introduction

In Section 1 we include some preliminaries. In Section 2 we describe the isolated points of an arbitrary topological space (Theorem 2.1). In particular, we describe the isolated points in a topological space (X,τ) such that the pre-ordered set $(X,\leq_\tau)$ has enough maximal elements, where $\leq_\tau$ is the τ-specialization pre-order on X (Theorem 2.2), and we apply this result to the prime spectrum of a ring (Corollary 2.1). In Section 3 we characterize the isolated points in an arbitrary subspace of the prime spectrum $\mathrm{Spec}(R)$ of a ring R (Theorem 3.1) and we apply this to the maximal and minimal spectrum of R (Theorems 3.4–3.5). Also, using these results, we characterize when each of these subspaces is a discrete space (Corollaries 3.2–3.3). Further, we characterize the points which are isolated points in its kernel (Theorem 3.6), as well as when $\mathrm{Spec}(R)$ is a scattered space (Corollary 3.5).

1. Preliminaries

We denote by $\mathbb{N} := \{0,1,2,\dots\}$ the set of natural numbers, a set X with a topology τ will be denoted by (X,τ) and we assume no separation axioms, thus a point p is *isolated* if it is simply an open point. For every subset Y of X, we denote by $\tau|_Y$ the subspace topology on Y, by $\overline{Y}^{\tau}$ the τ-closure of Y, by $\widehat{Y}^{\tau}$ the

τ-*kernel* of Y (the intersection of the τ-open subsets of X containing Y), and Y is said to be τ-*kerneled* if $Y = \widehat{Y}^{\tau}$. Also, the τ-*saturation* of Y is the set $\bigcup_{y\in Y}\overline{y}^{\tau}$, and we say Y is τ-*saturated* if it coincides with its τ-saturation. In particular, $\overline{x}^{\tau} := \overline{\{x\}}^{\tau}$ and $\widehat{x}^{\tau} := \widehat{\{x\}}^{\tau}$ for every $x \in X$.

Let R be a ring. We set $I \leq R$ to indicate that I is an ideal of R and we denote by $\mathrm{Spec}(R)$ (resp. $\mathrm{Max}(R)$, $\mathrm{Min}(R)$) the family of prime (resp. maximal, minimal prime) ideals of R. Recall that every proper ideal is contained in a maximal ideal and every prime ideal contains a minimal prime ideal ([2]). We set $J(R) := \bigcap \mathrm{Max}(R)$ the *Jacobson radical* of R, for every $I \leq R$, we denote by $\eta(I)$ the *prime radical* of I (the intersection of the prime ideals of R containing I) and we say I is a *radical ideal* if $I = \eta(I)$. In particular, $\eta(R) := \eta(0)$ is the *prime radical* of R, and R is called a *reduced ring* if $\eta(R) = \{0\}$. Note that $\eta(R) = \bigcap \mathrm{Min}(R)$ and we set $Ra := \{ra : r \in R\}$ and $(I : a) := \{r \in R : ra \in I\}$ for every $a \in R$.

Let I be an ideal of a ring R. We denote by $(I)_0$ the family of prime ideals of R containing I and by $D_0(I) := \mathrm{Spec}(R)\backslash(I)_0$. Also, $(a)_0 := (Ra)_0$ and $D_0(a) := D_0(Ra)$ for every $a \in R$. It is easy to see that the family $\{(I)_0 : I \leq R\}$ satisfies the axioms of closed sets for a topology t_Z on $\mathrm{Spec}(R)$, the *Zariski topology*, and the space $(\mathrm{Spec}(R), t_Z)$ is the *prime spectrum* of R. Note that $\overline{\{P\}}^{t_Z} = (P)_0$ and $\widehat{P}^{\tau} = \{Q \in \mathrm{Spec}(R) : Q \subseteq P\}$ for every $P \in \mathrm{Spec}(R)$, and in this work we consider the family $\mathrm{Spec}(R)$ as a space with the Zariski topology.

