Ratio Tauberian theorems for relatively bounded functions and sequences in Banach spaces

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Abstract. We prove ratio Tauberian theorems for relatively bounded functions and sequences in Banach spaces.

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1. Introduction

Let X be a Banach space and $u : [0,\infty) \to X$ be a locally integrable function. Let $g : [0,\infty) \to \mathbb{R}_+$ be a locally integrable function such that $\int_0^\infty g(t)\,dt > 0$, where $\mathbb{R}_+ := \{t \geq 0 : t \in \mathbb{R}\}$. We assume the condition

$$\frac{\int_0^t g(r)\,dr}{\int_0^s g(r)\,dr} \to 1 \quad \text{as} \quad t,s \to \infty \quad \text{with} \quad \frac{t}{s} \to 1,$$

and prove that if $\|u(t)\| = O(g(t))$, $t \to \infty$, then the following statements are equivalent:

(i) $x = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$;
(ii) $x = \lim_{\lambda\downarrow 0} \big(\int_0^\infty e^{-\lambda t}u(t)\,dt\big)/\big(\int_0^\infty e^{-\lambda t}g(t)\,dt\big)$.

This solves the open problem posed in [6]. Then particular choices of the function g will be considered, leading to some generalized Tauberian theorems. Discrete analogues are obtained as well.

2. Results for functions

Let X be a Banach space and $u : [0,\infty) \to X$ be a locally integrable function. The class of all such functions will be denoted by $L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$. For $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$, $\gamma \geq 1$ and $t > 0$ we define the γ-th order Cesàro integral $\mathfrak{s}_t^\gamma(u)$ over $[0,t]$ as

$$\mathfrak{s}_t^\gamma(u) := (k_\gamma * u)(t) = \int_0^t k_\gamma(t-s)u(s)\,ds, \tag{1}$$

where $k_\gamma(t) := t^{\gamma-1}/\Gamma(\gamma)$ for $t \in \mathbb{R}_+$. In particular we have $\mathfrak{s}_t^1(u) = \int_0^t u(s)\,ds$. The Laplace integral $\widehat{u}(\lambda)$ for $\lambda \in \mathbb{R}$ is defined as

$$\widehat{u}(\lambda) := \int_0^\infty e^{-\lambda t} u(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-\lambda t} u(t)\,dt \tag{2}$$

if the limit exists. It is known (see e.g. [1, Proposition 1.4.1]) that if $\widehat{u}(\lambda_0)$ exists then $\widehat{u}(\lambda)$ exists for all $\lambda > \lambda_0$. If μ is a locally finite positive measure on $\mathbb{R}_+$, then we use the notation $\widehat{\mu}(\lambda)$ to denote $\int_0^\infty e^{-\lambda t} d\mu(t)$ when $\int_0^\infty e^{-\lambda t} d\mu(t) < \infty$.

We begin with the following key lemma.

Lemma 2.1. *Let μ be a locally finite positive measure on $\mathbb{R}_+$ such that $\mu[0,\infty) > 0$. If*

$$\frac{\mu[0,t]}{\mu[0,s]} \to 1 \quad \text{as} \quad t, s \to \infty \quad \text{with} \quad \frac{t}{s} \to 1, \tag{C}$$

then

$$\liminf_{\lambda\downarrow 0} \frac{\mu[0,1/\lambda]}{\widehat{\mu}(\lambda)} = \liminf_{\lambda\downarrow 0} \frac{\mu[0,1/\lambda]}{\int_0^\infty e^{-\lambda t} d\mu(t)} > 0. \tag{C1}$$

Proof: By hypothesis there are two constants $G > 1$ and $\delta > 0$ such that if $t > s > G$ and $t/s \le 1+\delta$ then

$$0 \le \frac{\mu(s,t]}{\mu[0,s]} < 1.$$

Thus for $\lambda > 0$ with $1/\lambda > G$ we have $\mu(1/\lambda,\, (1+\delta)/\lambda] < 2^0 \mu[0,1/\lambda]$, and

$$\mu((1+\delta)/\lambda,\, (1+\delta)^2/\lambda] < \mu[0,(1+\delta)/\lambda] < 2^1 \mu[0,1/\lambda].$$

Then for $n \ge 2$ we have inductively

$$\begin{aligned} \mu((1+\delta)^n/\lambda,\, (1+\delta)^{n+1}/\lambda] &< \mu[0,(1+\delta)^n/\lambda] \\ &= \mu[0,1/\lambda] + \sum_{k=0}^{n-1} \mu((1+\delta)^k/\lambda,\, (1+\delta)^{k+1}/\lambda] \\ &< \Big(1 + \sum_{k=0}^{n-1} 2^k\Big)\mu[0,1/\lambda] = 2^n \mu[0,1/\lambda]. \end{aligned}$$

