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On meager function spaces, network character and meager convergence in topological spaces

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Abstract. For a non-isolated point x of a topological space X let $\mathrm{nw}_\chi(x)$ be the smallest cardinality of a family $\mathcal{N}$ of infinite subsets of X such that each neighborhood $O(x) \subset X$ of x contains a set $N \in \mathcal{N}$. We prove that

- each infinite compact Hausdorff space X contains a non-isolated point x with $\mathrm{nw}_\chi(x) = \aleph_0$;
- for each point $x \in X$ with $\mathrm{nw}_\chi(x) = \aleph_0$ there is an injective sequence $(x_n)_{n\in\omega}$ in X that $\mathcal{F}$-converges to x for some meager filter $\mathcal{F}$ on ω;
- if a functionally Hausdorff space X contains an $\mathcal{F}$-convergent injective sequence for some meager filter $\mathcal{F}$, then for every path-connected space Y that contains two non-empty open sets with disjoint closures, the function space $C_p(X, Y)$ is meager.

Also we investigate properties of filters $\mathcal{F}$ admitting an injective $\mathcal{F}$-convergent sequence in $\beta\omega$.

Keywords: network character, meager convergent sequence, meager filter, meager space, function space

Classification: Primary 54A20, 54C35; Secondary 54E52

This paper was motivated by a question of the second author who asked if the function space $C_p(\omega^*, 2)$ is meager. Here $\omega^* = \beta\omega\backslash\omega$ is the remainder of the Stone-Čech compactification of the discrete space of finite ordinals ω and $2 = \{0, 1\}$ is the doubleton endowed with the discrete topology. According to Theorem 4.1 of [13] this question is closely related to the so-called meager convergence of sequences in ω^*.

A filter $\mathcal{F}$ on ω is *meager* if it is meager (i.e., of the first Baire category) in the power-set $\mathcal{P}(\omega) = 2^\omega$ endowed with the usual compact metrizable topology. The simplest example of a meager filter is the Fréchet filter $\mathfrak{Fr} = \{A \subset \omega : \omega \setminus A$ is finite$\}$ of all cofinite subsets of ω. By the Talagrand characterization [18], a free filter $\mathcal{F}$ on ω is meager if and only if $\xi(\mathcal{F}) = \mathfrak{Fr}$ for some finite-to-one function $\xi : \omega \to \omega$. A function $\xi : \omega \to \omega$ is *finite-to-one* if for each point $y \in \omega$ the preimage $\xi^{-1}(y)$ is finite and non-empty. A filter $\mathcal{F}$ on ω is defined to be ξ-*meager* for a surjective function $\xi : \omega \to \omega$ if $\xi(\mathcal{F}) = \mathfrak{Fr}$.

We shall say that for a filter $\mathcal{F}$ on ω, a sequence $(x_n)_{n\in\omega}$ of points of a topological space X $\mathcal{F}$-*converges* to a point $x_\infty \in X$ if for each neighborhood

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$O(x_\infty) \subseteq X$ of x_∞ the set $\{n \in \omega : x_n \in O(x_\infty)\}$ belongs to the filter $\mathcal{F}$. Observe that the usual convergence of sequences coincides with the $\mathfrak{Fr}$-convergence for the Fréchet filter $\mathfrak{Fr}$. The filter convergence of sequences has been actively studied both in Analysis [1], [4] and Topology [5]. A sequence $(x_n)_{n\in\omega}$ will be called *meager-convergent* if it is $\mathcal{F}$-convergent for some meager filter $\mathcal{F}$ on ω. A sequence $(x_n)_{n\in\omega}$ is called *injective* if $x_n \neq x_m$ for all $n \neq m$.

We shall prove that for a zero-dimensional Hausdorff space X the function space $C_p(X, 2)$ is meager if X contains an injective meager-convergent sequence. We recall that a topological space X is *functionally Hausdorff* if for any distinct points $x, y \in X$ there is a continuous function $\lambda : X \to \mathbb{I}$ such that $\lambda(x) \neq \lambda(y)$. Here $\mathbb{I} = [0, 1]$ is the unit interval. For topological spaces X, Y by $C_p(X, Y)$ we denote the space of continuous functions endowed with the topology of pointwise convergence.