2. Isolated points

Let (X,τ) be a space. A point $x \in X$ is called a *kerneled* (resp. *isolated*, *Alexandroff*) *point* of (X,τ) if $\{x\} = \widehat{x}^{\tau}$ (resp. $\{x\} \in \tau$, $\widehat{x}^{\tau} \in \tau$). The kerneled points of (X,τ) are the maximal elements in the pre-ordered set $(X, \leq_\tau)$, where $\leq_\tau$ is the τ-*specialization pre-order* on X, this is, $x \leq_\tau y$ in X if $x \in \overline{y}^{\tau}$, or equivalently, $y \in \widehat{x}^{\tau}$. Note that (X,τ) is a T_0-space if and only if $\leq_\tau$ is a partial order on X.

Let $(X,\leq)$ be a pre-ordered set. We denote by $\mathrm{Max}(X,\leq)$ the set of maximal elements in $(X,\leq)$, and we say $(X,\leq)$ *has enough maximal elements* if for every $x \in X$, there exists $y \in \mathrm{Max}(X,\leq)$ such that $x \leq y$. Dually, we define the set $\mathrm{Min}(X,\leq)$.

The following result is well known, but we present it here for further reference in this paper.

Theorem 2.1. *Let (X,τ) be a space and $x \in X$. Then, the following conditions are equivalent.*

(a) *x is an isolated point of (X,τ).*

(b) *Whenever $A \subseteq X$ with $x \in \overline{A}^{\tau}$, we have $x \in A$.*

(c) *x is both an Alexandroff point of (X,τ) and a maximal element in $(X,\leq_\tau)$.*

PROOF: It is clear that (a)$\Rightarrow$(b) and since $\mathrm{Max}(X,\leq_\tau)$ is the set of kerneled points of (X,τ), we have (c)$\Rightarrow$(a). To prove that (b)$\Rightarrow$(c), let $y \in \widehat{x}^{\tau}$. Then, $x \in \overline{y}^{\tau}$ and thus, $x = y$. Hence, $\widehat{x}^{\tau} = \{x\}$ and by hypothesis, the set $A = X\backslash\{x\}$ is τ-closed (otherwise, A is τ-dense and $x \in A$ which is a contradiction). Therefore, $\{x\} \in \tau$ and (c) holds. $\square$

Theorem 2.2. *Let (X, τ) be a space such that $(X, \leq_\tau)$ has enough maximal elements and $x \in X$. Then, x is an isolated point of (X, τ) if and only if x is both an isolated point in $\mathrm{Max}(X, \leq_\tau)$ and in $\overline{x}^{\tau}$.*

PROOF: The necessary condition is clear. Suppose the sufficiency condition and let $Y = \mathrm{Max}(X, \leq_\tau)$ and $Z = \overline{x}^{\tau}$. Then, $\{x\} = Y \cap U = Z \cap V$ for some pair $U, V \in \tau$. Note that $Y \cap Z = \{x\}$, since if $y \in Y \cap Z$ then $y \leq_\tau x$ and by maximality, we have $y = x$. Hence, $\{x\} = W \cap \{x\}$ where $W = U \cap V \in \tau$. We will show that $\{x\} = W$, for if $y \in W$ then, by hypothesis, there exists $z \in Y$ such that $y \leq_\tau z$ and thus, $y \in \overline{z}^{\tau}$ and since $y \in U$, we have $z \in U \cap Y = \{x\}$ and thus, $z = x$ and $y \leq_\tau x$. Hence, $y \in Z \cap V = \{x\}$ and $y = x$. $\square$

Corollary 2.1. *A prime ideal P of a ring R is an isolated point of the prime spectrum of R if and only if P is an isolated point in $\mathrm{Min}(R)$ and in the Zariski-closure of $\{P\}$.*

PROOF: Use Theorem 2.2, since $\mathrm{Min}(R) = \mathrm{Max}(\mathrm{Spec}(R), \leq_{tz})$ and $\leq_{tz} = \supseteq$. $\square$

Note that Corollary 2.1 is part (2) of Proposition 3 in [4], and in the next section we study each of the two sufficient conditions in Corollary 2.1.