Hence

$$\begin{aligned} 0 < \int_0^\infty e^{-\lambda t} d\mu(t) &= \int_{[0,1/\lambda]} e^{-\lambda t} d\mu(t) + \sum_{n=0}^\infty \int_{((1+\delta)^n/\lambda,\, (1+\delta)^{n+1}/\lambda]} e^{-\lambda t}\, d\mu(t) \\ &\le \mu[0,1/\lambda] + \sum_{n=0}^\infty 2^n \mu[0,1/\lambda] e^{-(1+\delta)^n} < \infty. \end{aligned}$$

Therefore

$$\frac{\mu[0,1/\lambda]}{\widehat{\mu}(\lambda)} = \frac{\mu[0,1/\lambda]}{\int_0^\infty e^{-\lambda t} d\mu(t)} \geq \Big(1+\sum_{n=0}^\infty 2^n e^{-(1+\delta)^n}\Big)^{-1} > 0,$$

completing the proof. □

Theorem 2.2 (cf. [2, Theorem 2.2]). *Suppose $0 \neq g \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}_+)$ satisfies condition* (C) *with $\mu := g(t)\,dt$. Then for any $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+,X)$ with $\|u(t)\| = O(g(t))$, $t\to\infty$, the following statements are equivalent:*

(i) $x = \lim_{t\to\infty} \mathfrak{s}^1_t(u)/\mathfrak{s}^1_t(g) = \lim_{t\to\infty}\big(\int_0^t u(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$;

(ii) $x = \lim_{t\to\infty} \mathfrak{s}^\beta_t(u)/\mathfrak{s}^\beta_t(g)$
$= \lim_{t\to\infty}\big(\int_0^t (t-s)^{\beta-1}u(s)\,ds\big)/\big(\int_0^t (t-s)^{\beta-1}g(s)\,ds\big)$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0} \widehat{u}(\lambda)/\widehat{g}(\lambda) = \lim_{\lambda\downarrow 0}\big(\int_0^\infty e^{-\lambda t}u(t)\,dt\big)/\big(\int_0^\infty e^{-\lambda t}g(t)\,dt\big)$.

Proof: "(i) ⇒ (ii) ⇒ (iii)" follows from [2, Theorem 2.1].

(iii) ⇒ (i): We first note that if $P(t) = \sum_{n=0}^N a_n t^n$ is a polynomial function such that

$$P(t) \geq d > 0 \quad \text{on} \quad [0,1], \tag{3}$$

then

$$x = \lim_{\lambda\downarrow 0} \frac{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})g(t)\,dt}. \tag{4}$$

To see this, put $\widetilde{P}(\lambda) := \int_0^\infty e^{-\lambda t}P(e^{-\lambda t})g(t)\,dt$. Then

$$\begin{aligned}&\frac{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})g(t)\,dt}\\ &\qquad = \frac{1}{\widetilde{P}(\lambda)}\sum_{n=0}^N a_n\Big(\int_0^\infty e^{-\lambda(n+1)t}g(t)\,dt\Big)\cdot\frac{\int_0^\infty e^{-\lambda(n+1)t}u(t)\,dt}{\int_0^\infty e^{-\lambda(n+1)t}g(t)\,dt}.\end{aligned}$$

Here

$$\lim_{\lambda\downarrow 0}\frac{\int_0^\infty e^{-\lambda(n+1)t}u(t)\,dt}{\int_0^\infty e^{-\lambda(n+1)t}g(t)\,dt} = x \qquad \text{(by (iii))}, \tag{5}$$

and

$$0 < \frac{\int_0^\infty e^{-\lambda(n+1)t}g(t)\,dt}{\widetilde{P}(\lambda)} \leq \frac{1}{d} \qquad \text{(by (3))}. \tag{6}$$

Thus

$$\lim_{\lambda\downarrow 0}\frac{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})g(t)\,dt} = \lim_{\lambda\downarrow 0}\frac{1}{\widetilde{P}(\lambda)}\sum_{n=0}^{N} a_n\Big(\int_0^\infty e^{-\lambda(n+1)t}g(t)\,dt\Big)x = x.$$

Next we write

$$\frac{\int_0^{1/\lambda} u(t)\,dt}{\int_0^{1/\lambda} g(t)\,dt} = \frac{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt}, \tag{7}$$

where

$$h(t) := \begin{cases} 0 & \text{if } 0 \le t < e^{-1}, \\ t^{-1} & \text{if } e^{-1} \le t \le 1. \end{cases}$$

For the proof we may assume without loss of generality that $\|u(t)\| \le g(t)$ for all $t \ge 0$. By condition (C), given an $\varepsilon > 0$, there are two constants $G > 1$ and $\delta > 0$ such that

$$0 \le \frac{\mu(s,t]}{\mu[0,s]} < \varepsilon \quad \text{if} \quad t > s > G \quad \text{and} \quad \frac{t}{s} \le 1+\delta. \tag{8}$$

It is standard to see that there exists a polynomial function $P(t) = \sum_{n=0}^N a_n t^n$ such that

(a) $h(t) < P(t) \le \varepsilon$ on $[0, e^{-(1+\delta)}]$,
(b) $h(t) < P(t) \le h(e^{-1}) + \varepsilon$ on $[e^{-(1+\delta)},\, e^{-1}]$,
(c) $h(t) < P(t) \le h(t) + \varepsilon$ on $[e^{-1},\, 1]$.