Theorem 1. *Let X be a functionally Hausdorff space and let Y be a topological space that contains two open non-empty subsets with disjoint closures. Assume that X is zero-dimensional or Y is path-connected. If X contains an injective meager-convergent sequence, then the function space $C_p(X, Y)$ is meager.*

Proof: Let $(x_n)_{n\in\omega}$ be a sequence in X that $\mathcal{F}$-converges to $x_\infty \in X$ for some meager filter $\mathcal{F}$ in ω. Then there is a finite-to-one surjection $\xi : \omega \to \omega$ such that $\xi(\mathcal{F}) = \mathfrak{Fr}$. By our assumption, Y contains two non-empty open subsets W_0, W_1 with disjoint closures. For every $n \in \omega$ consider the subset $\mathcal{C}_n = \{f \in C_p(X, Y) : \forall i \in \{0, 1\}\ (f(x_\infty) \notin \overline{W}_i \Rightarrow \forall m \geq n\ \exists k \in \xi^{-1}(m)\ \ (f(x_k) \notin \overline{W}_i))\}$.

The fact that $C_p(X, Y)$ is meager will follow as soon as we check that $C_p(X, Y) = \bigcup_{n\in\omega} \mathcal{C}_n$ and each set $\mathcal{C}_n$ is nowhere dense in $C_p(X, Y)$.

To show that $C_p(X, Y) = \bigcup_{n\in\omega} \mathcal{C}_n$, fix any continuous function $f \in C_p(X, Y)$. Since $Y = (Y \setminus \overline{W}_0) \cup (Y \setminus \overline{W}_1)$, there is $i \in \{0, 1\}$ such that $f(x_\infty) \notin \overline{W}_i$. Since (x_n) is $\mathcal{F}$-convergent to x_∞ and $f^{-1}(Y \setminus \overline{W}_i)$ is an open neighborhood of x_∞, the set $F = \{n \in \omega : f(x_n) \notin \overline{W}_i\}$ belongs to the filter $\mathcal{F}$ and thus the image $\xi(F)$, being cofinite in ω, contains the set $\{m \in \omega : m \geq n\}$ for some $n \in \omega$. Then $f \in \mathcal{C}_n$ by the definition of the set $\mathcal{C}_n$.

Next, we show that each set $\mathcal{C}_n$ is nowhere dense in $C_p(X, Y)$. Fix any non-empty open set $\mathcal{U} \subseteq C_p(X, Y)$. Without loss of generality, $\mathcal{U}$ is a basic open set of the following form:

$$\mathcal{U} = \{f \in C_p(X, Y) : \forall z \in Z\ \ f(z) \in U_z\}$$

for some finite set $Z \subseteq X$ and non-empty open sets $U_z \subseteq Y$, $z \in Z$. We can additionally assume that $x_\infty \in Z$. We need to find a non-empty open set $\mathcal{V} \subseteq C_p(X, Y)$ such that $\mathcal{V} \subseteq \mathcal{U} \setminus \mathcal{C}_n$. If $\mathcal{U} \cap \mathcal{C}_n$ is empty, then put $\mathcal{V} = \mathcal{U}$. So we assume that $\mathcal{U} \cap \mathcal{C}_n$ contains some function f_0. For this function we can find $i \in \{0, 1\}$ such that $f_0(x_\infty) \notin \overline{W}_i$. Since $f_0(x_\infty) \in U_{x_\infty}$, we lose no generality assuming that $U_{x_\infty} \subseteq Y \setminus \overline{W}_i$.

Since the sequence $(x_n)_{n\in\omega}$ is injective, we can find $m \geq n$ such that the set $X_m = \{x_k : k \in \xi^{-1}(m)\}$ does not intersect the finite set Z. Choose any function $g : Z \cup X_m \to Y$ such that $g(z) = f_0(z)$ for all $z \in Z$ and $g(x) \in W_{1-i}$ for all $x \in X_m$.