3. Redundancy and scattered spectral spaces

Let Y be a nonempty family of prime ideals of a ring R and $P \in Y$. An ideal I of R is *absolutely Y-irreducible* if whenever $\mathcal{F} \subseteq Y$ with $\bigcap \mathcal{F} \subseteq I$, there exists $Q \in \mathcal{F}$ such that $Q \subseteq I$. If $Y = \mathrm{Min}(R)$ then I is said to be *absolutely minimal-irreducible*, and if $Y = \mathrm{Max}(R)$ then I is said to be *absolutely maximal-irreducible* ([5]). Let $I(Y) := \bigcap Y$ be the radical ideal of Y and $I_P(Y) := \bigcap\{Q \in Y : Q \neq P\}$. Then, $\overline{Y}^{tz} = (I(Y))_0$ and we say P is *Y-redundant* if $I(Y) = I_P(Y)$. In particular, if $Y = \mathrm{Spec}(R)$ we have the *weak η-redundancy* studied in [5]. Also, if $Y = \mathrm{Min}(R)$ we speak of *η-redundancy* and if $Y = \mathrm{Max}(R)$ we speak of *J-redundancy*. We now give a description of the isolated points in an arbitrary subspace of $\mathrm{Spec}(R)$ with at least two points. We denote by $\mathrm{Min}(Y, \subseteq)$ the set of minimal elements in the poset $(Y, \subseteq)$.

Theorem 3.1. *Let R be a ring, Y a non-empty subset of $\mathrm{Spec}(R)$, $I = \bigcap Y$ the radical ideal of Y and $P \in Y$. Then, the following conditions are equivalent.*

(a) *P is an isolated point of Y (as subspace of $\mathrm{Spec}(R)$).*
(b) *P is an absolutely Y-irreducible ideal of R and $P \in \mathrm{Min}(Y, \subseteq)$.*
(c) *Whenever $\mathcal{F} \subseteq Y$ with $\bigcap \mathcal{F} \subseteq P$, we have $P \in \mathcal{F}$.*
(d) *There exists $a \in I_P(Y)\backslash P$ such that $P = (I : a)$.*
(e) *P is not Y-redundant.*

Further, in such a case, $P = (I : a)$ for every $a \in I_P(Y)\backslash P$.

Proof: Let $t = t_Z|_Y$. To show that (a)$\Rightarrow$(b), let $Q \in Y$ with $Q \subseteq P$. Then, $Q \in \widehat{P}^{tz} \cap Y = \widehat{P}^{t} = \{P\}$ and thus, $Q = P$ and $P \in \mathrm{Min}(Y, \subseteq)$. Now, let $\mathcal{F} \subseteq Y$ with $\bigcap \mathcal{F} \subseteq P$. Then, $P \in \overline{\mathcal{F}}^{tz} \bigcap Y = \overline{\mathcal{F}}^{t}$ and by hypothesis, $\{P\} \bigcap \mathcal{F} \neq \emptyset$. Hence, $P \in \mathcal{F}$. The implication (b)$\Rightarrow$(c) is clear. Let us show that (c)$\Rightarrow$(e), and let $\mathcal{F} = Y \backslash \{P\}$. Then $I(Y) \neq I_P(Y)$ (otherwise, we will have $P \in \mathcal{F}$ which is a contradiction). Hence, P is not Y-redundant. Let us show that (e)$\Rightarrow$(a), and let $H = I_P(Y)$. Then $\{P\} = D_0(H) \bigcap Y \in t$. In fact, by hypothesis, $P \in D_0(H) \bigcap Y$ and if $Q \in D_0(H) \bigcap Y$ then $Q = P$ (otherwise, $H \subseteq Q$ which is a contradiction). Let us see (e)$\Rightarrow$(d), and let $a \in I_P(Y) \backslash P$. Let us show that $P = (I : a)$. In fact it is clear that $(I : a) \subseteq P$ and if $x \in P$ such that $ax \notin I$ then there exists $Q \in Y$ with $ax \notin Q$. But then, $Q \neq P$ (since $x \in P$) and thus, $a \in Q$ which is a contradiction. Finally, (d)$\Rightarrow$(a) since if $a \in I_P(Y) \backslash P$ with $P = (I : a)$ then $\{P\} = Y \bigcap D_0(a) \in t$. □

Corollary 3.1. *A prime ideal P of a ring R is an isolated point of the Zariski-closure of $\{P\}$ if and only if P is not the intersection of the prime ideals which strictly contain it.*

Corollary 3.1 is part (2) of Proposition 3 in [4] under the assumption that R is a reduced ring. Recall that a prime ideal P of a ring R is a minimal prime ideal if and only if $P = \bigcup_{a \in R \backslash P} (\eta(R) : a)$ ([2, Lemma 1.1]). Thus it is natural to ask when this last property holds for a non-empty family of prime ideals of R.