Then

$$\begin{aligned}
&\frac{\int_0^{1/\lambda} u(t)\,dt}{\int_0^{1/\lambda} g(t)\,dt} - x \\
&= \frac{\int_0^\infty e^{-\lambda t}\big(h(e^{-\lambda t}) - P(e^{-\lambda t})\big)u(t)\,dt + \int_0^\infty e^{-\lambda t}P(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt} - x \\
&=: I_\lambda + II_\lambda - x,
\end{aligned}$$

and

$$\begin{aligned}
I_\lambda &= \frac{\Big(\int_0^{1/\lambda} + \int_{1/\lambda}^{(1+\delta)/\lambda} + \int_{(1+\delta)/\lambda}^\infty\Big)e^{-\lambda t}\big(h(e^{-\lambda t}) - P(e^{-\lambda t})\big)u(t)\,dt}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt} \\
&=: \frac{I_\lambda(1) + I_\lambda(2) + I_\lambda(3)}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt},
\end{aligned}$$

where

$$\|I_\lambda(1)\| < \int_0^{1/\lambda} e^{-\lambda t}\varepsilon\, g(t)\,dt \le \varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\,dt \tag{9}$$

by (c) and the assumption that $\|u(t)\| \le g(t)$ for all $t \ge 0$. On the other hand, (b) implies

$$\|I_\lambda(2)\| < \int_{1/\lambda}^{(1+\delta)/\lambda} e^{-\lambda t}\big(h(e^{-1}) + \varepsilon\big) g(t)\,dt \le (e+\varepsilon)\int_{1/\lambda}^{(1+\delta)/\lambda} g(t)\,dt,$$

where if $\lambda > 0$ is sufficiently small, then by (8)

$$\int_{1/\lambda}^{(1+\delta)/\lambda} g(t)\,dt < \varepsilon \int_0^{1/\lambda} g(t)dt = \varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\,dt,$$

so that

$$\|I_\lambda(2)\| < (e+\varepsilon)\,\varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\,dt \tag{10}$$

for all sufficiently small $\lambda > 0$. Finally (a) implies

$$\|I_\lambda(3)\| < \int_{(1+\delta)/\lambda}^\infty e^{-\lambda t}\varepsilon\, g(t)\,dt \le \varepsilon \int_0^\infty e^{-\lambda t} g(t)\,dt.$$

We apply Lemma 2.1 to infer that there exists a constant $\eta > 0$ such that

$$\liminf_{\lambda\downarrow 0} \frac{\int_0^{1/\lambda} g(t)\,dt}{\int_0^\infty e^{-\lambda t} g(t)\,dt} > \eta.$$

Thus if $\lambda > 0$ is sufficiently small, then

$$\int_0^{1/\lambda} g(t)\,dt > \eta \int_0^\infty e^{-\lambda t} g(t)\,dt,$$

so that

$$\|I_\lambda(3)\| < \frac{\varepsilon}{\eta} \int_0^{1/\lambda} g(t)\,dt = \frac{\varepsilon}{\eta}\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\,dt. \tag{11}$$

Consequently

$$\limsup_{\lambda\downarrow 0} \|I_\lambda\| < \varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\,. \tag{12}$$

Now we write

$$II_\lambda = \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\,dt} \cdot \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt}\,.$$

Similarly as in (12) one may show that if $\lambda > 0$ is sufficiently small, then

$$1 \le \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\, dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt} < 1 + \varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}.$$

Since $P(t) \ge d > 0$ on $[0,1]$ for some $d > 0$, it follows that

$$\lim_{\lambda \downarrow 0} \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\, dt} = x.$$