We claim that the function g has a continuous extension $\bar{g} : X \to Y$. By our assumption, X is zero-dimensional or Y path-connected. In the first case we can find a retraction $r : X \to Z \cup X_m$ and put $\bar{g} = g \circ r$. If Y is path-connected, then take any injective function $\phi : g(Z \cup X_m) \to \mathbb{I}$ and extend the function $\phi \circ g : Z \cup X_m \to \mathbb{I}$ to a continuous map $\lambda : X \to \mathbb{I}$ using the functional Hausdorff property of X. Since Y is path-connected, the map $\phi^{-1} : (\phi \circ g)(Z \cup X_m) \to Y$ extends to a continuous map $\psi : \mathbb{I} \to Y$. Then the continuous map $\bar{g} = \psi \circ \lambda : X \to Y$ is a required continuous extension of g.

In both cases the set

$$\mathcal{V} = \{f \in C_p(X,Y) : \forall\, z \in Z\ f(z) \in U_z, \ \text{ and } \ \forall\, x \in X_m\ f(x) \in W_{1-i}\}$$

is an open neighborhood of $\bar{g}$ that lies in $\mathcal{U} \setminus \mathcal{C}_n$, witnessing that the set $\mathcal{C}_n$ is nowhere dense in $C_p(X,Y)$. □

Theorem 1 motivates the problem of detecting topological spaces that contain injective meager-convergent sequences. This will be done for spaces containing points with countable network character.

A family $\mathcal{N}$ of subsets of a topological space X is called a *π-network* at a point $x \in X$ if each neighborhood $O(x) \subset X$ of x contains some set $N \in \mathcal{N}$. If each set $N \in \mathcal{N}$ is infinite, then $\mathcal{N}$ will be called an *i-network* at x. An i-network at x exists if and only if each neighborhood of x in X is infinite. In this case let $\mathrm{nw}_\chi(x;X)$ denote the smallest cardinality $|\mathcal{N}|$ of an i-network $\mathcal{N}$ at x. If some neighborhood of x in X is finite, then let $\mathrm{nw}_\chi(x;X) = 1$. If the space X is clear from the context, then we write $\mathrm{nw}_\chi(x)$ instead of $\mathrm{nw}_\chi(x;X)$ and call this cardinal the *network character* of x in X. If X is a T_1-space, then $\mathrm{nw}_\chi(x) \geq \aleph_0$ if and only if the point x is not isolated in X. The cardinal $\mathrm{hnw}_\chi(x) = \sup\{\mathrm{nw}_\chi(x;A) : x \in A \subset X\}$ is called the *hereditary network character* at x. Points $x \in X$ with $\mathrm{hnw}_\chi(x) \leq \aleph_0$ are called *Pytkeev points*, see [11].

Theorem 2. *If some point x of a topological space X has $\mathrm{nw}_\chi(x) = \aleph_0$, then for each finite-to-one function $\xi : \omega \to \omega$ with $\lim_{n\to\infty} |\xi^{-1}(n)| = \infty$ there is an injective sequence $(x_n)_{n\in\omega}$ in X that $\mathcal{F}$-converges to x for some ξ-meager filter $\mathcal{F}$.*

Proof: Let $(N_i)_{i\in\omega}$ be a countable i-network at x. Since each set N_i is infinite, we can choose an injective sequence $(x_k)_{k\in\omega}$ in X such that for every $n \in \omega$ and $0 \leq i < |\xi^{-1}(n)|$ the set N_i meets the set $\{x_k : k \in \xi^{-1}(n)\}$.