Theorem 3.2. *Let Y be a nonempty family of prime ideals of a ring R, $I = \bigcap Y$ and $P \in Y$. If $P = \bigcup_{a \in R \backslash P} (I : a)$ then $P \in \mathrm{Min}(Y, \subseteq)$. Further, the converse holds in any one of the following cases:* (a) *if Y is a Zariski-kerneled set;* (b) *the Zariski-closure of Y coincides with its Zariski-saturation.*

Proof: Suppose $P = \bigcup_{a \in R \backslash P} (I : a)$ and that $P \notin \mathrm{Min}(Y, \subseteq)$. Then, there exists $Q \in Y$ such that $Q \subsetneq P$. Let $x \in P \backslash Q$ and $a \in R \backslash P$ such that $x \in (I : a)$. Then, $ax \in I \subseteq Q$ which is a contradiction. Conversely, suppose $P \in \mathrm{Min}(Y, \subseteq)$ and either (a) or (b) holds. It is clear that $\bigcup_{a \in R \backslash P} (I : a) \subseteq P$. Now, suppose $x \in P$ such that $ax \notin I$ for every $a \in R \backslash P$. Let $S = R \backslash P$ and $T = \bigcup_{n \in \mathbb{N}} Sx^n$. Then, $S \subsetneq T$ and T is a multiplicatively closed subset of R such that $1 \in T$. Note that $0 \notin T$ (otherwise, $sx^n = 0 \in I$ for some $s \in S$ and $n \in \mathbb{N}$ and since I is radical and $s \notin I$, we will have $x^m \in I$ for some integer $m \geq 2$ and thus, $x \in I$ which is a contradiction). Hence, $0 \notin T$ and $I \cap T = \emptyset$, and by Krull's Lemma (Theorem 2.2 in Chapter VIII of [3]), there exists a prime ideal Q of R such that $I \subseteq Q$ and $T \cap Q = \emptyset$. But then, $Q \in \overline{Y}^{tz}$ and $Q \subseteq P$. Now, if (a) holds then Y a lower segment of $(\mathrm{Spec}(R), \subseteq)$ and thus $Q \in Y$ and by minimality, $P = Q$ which is a contradiction (since $x \in P$ and $x \notin Q$). On the other hand, if (b) holds then $Q \in \overline{Y}^{tz} = \bigcup_{H \in Y} (H)_0$ and there exists $H \in Y$ such that $H \subseteq Q$ and as above, $P = H = Q$ obtaining a contradiction. □

Note that condition (b) in Theorem 3.2 is satisfied if Y is either Zariski-closed or dense with respect to the Alexandroff closure of the Zariski topology, denoted by $\overline{t_Z}$. Also, condition (b) is equivalent to the following: for every prime ideal P of R such that $\bigcap Y \subseteq P$, there exists $Q \in Y$ such that $Q \subseteq P$. In particular, any finite subset of $\mathrm{Spec}(R)$ satisfies this property. Moreover, the conditions (a) and (b) are independent. In fact, if R is a zero-dimensional ring with infinite prime ideals then $\mathrm{Spec}(R)$ is not a discrete space and there exists a Zariski-open set Y which is not Zariski-closed (otherwise, the Zariski topology will be an Alexandroff T_1-topology which is discrete). On the other hand, if R is not a zero-dimensional ring then there exists a maximal nonminimal ideal P of R and thus, the set $Y = \{P\}$ satisfies trivially condition (b) but not condition (a).