Hence

$$\begin{aligned} \|II_\lambda - x\| &\le \left\| \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\, dt} - x \right\| \\ &\quad + \Big(\varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\Big) \left\| \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\, dt} \right\| \\ &\longrightarrow \Big(\varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\Big)\|x\| \end{aligned} \tag{13}$$

as $\lambda \downarrow 0$. Combining this with (12) yields

$$\limsup_{\lambda \downarrow 0} \left\| \frac{\int_0^{1/\lambda} u(t)\, dt}{\int_0^{1/\lambda} g(t)\, dt} - x \right\| < \Big(\varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\Big)(1 + \|x\|) \tag{14}$$

which completes the proof, since $\varepsilon > 0$ is arbitrary. □

Theorem 2.3 (cf. [2, Theorem 4.2], [5, Proposition 3.4])**.** *Let $\alpha \ge 0$. Suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{\alpha-1})$, $t \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{t\to\infty} \big(\Gamma(\alpha+1)/t^\alpha\big) \int_0^t u(s)\, ds$;

(ii) $x = \lim_{t\to\infty} \big(\Gamma(\alpha+\beta)/\Gamma(\beta) t^{\alpha+\beta-1}\big) \int_0^t (t-s)^{\beta-1} u(s)\, ds$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0} \lambda^\alpha \widehat{u}(\lambda) = \lim_{\lambda\downarrow 0} \lambda^\alpha \int_0^\infty e^{-\lambda t} u(t)\, dt$.

Proof: "(i) ⇒ (ii) ⇒ (iii)" follows from [2, Theorem 4.1].

(iii) ⇒ (i): Suppose $\alpha > 0$. Then define $g(t) := k_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t \in \mathbb{R}_+$ and $\mu := g(t)\, dt$. It follows that $\|u(t)\| = O(g(t))$, $t \to \infty$, that $\widehat{g}(\lambda) = \lambda^\alpha$ for all $\lambda > 0$, and that $\mu[0,t] = \int_0^t k_\alpha(s)\, ds = (k_1 * k_\alpha)(t) = k_{\alpha+1}(t) = t^\alpha/\Gamma(\alpha+1)$. Hence μ satisfies condition (C), and so (i) follows from Theorem 2.2.

Next suppose $\alpha = 0$. Since $\|u(t)\| = O(t^{-1})$, $t \to \infty$, it follows from standard calculations (see e.g. [8, pp. 204, 206]) that the function $U(t) := \int_0^t u(s)\, ds$ is bounded and feebly oscillating (i.e. $\|U(t) - U(s)\| \to 0$ as t and $s \to \infty$ in such a way that $t/s \to 1$). Thus (i) follows from [5, Proposition 3.4]. The proof is complete. □

Remark. The special case $\alpha = 1$ of Theorem 2.3 states that, under the assumption that u is bounded, the Cesàro limit $\lim_{t\to\infty}(1/t)\int_0^t u(s)\,ds$ exists if and only if the Abel limit $\lim_{\lambda\downarrow 0}\lambda\,\widehat{u}(\lambda)$ exists and they are both equal. This is a classical Tauberian theorem (see e.g. [1, Theorem 4.2.7]). The special case $\alpha = 0$ of Theorem 2.3 states that, under the assumption that $\|u\| = O(t^{-1})$, $t\to\infty$, the limit $\lim_{t\to\infty}\int_0^t u(s)\,ds$ exists if and only if the limit $\lim_{\lambda\downarrow 0}\widehat{u}(\lambda)$ exists and they are both equal. This is another classical Tauberian theorem (see e.g. [1, Theorem 4.2.9]).

Theorem 2.4. *Suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{-1})$, $t\to\infty$. Then the following statements are equivalent:*

(i) $x = \lim_{t\to\infty}\big(1/\log t\big)\int_0^t u(s)\,ds$;

(ii) $x = \lim_{t\to\infty}\big(1/t^{\beta-1}\log t\big)\int_0^t (t-s)^{\beta-1}u(s)\,ds$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0}\big(1/-\log\lambda\big)\widehat{u}(\lambda) = \lim_{\lambda\downarrow 0}\big(1/-\log\lambda\big)\int_0^\infty e^{-\lambda t}u(t)\,dt$.

Proof: Let

$$g(t) := \begin{cases} 0 & \text{if } 0 \le t < 1, \\ t^{-1} & \text{if } t \ge 1. \end{cases}$$

An approximation argument yields that

$$\tag{15} \begin{aligned} \lim_{t\to\infty}\frac{\int_0^t (t-s)^{\gamma-1}g(s)\,ds}{t^{\gamma-1}\log t} &= \lim_{t\to\infty}\frac{\int_1^t (t-s)^{\gamma-1}s^{-1}\,ds}{t^{\gamma-1}\log t} \\ &= \lim_{t\to\infty}\frac{\int_{1/t}^1 (1-s)^{\gamma-1}s^{-1}\,ds}{\log t} = 1 \end{aligned}$$

for all $\gamma \ge 1$ and that

$$\tag{16} \lim_{\lambda\downarrow 0}\frac{\widehat{g}(\lambda)}{-\log\lambda} = \lim_{\lambda\downarrow 0}\frac{\int_1^\infty e^{-\lambda t}t^{-1}\,dt}{-\log\lambda} = \lim_{\lambda\downarrow 0}\frac{\int_\lambda^\infty e^{-t}t^{-1}\,dt}{-\log\lambda} = 1.$$