It is clear that the sequence $(x_n)_{n\in\omega}$ $\mathcal{F}$-converges to x for the filter

$$\mathcal{F} = \big\{\{n \in \omega : x_n \in O(x)\} : O(x) \text{ is a neighborhood of } x \text{ in } X\big\}.$$

It remains to check that the filter $\mathcal{F}$ is ξ-meager. Given any neighborhood $O(x) \subset X$ of x we need to find $n \in \omega$ such that for every $m \geq n$ there is $k \in \xi^{-1}(m)$

with $x_k \in O(x)$. Since $(N_i)_{i\in\omega}$ is a network at x, there is $i \in \omega$ such that $N_i \subset O(x)$. Taking into account that $\lim_{n\to\infty} |\xi^{-1}(n)| = \infty$, find $n \in \omega$ such that $|\xi^{-1}(m)| > i$ for all $m \geq n$. Now the choice of the sequence (x_k) guarantees that for every $m \geq n$ there is $k \in \xi^{-1}(m)$ with $x_k \in N_i \subset O(x)$. $\square$

Theorem 2 shows that it is important to detect points x with countable network character $\mathrm{nw}_\chi(x)$. Let us recall that the *character* $\chi(x)$ (resp. the π*-character* $\pi\chi(x)$) of a point x in a topological space X is equal to the smallest cardinality of a neighborhood base (resp. a π-base) at x. A π*-base* at x is any π-network at x consisting of non-empty open subsets of X. These definitions imply the following simple:

Proposition 3. *For any non-isolated point x of a T_1-space X,*

(1) $\mathrm{nw}_\chi(x) \leq \chi(x)$*;*
(2) $\mathrm{nw}_\chi(x) \leq \pi\chi(x)$ *provided that x has a neighborhood containing no isolated point of X;*
(3) $\mathrm{nw}_\chi(x) = \aleph_0$ *if x is the limit of an injective $\mathfrak{Fr}$-convergent sequence in X.*

The following simple example shows that the usual convergence of the injective sequence in Proposition 3(3) cannot be replaced by the meager convergence. It also shows that Theorem 2 cannot be reversed.

Example 4. Let $\mathcal{F}$ be the meager filter on ω consisting of the sets $F \subset \omega$ such that

$$\lim_{n\to\infty} \frac{|F \cap [2^n, 2^{n+1})|}{2^n} = 1.$$

On the space $X = \omega \cup \{\infty\}$ consider the topology in which all points $n \in \omega$ are isolated while the sets $F \cup \{\infty\}$, $F \in \mathcal{F}$, are neighborhoods of ∞. It is clear that the sequence $x_n = n$, $n \in \omega$, $\mathcal{F}$-converges to ∞ in X. On the other hand, a simple diagonal argument shows that $\mathrm{nw}_\chi(\infty; X) > \aleph_0$.

Theorem 5. *Each infinite compact Hausdorff space X contains a point $x \in X$ with $\mathrm{nw}_\chi(x) = \aleph_0$.*

Proof: Theorem trivially holds if X contains a non-trivial convergent sequence. So we assume that X contains no non-trivial convergent sequence. Then X contains a closed subset $C \subset X$ that admits a continuous map $g : C \to \mathbb{I}$ onto the unit interval $\mathbb{I} = [0, 1]$, see [7, p.172]. Replacing C by a smaller subset, we can assume that the map $g : C \to \mathbb{I}$ is irreducible, which means that $g(C') \neq \mathbb{I}$ for any proper closed subset $C' \subset C$. Fix any countable base $\mathcal{B}$ of the topology of $\mathbb{I}$. The irreducibility of the map $g : C \to \mathbb{I}$ implies that the space C has no isolated points. Also the irreducibility of g implies that the countable family $\mathcal{N} = \{g^{-1}(U) : U \in \mathcal{B}\}$ of open infinite subsets of C is an i-network at each point $x \in C$. Consequently, $\mathrm{nw}_\chi(x) = \aleph_0$ for each point $x \in C$. $\square$

Theorems 1–5 imply:

Corollary 6. *For each infinite zero-dimensional compact Hausdorff space X and each topological space Y containing two non-empty open sets with disjoint closures the function space $C_p(X,Y)$ is meager. In particular, the function space $C_p(\omega^*,2)$ is meager.*

Also Theorems 2 and 5 imply

Corollary 7. *Let $\xi:\omega\to\omega$ be a finite-to-one function with $\lim_{n\to\infty}|\xi^{-1}(n)|=\infty$. Each infinite compact Hausdorff space X contains an injective $\mathcal{F}$-convergent sequence for some ξ-meager filter $\mathcal{F}$ on ω.*

In fact, the condition $\lim_{n\to\infty}|\xi^{-1}(n)|=\infty$ in Corollary 7 cannot be weakened.