Theorem 3.3. *Let Y be a nonempty family of prime ideals of a ring R, $P \in Y$ and $a \in R \backslash P$. If $P = (I(Y) : a)$ then $\{P\} = \mathrm{Min}(Y, \subseteq) \bigcap D_0(a)$ and the converse holds if the poset $(Y, \subseteq)$ has enough minimal elements.*

Proof: If $Q \in Y$ with $Q \subseteq P$ then $Pa \subseteq I(Y) \subseteq Q$ and since $a \notin Q$, we have $P \subseteq Q$ and $P = Q$. Hence, $\{P\} \subseteq \mathrm{Min}(Y, \subseteq) \bigcap D_0(a)$. Conversely, if $Q \in \mathrm{Min}(Y, \subseteq) \bigcap D_0(a)$ then $Pa \subseteq I(Y) \subseteq Q$ and thus, $P \subseteq Q$ and $P = Q$. On the other hand, suppose that the poset $(Y, \subseteq)$ has enough minimal elements and $\{P\} = \mathrm{Min}(Y, \subseteq) \bigcap D_0(a)$. It is clear that $(I(Y) : a) \subseteq P$ and if $x \in P$ with $ax \notin I(Y)$ then there exists $Q \in Y$ such that $ax \notin Q$. Now, if $Q_0 \in \mathrm{Min}(Y, \subseteq)$ such that $Q_0 \subseteq Q$ then $ax \notin Q_0$ and $P = Q_0$ which is a contradiction. Hence, $P = (I(Y) : a)$. □

We now prove some more consequences of Theorems 3.1 and 3.3.

Theorem 3.4. *Let R be a ring and P a prime ideal of R. Then, the following conditions are equivalent.*

(a) *P is an isolated point of* $\mathrm{Min}(R)$.
(b) *P is both an absolutely minimal-irreducible and minimal prime ideal of R.*
(c) *There exists $a \in R \backslash P$ such that $P \subseteq Q$ for every $Q \in D_0(a)$.*
(d) *There exists $a \in R \backslash P$ such that $P = (\eta(R) : a)$.*
(e) *P is not η-redundant.*
(f) *P is a minimal prime ideal of R and there exists $a \in R \backslash P$ such that $(\eta(R) : x) \subseteq (\eta(R) : a)$ for every $x \in R \backslash P$.*

Proof: By Theorems 3.1 and 3.3, we have (a)⇔(b)⇔(d)⇔(e). Let us show that (d)⇒(c). Suppose $a \in R \backslash P$ with $P = (\eta(R) : a)$ and let $Q \in D_0(a)$. Then $a \notin Q$ and if $x \in P$ then $ax \in \eta(R) \subseteq Q$ and thus, $x \in Q$ and $P \subseteq Q$. We see (c)⇒(b). Suppose $a \in R \backslash P$ such that $P \subseteq Q$ for every $Q \in D_0(a)$, and let $Q_0 \in \mathrm{Min}(R)$ with $Q_0 \subseteq P$. If $Q_0 \neq P$ then $a \in Q_0$ (otherwise $P \subseteq Q_0$ and thus $P = Q_0$ which is a contradiction). Hence, $P \in \mathrm{Min}(R)$. Now, let $\mathcal{F} \subseteq \mathrm{Min}(R)$ with $\bigcap \mathcal{F} \subseteq P$. Then $a \notin \bigcap \mathcal{F}$ and there exists $Q \in \mathcal{F}$ with $a \notin Q$ and by hypothesis, $P \subseteq Q$ and by minimality, $P = Q$. Finally, for (f)⇔(d) use that $P = \bigcup_{a \in R \backslash P} (\eta(R) : a)$ ([2]). □

Corollary 3.2. *Let R be a ring. Then,* $\mathrm{Min}(R)$ *is a discrete space if and only if every minimal prime ideal of R is not η-redundant if and only if the prime radical of R is the irredundant intersection of the minimal prime ideals of R.*

Condition (d) in Theorem 3.4 is an extension of part (3) of Proposition 4 in [4]. Dually, we have the following two results.

Theorem 3.5. *Let R be a ring and P a maximal ideal of R. Then, the following conditions are equivalent.*

(a) *P is an isolated point of* $\mathrm{Max}(R)$.
(b) *P is an absolutely maximal-irreducible ideal of R.*
(c) *If $\mathcal{F} \subseteq \mathrm{Max}(R)$ such that $\bigcap \mathcal{F} \subseteq P$ then $P \in \mathcal{F}$.*
(d) *There exists $a \in I_P(\mathrm{Max}(R))\backslash P$ such that $P = (J(R) : a)$.*
(e) *There exists $a \in R\backslash P$ such that $P = (J(R) : a)$.*
(f) *P is not J-redundant.*