Since $\|u(t)\| = O(g(t))$, $t\to\infty$, and the measure $\mu := g(t)\,dt$ satisfies condition (C), the desired result follows from Theorem 2.2. □

Remark. If X is a Banach lattice with positive cone X_+ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then statements (i), (ii) and (iii) in Theorem 2.4 are also equivalent. This follows from [2, Theorem 2.2]. (We note that if $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then statement (ii) in Theorem 2.4 implies that $\widehat{u}(\lambda)$ exists for all $\lambda > 0$ (see [3, Lemma 2.5]).)

Fact 2.5. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$. Consider the following three statements:*

(i) $x = \lim_{t\to\infty}\int_0^t u(s)\,ds$;

(ii) $\widehat{u}(\lambda)$ *exists for all* $\lambda > 0$ *and* $x = \lim_{t\to\infty}\big(1/t^{\beta-1}\big)\int_0^t (t-s)^{\beta-1}u(s)\,ds$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0}\widehat{u}(\lambda) = \lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}u(t)\,dt$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROOF: Letting $g(t) := \chi_{[0,1]}(t)$ we have

$$\lim_{t\to\infty} \frac{\int_0^t (t-s)^{\gamma-1} g(s)\,ds}{t^{\gamma-1}} = \lim_{t\to\infty} \frac{\int_0^1 (t-s)^{\gamma-1}\,ds}{t^{\gamma-1}} = 1$$

for all $\gamma \geq 1$ and

$$\lim_{\lambda\downarrow 0} \widehat{g}(\lambda) = \lim_{\lambda\downarrow 0} \int_0^1 e^{-\lambda t}\,dt = 1.$$

Thus the desired result follows from [2, Theorem 2.1]. □

Remarks. (a) If $\int_0^\infty \|u(t)\|\,dt < \infty$, then clearly both (i) and (iii) in Fact 2.5 hold. In general (iii) does not imply (i). (For example let $u(t) := \sin t$.) If $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$, or if X is a Banach lattice and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then (iii) implies (i). (See Theorem 2.3 and [2, Theorem 4.2], respectively.)

(b) There exists a continuous function $u : [0,\infty) \to \mathbb{R}$ such that $\inf\{\lambda \in \mathbb{R} : \widehat{u}(\lambda) \text{ exists}\} = 1$ and also such that $\lim_{t\to\infty} (1/t) \int_0^t (t-s)u(s)\,ds\ (\in \mathbb{R})$ exists (see the Remark over Theorem 2.4 in [3], or [7, Example 5]). Thus the hypothesis that $\widehat{u}(\lambda)$ exists for all $\lambda > 0$ cannot be omitted from (ii) in Fact 2.5.

3. Results for sequences

Let $\{x_n\} := \{x_n\}_{n=0}^\infty$ be a sequence in a Banach space X. For $\gamma \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, we define the γ-th order Cesàro sum $\mathfrak{s}_n^\gamma(\{x_i\})$ as

$$\mathfrak{s}_n^\gamma(\{x_i\}) := \sum_{k=0}^{n} \binom{n-k+\gamma-1}{n-k} x_k, \tag{17}$$

where $\binom{r}{0} := 1$ and $\binom{r}{n} := r(r-1)\dots(r-n+1)/n!$ for $r \in \mathbb{R}$ and $n \geq 1$. Thus $\mathfrak{s}_0^\gamma(\{x_i\}) = x_0$ for all $\gamma \in \mathbb{R}$, $\mathfrak{s}_n^0(\{x_i\}) = x_n$ and $\mathfrak{s}_n^1(\{x_i\}) = \sum_{k=0}^n x_k$ for all $n \in \mathbb{N}_0$. The Abel sum $\widehat{\{x_i\}}(r)$ of $\{x_n\}$ is defined as

$$\widehat{\{x_i\}}(r) := \sum_{n=0}^{\infty} r^n x_n, \qquad 0 < r < \Big(\limsup_{n\to\infty} \|x_n\|^{1/n}\Big)^{-1}. \tag{18}$$

Clearly $\widehat{\{x_i\}}(r)$ exists for all $0 < r < 1$ if and only if $\limsup_{n\to\infty} \|x_n\|^{1/n} \leq 1$. Let $\{a_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers such that $\sum_{n=0}^\infty a_n > 0$. We define $u(t) := x_{[t]}$ and $g(t) := a_{[t]}$ for $t \geq 0$, where $[t]$ denotes the largest integer less than or equal to t. Then we have the following

Lemma 3.1. (i) $x = \lim_{n\to\infty} \big(\sum_{k=0}^n x_k\big)/\big(\sum_{k=0}^n a_k\big)$ *if and only if* $x = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$.