Let us recall that an infinite subset A is called a *pseudointersection* of a family of sets $\mathcal{F}$ if $A\subseteq^* F$ for all $F\in\mathcal{F}$ where $A\subseteq^* F$ means that $A\setminus F$ is finite. If a sequence $(x_n)_{n\in\omega}$ in a topological space $\mathcal{F}$-converges to a point x_∞ for some filter $\mathcal{F}$ with infinite pseudointersection $A\subseteq\omega$, then the subsequence $(x_k)_{k\in A}$ converges to x_∞ in the standard sense.

Lemma 8. *Let I be a countable set and $C=\bigcup_{i\in I}C_i$, where the sets C_i are nonempty and mutually disjoint, and $\sup_{i\in I}|C_i|<\omega$. If $\mathcal{H}$ is a filter on C all of whose elements intersect all but finitely many C_i's, then $\mathcal{H}$ has an infinite pseudointersection.*

Proof: The proposition will be proved by induction on $n=\sup_{i\in I}|C_i|$. In case $n=1$ there is nothing to prove. Suppose that it is true for all $k<n$ and let I, $\{C_i:i\in I\}$, $\mathcal{H}$ be as above with $\max\{|C_i|:i\in I\}=n$. If for every $H\in\mathcal{H}$ the set $\{i\in I:|C_i\cap H|<n\}$ is finite, then C itself is a pseudointersection of $\mathcal{H}$. So suppose that $J=\{i\in I:|C_i\cap H_0|<n\}$ is infinite for some $H_0\in\mathcal{H}$. In this case we may use our inductive hypothesis for J, $\{C_i\cap H_0:i\in J\}$, $\mathcal{G}=\mathcal{H}\restriction(\bigcup_{i\in J}C_i\cap H_0)$, and $n-1$. Thus $\mathcal{G}$ has an infinite pseudointersection, and hence so does $\mathcal{H}$. □

Proposition 9. *If $\mathcal{F}$ is a ξ-meager filter on ω for some surjective function $\xi:\omega\to\omega$ with $\underline{\lim}_{n\to\infty}|\xi^{-1}(n)|<\infty$, then any sequence $(x_n)_{n\in\omega}$ in a topological space X that $\mathcal{F}$-converges to a point $x_\infty\in X$ contains a subsequence $(x_{n_k})_{k\in\omega}$ that converges to x_∞.*

Proof: Choose an infinite set $I\subseteq\omega$ such that $\sup_{i\in I}|\xi^{-1}(i)|<\omega$. Let $C_i=\xi^{-1}(i)$ for every $i\in I$, $C=\bigcup_{i\in I}C_i$ and $\mathcal{H}=\{F\cap C:F\in\mathcal{F}\}$. According to Lemma 8 there exists an infinite set $D\subseteq C$ such that $D\subseteq^* H$ for every $H\in\mathcal{H}$. Then the subsequence $(x_i)_{i\in D}$ converges to x_∞. □

Now let us compare two facts:

(1) the compact Hausdorff space $\beta\omega$ contains no injective $\mathfrak{Fr}$-convergent sequences;
(2) each infinite compact Hausdorff space X contains an injective $\mathcal{F}$-convergent sequence for some meager filter $\mathcal{F}$.

These two facts suggest a problem of finding the borderline between filters $\mathcal{F}$ that admit an injective $\mathcal{F}$-convergent sequence in $\beta\omega$ and filters that admit no such sequences. We hope that this borderline passes near analytic filters. Let us recall the definitions of some properties of filters.

A filter $\mathcal{F}$ is *analytic* (resp. an *F_σ-filter*, *$F_{\sigma\delta}$-filter*) if $\mathcal{F}$ is an analytic subset (resp. F_σ-subset, $F_{\sigma\delta}$-subset) of the power-set $\mathcal{P}(\omega) = 2^\omega$ endowed with the natural compact metrizable topology.