Corollary 3.3. *Let R be a ring. Then,* $\mathrm{Max}(R)$ *is a discrete space if and only if every maximal ideal of R is not J-redundant if and only if the Jacobson radical of R is the irredundant intersection of the maximal ideals of R.*

Every maximal ideal of a ring R generated by an idempotent element of R is an isolated point in $\mathrm{Max}(R)$ and the converse holds if R is a *semiprimitive ring*, this is, $J(R) = \{0\}$ (Lemma 2.1 in [4]). Note that if $J = J(R)$ then the quotient ring R/J is semiprimitive and the spaces $\mathrm{Max}(R)$ and $\mathrm{Max}(R/J)$ are homeomorphic. Hence, in general, the isolated points of the space $\mathrm{Max}(R)$ are the maximal ideals P of R for which P/J is an ideal of R/J generated by an idempotent element of R/J.

Theorem 3.6. *Let P be a prime ideal of a ring R and $Y = \widehat{P}^{tz}$. Then, the following conditions are equivalent.*

(a) *P is an isolated point of Y.*
(b) *P is a minimal element in the poset $(Y, \subseteq)$.*
(c) *P is a minimal prime ideal of R.*
(d) *$P = (I(Y) : a)$ for some $a \in R\backslash P$.*

Proof: By Theorem 3.1, (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) and (a)$\Rightarrow$(d). To show that (d)$\Rightarrow$(c), let $Q \in \mathrm{Min}(R)$ such that $Q \subseteq P$. Since $Q \in Y$ and $a \notin Q$, we have $Pa \subseteq I(Y) \subseteq Q$ and thus, $P \subseteq Q$ and $P = Q$. $\square$

By Corollary 3.1, the property (a) in Theorem 3.6 is not dual. Also, for every prime ideal P of a ring R, the family $Y = \widehat{P}^{tz}$ has enough minimal elements and the radical ideal $I(Y)$ is the intersection of the minimal prime ideals of R contained in P (see Theorem 3.3).

Theorem 3.7. *Let P and Q be prime ideals of a ring R. Then P is an isolated point of $\overline{Q}^{tz}$ if and only if $P = Q$ and P is not the intersection of the prime ideals which strictly contain it.*

Proof: The sufficiency condition is a consequence of Corollary 3.1. Conversely, suppose P is an isolated point of $\overline{Q}^{tz} = (Q)_0$. By Theorem 3.1, for every $\mathcal{F} \subseteq (Q)_0$ with $\bigcap \mathcal{F} \subseteq P$, we have $P \in \mathcal{F}$. In particular, if $\mathcal{F} = \{Q\}$ then $P = Q$. For the last part, use Corollary 3.1. □

Corollary 3.4. *Let P be a prime ideal of a ring R. If the (irreducible Zariski-closed) set $(P)_0$ has an isolated point then this point is P.*

Recall that (X, τ) is a *scattered space* if every nonempty subset of X contains a point that is isolated in the relative topology. By Corollary 3.4, if $\mathrm{Spec}(R)$ is a scattered space and $P \in \mathrm{Spec}(R)$ then P is the unique isolated point of $(P)_0$. By Theorem 2.8 in [1], if R is a zero-dimensional ring then $\mathrm{Spec}(R)$ is a scattered space if and only if every radical ideal of R is an irredundant intersection of (maximal) prime ideals of R. Compare this last result with Corollary 3.2. Further, by Theorem 3.1, we have the following result.

Corollary 3.5. *Let R be a ring. Then, the following conditions are equivalent.*

(a) $\mathrm{Spec}(R)$ *is a scattered space.*
(b) *For every nonempty family Y of prime ideals of R, there is an absolutely Y-irreducible ideal of R which is a minimal element in $(Y, \subseteq)$.*
(c) *For every nonempty family Y of prime ideals of R, there exists $P \in Y$ such that if $\mathcal{F} \subseteq Y$ with $\bigcap \mathcal{F} \subseteq P$, then $P \in \mathcal{F}$.*
(d) *For every nonempty family Y of prime ideals of R, there are $P \in Y$ and $a \in I_Y(P) \backslash P$ with $P = (I(Y) : a)$.*
(e) *For every nonempty family Y of prime ideals of R, there exists $P \in Y$ which is not Y-redundant.*

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