(ii) *Suppose* $\widehat{\{x_i\}}(r)$ *and* $\widehat{\{a_i\}}(r)$ *exist for all* $0 < r < 1$. *Then*

$$x = \lim_{r\uparrow 1} \frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \lim_{r\uparrow 1} \frac{\sum_{n=0}^{\infty} r^n x_n}{\sum_{n=0}^{\infty} r^n a_n}$$

if and only if

$$x = \lim_{\lambda\downarrow 0} \frac{\widehat{u}(\lambda)}{\widehat{g}(\lambda)} = \lim_{\lambda\downarrow 0} \frac{\int_0^\infty e^{-\lambda t} u(t)\,dt}{\int_0^\infty e^{-\lambda t} g(t)\,dt}.$$

Proof: (i) Putting $\delta(t) := t - [t]$ we have $0 \le \delta(t) < 1$, and

$$\frac{\int_0^t u(s)\,ds}{\int_0^t g(s)\,ds} = \frac{\big(1-\delta(t)\big)\sum_{k=0}^{[t]-1} x_k + \delta(t)\sum_{k=0}^{[t]} x_k}{\big(1-\delta(t)\big)\sum_{k=0}^{[t]-1} a_k + \delta(t)\sum_{k=0}^{[t]} a_k},$$

so that the first condition of (i) implies the second condition. The converse implication is obvious.

(ii) By an elementary calculation we have

$$\frac{\int_0^\infty e^{-\lambda t} u(t)\,dt}{\int_0^\infty e^{-\lambda t} g(t)\,dt} = \frac{\sum_{n=0}^\infty e^{-\lambda n} x_n}{\sum_{n=0}^\infty e^{-\lambda n} a_n}, \qquad \lambda > 0,$$

whence (ii) follows. □

Theorem 3.2 (cf. [2, Theorem 3.2]). *Let* $\{a_n\}_{n=0}^\infty$ *be a sequence of nonnegative real numbers such that* $\sum_{n=0}^\infty a_n > 0$. *Suppose*

$$\text{(D)} \qquad \frac{\sum_{k=0}^m a_k}{\sum_{k=0}^n a_k} \to 1 \quad \text{as} \quad m,n\to\infty \quad \text{with} \quad \frac{m}{n} \to 1.$$

Then for any sequence $\{x_n\}_{n=0}^\infty$ *in* X, *with* $\|x_n\| = O(a_n)$, $n\to\infty$, *the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \mathfrak{s}_n^1(\{x_i\})/\mathfrak{s}_n^1(\{a_i\}) = \lim_{n\to\infty} \big(\sum_{k=0}^n x_k\big)/\big(\sum_{k=0}^n a_k\big)$;

(ii) $x = \lim_{n\to\infty} \mathfrak{s}_n^\beta(\{x_i\})/\mathfrak{s}_n^\beta(\{a_i\})$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{r\uparrow 1} \widehat{\{x_i\}}(r)/\widehat{\{a_i\}}(r) = \lim_{r\uparrow 1} \big(\sum_{n=0}^\infty r^n x_n\big)/\big(\sum_{n=0}^\infty r^n a_n\big)$.

Proof: Condition (D) implies that the function $g(t) = a_{[t]}$ satisfies condition (C) with $\mu := g(t)\,dt$. Hence $\widehat{\{a_i\}}(r)$ and $\widehat{\{x_i\}}(r)$ exist for all $0 < r < 1$. Then "(i) ⇒ (ii) ⇒ (iii)" follows from [2, Theorem 3.1].

(iii) ⇒ (i): By Lemma 3.1 and Theorem 2.2 we have

$$x = \lim_{r\uparrow 1} \frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \lim_{\lambda\downarrow 0} \frac{\widehat{u}(\lambda)}{\widehat{g}(\lambda)} = \lim_{t\to\infty} \frac{\mathfrak{s}_t^1(u)}{\mathfrak{s}_t^1(g)} = \lim_{n\to\infty} \frac{\mathfrak{s}_n^1(\{x_i\})}{\mathfrak{s}_n^1(\{a_i\})},$$

which completes the proof. □

Theorem 3.3 (cf. [2, Theorem 4.4], [5, Proposition 3.6]). *Let $\alpha \geq 0$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\|x_n\| = O(n^{\alpha-1})$, $n \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \big(\Gamma(\alpha+1)/(n+1)^{\alpha}\big) \sum_{k=0}^{n} x_k$;
(ii) $x = \lim_{n\to\infty} \big(\Gamma(\alpha+\beta)/(n+1)^{\alpha+\beta-1}\big) \mathfrak{s}_n^{\beta}(\{x_i\})$ *for some/all* $\beta > 1$;
(iii) $x = \lim_{r\uparrow 1} (1-r)^{\alpha} \widehat{\{x_i\}}(r) = \lim_{r\uparrow 1} (1-r)^{\alpha} \sum_{n=0}^{\infty} r^n x_n$.

Proof: "(i) ⇒ (ii) ⇒ (iii)" follows from [2, Theorem 4.3].

(iii) ⇒ (i): Suppose $\alpha > 0$. Then define $a_n := \binom{n+\alpha-1}{n}$ for $n \geq 0$. It follows (cf. [9, pp. 76–77]) that $(1-r)^{-\alpha} = \sum_{n=0}^{\infty} r^n a_n$ for $0 < r < 1$, and $a_n = n^{\alpha-1}(1+o(1))/\Gamma(\alpha)$, $n \to \infty$. Thus $\|x_n\| = O(a_n)$, $n \to \infty$. Since

$$\sum_{k=0}^{n} a_k = \binom{n+\alpha}{n} = \frac{n^{\alpha}}{\Gamma(\alpha+1)}\big(1+o(1)\big), \quad n \to \infty,$$

$\{a_n\}_{n=0}^{\infty}$ satisfies condition (D). Hence (i) follows from Theorem 3.2.

Next suppose $\alpha = 0$. Then the function $u(t) = x_{[t]}$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$, and (iii) implies that $x = \lim_{\lambda\downarrow 0} \int_0^{\infty} e^{-\lambda t} u(t)\, dt$. Hence, by Theorem 2.3, $x = \lim_{t\to\infty} \int_0^t u(s)\, ds = \lim_{n\to\infty} \sum_{k=0}^{n} x_k$. This completes the proof. □

Remark. The special cases $\alpha = 1$ and $\alpha = 0$ of Theorem 3.3 are classical results for sequences corresponding to $\alpha = 1$ and $\alpha = 0$ of Theorem 2.3, respectively. (See e.g. [4, Theorem 3.1], [1, Theorem 4.2.17].)

Theorem 3.4. *Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\|x_n\| = O(n^{-1})$, $n \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \big(1/\log(n+1)\big) \sum_{k=0}^{n} x_k$;
(ii) $x = \lim_{n\to\infty} \big(\Gamma(\beta)/(n+1)^{\beta-1} \log(n+1)\big) \mathfrak{s}_n^{\beta}(\{x_i\})$ *for some/all* $\beta > 1$;
(iii) $x = \lim_{\lambda\downarrow 0} \big(1/-\log\lambda\big) \widehat{\{x_i\}}(e^{-\lambda}) = \lim_{\lambda\downarrow 0} \big(1/-\log\lambda\big) \sum_{n=0}^{\infty} e^{-\lambda n} x_n$.

Proof: Define $a_0 := 0$ and $a_n := n^{-1}$ for $n \geq 1$. Hence $\|x_n\| = O(a_n)$, $n \to \infty$, and $\sum_{k=0}^{n} a_k = \log n + O(1)$, $n \to \infty$. It follows that $\{a_n\}_{n=0}^{\infty}$ satisfies condition (D). If $\beta > 1$, then

$$\mathfrak{s}_n^{\beta}(\{a_i\}) = \sum_{k=1}^{n} \binom{n-k+\beta-1}{n-k} \frac{1}{k}. \tag{19}$$

Since

$$\binom{n+\beta-1}{n} = \frac{n^{\beta-1}}{\Gamma(\beta)}\big(1+o(1)\big), \quad n \to \infty,$$

it follows by an approximation argument that

$$\begin{aligned}\mathfrak{s}_n^\beta(\{a_i\}) &= \int_1^n \frac{(n-s)^{\beta-1}}{\Gamma(\beta)} s^{-1}\,ds \cdot \bigl(1+o(1)\bigr)\\ &= \frac{n^{\beta-1}\log n}{\Gamma(\beta)} \cdot \bigl(1+o(1)\bigr), \quad n\to\infty \qquad \text{(by (15))}.\end{aligned}$$

Similarly

$$\begin{aligned}\widehat{\{a_i\}}(e^{-\lambda}) &= \sum_{n=1}^{\infty} e^{-\lambda n} n^{-1} = \int_1^\infty e^{-\lambda t} t^{-1}\,dt \cdot \bigl(1+o(1)\bigr)\\ &= -\log\lambda \cdot \bigl(1+o(1)\bigr), \quad \lambda\downarrow 0 \qquad \text{(by (16))}.\end{aligned}$$

Hence the desired result follows from Theorem 3.2. □

Remark. If X is a Banach lattice and $\{x_n\}_{n=0}^\infty \subset X_+$, then statements (i), (ii) and (iii) in Theorem 3.4 are also equivalent. This follows from [2, Theorem 3.2]. (We note that statement (ii) in Theorem 3.4 implies that $\widehat{\{x_i\}}(r)$ exists for all $0<r<1$.)

Fact 3.5. *Let $\{x_n\}_{n=0}^\infty$ be a sequence in X. Consider the following three statements:*

(i) $x = \lim_{n\to\infty} \sum_{k=0}^n x_k$;
(ii) $x = \lim_{n\to\infty} \bigl(\Gamma(\beta)/(n+1)^{\beta-1}\bigr)\, \mathfrak{s}_n^\beta(\{x_i\})$ *for some/all* $\beta>1$;
(iii) $x = \lim_{r\uparrow 1} \widehat{\{x_i\}}(r) = \lim_{r\uparrow 1} \sum_{n=0}^\infty r^n x_n$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof: By letting $a_0 := 1$ and $a_n := 0$ for $n \ge 1$, the desired result follows as in Fact 2.5. We may omit the details. □

Remark. In general (iii) does not imply (i) in Fact 3.5. (For example let $x_n := (-1)^n$.) If $\{x_n\}$ satisfies $\|x_n\| = O(n^{-1})$, $n\to\infty$, or if X is a Banach lattice and $\{x_n\}\subset X_+$, then (iii) implies (i). (See Theorem 3.3 and [2, Theorem 4.4], respectively.)

4. A counterexample

The following example shows that condition (D) is essential in Theorem 3.2. (See also Example 3 in [6].)

Example. Define $\{a_n\}_{n=0}^\infty$ by

$$a_n := \begin{cases} n & \text{if } n\in\{2^k, 2^k+1\} \text{ for some } k\ge 1,\\ 0 & \text{otherwise.}\end{cases}$$

Thus $\{a_n\}$ does not satisfy condition (D). Next define $\{x_n\}_{n=0}^{\infty}$ by

$$x_n := \begin{cases} n & \text{if } \ n = 2^k \ \text{ for some } \ k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $x_n = O(a_n)$, $n \to \infty$. An elementary calculation yields

$$\frac{1}{2} = \liminf_{n\to\infty} \frac{\sum_{k=0}^{n} x_k}{\sum_{k=0}^{n} a_k} < \limsup_{n\to\infty} \frac{\sum_{k=0}^{n} x_k}{\sum_{k=0}^{n} a_k} = \frac{2}{3},$$

so that $\lim_{n\to\infty} \mathfrak{s}_n^1(\{x_i\})/\mathfrak{s}_n^1(\{a_i\})$ does not exist. Nevertheless we have

$$\frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \frac{\sum_{n=1}^{\infty} 2^n r^{2^n}}{(1+r)\sum_{n=1}^{\infty} 2^n r^{2^n} + r\sum_{n=1}^{\infty} r^{2^n}} \to \frac{1}{2} \quad \text{as } \ r \uparrow 1.$$

Remark. Let $0 \neq g \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $\widehat{g}(\lambda)$ exists for all $\lambda > 0$ and that $x = \lim_{\lambda\downarrow 0} \widehat{u}(\lambda)/\widehat{g}(\lambda)$ implies $x = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$ for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ with $\|u(t)\| = O(g(t))$, $t \to \infty$. Then in view of Theorem 2.2 it would be natural to ask the following question: Does the measure $\mu := g(t)\,dt$ satisfy condition (C) of Lemma 2.1? The author could not solve this problem.

References

[1] Arendt W., Batty C.J.K., Hieber M., Neubrander F., *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics Vol. 96, Birkhäuser, Basel, 2001.

[2] Chen J.-C., Sato R., *Ratio limit theorems and Tauberian theorems for vector-valued functions and sequences*, J. Math. Anal. Appl. **367** (2010), 108–115.

[3] Chen J.-C., Sato R., Shaw S.-Y., *Growth orders of Cesàro and Abel means of functions in Banach spaces*, Taiwanese J. Math. **14** (2010), 1201–1248.

[4] Emilion R., *Mean-bounded operators and mean ergodic theorems*, J. Funct. Anal. **61** (1985), 1–14.

[5] Li Y.-C., Sato R., Shaw S.-Y., *Convergence theorems and Tauberian theorems for functions and sequences in Banach spaces and Banach lattices*, Israel J. Math. **162** (2007), 109–149.

[6] Li Y.-C., Sato R., Shaw S.-Y., *Ratio Tauberian theorems for positive functions and sequences in Banach lattices*, Positivity **11** (2007), 433–447.

[7] Sato R., *On means of Banach-space-valued functions*, Math. J. Okayama Univ., to appear.

[8] Widder D.V., *An Introduction to Transform Theory*, Academic Press, New York and London, 1971.

[9] Zygmund A., *Trigonometric Series. Vol. I*, Cambridge University Press, Cambridge, 1959.

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