A filter $\mathcal{F}$ is *measurable* (resp. *null*) if is it measurable (resp. has measure zero) with respect to the Haar measure on the Cantor cube 2^ω considered as the countable product of 2-element groups. It is well-known that a filter is measurable if and only if it is null. The relations between meager and null filters are not trivial and were investigated in [18] and [2]. Since each analytic filter is meager and null, we get the following chain of properties of filters:

$$F_\sigma \ \Rightarrow \ \text{analytic} \ \Rightarrow \ \text{meager \& null}.$$

We are going to show that some meager and null filter $\mathcal{F}$ admits an injective $\mathcal{F}$-convergent sequence in $\beta\omega$ while no F_σ-filer $\mathcal{F}$ admits such a sequence. The latter fact holds more generally for analytic P^+-filters.

A filter $\mathcal{F}$ on ω is called a *P-filter* (resp. a *P^+-filter*) if each countable subfamily $\mathcal{C} \subset \mathcal{F}$ has a pseudointersection A that belongs to $\mathcal{F}$ (resp. to $\mathcal{F}^+$). Here

$$\mathcal{F}^+ = \{A \subset \omega : \forall\, F \in \mathcal{F}\ A \cap F \neq \emptyset\}$$

coincides with the union of all filters that contain $\mathcal{F}$. It is clear that each P-filter is a P^+-filter. In particular, the Fréchet filter $\mathcal{F}$ is both a P-filter and P^+-filter.

For a filter $\mathcal{F}$ on ω by $\chi(\mathcal{F})$ we denote its *character*. It is equal to the smallest cardinality $|\mathcal{B}|$ of the base $\mathcal{B} \subset \mathcal{F}$ that generates $\mathcal{F}$ in the sense that $\mathcal{F} = \{F \subset \omega : \exists B \in \mathcal{B}\ \ B \subset F\}$. It is well-known that the character of each free ultrafilter on ω is uncountable. The uncountable cardinal $\mathfrak{u} = \min\{\chi(\mathcal{U}) : \mathcal{U} \in \beta\omega \setminus \omega\}$ is called the *ultrafilter number*, see [3], [20]. The *dominating number* $\mathfrak{d}$ is the smallest cardinality $|D|$ of a cofinal subset D in the partially ordered set $(\omega^\omega, \leq)$, see [3], [20]. By Ketonen's Theorem [10], *each filter $\mathcal{F}$ on ω with character $\chi(\mathcal{F}) < \mathfrak{d}$ is a P^+-filter.*

Now we can establish some properties of filters $\mathcal{F}$ admitting injective $\mathcal{F}$-convergent sequences in $\beta\omega$.

Theorem 10. *Assume that a filter $\mathcal{F}$ admits an injective $\mathcal{F}$-convergent sequence $(x_n)_{n\in\omega}$ in $\beta\omega$.*

1. *If $\mathcal{F}$ is a P^+-filter, then for some set $A \in \mathcal{F}^+$ the filter $\mathcal{F}|A = \{F \cap A : F \in \mathcal{F}\}$ on A is an ultrafilter.*
2. *$\chi(\mathcal{F}) \geq \min\{\mathfrak{d}, \mathfrak{u}\}$;*
3. *$\mathcal{F}$ is not an analytic P^+-filter;*
4. *$\mathcal{F}$ is not an F_σ-filter.*

Proof: 1. Assume that $\mathcal{F}$ is a P^+-filter. Let x_∞ be the $\mathcal{F}$-limit of the $\mathcal{F}$-convergent sequence $(x_n)_{n\in\omega}$ in $\beta\omega$. Since the sequence (x_n) is injective, there is $m\in\omega$ such that for every $n\geq m$ $x_n\neq x_\infty$ and hence we can fix a neighborhood U_n of x_∞ whose closure does not contain the point x_n. Since the sequence (x_k) $\mathcal{F}$-converges to x_∞, for every $n\geq m$ the set $F_n=\{k\in\omega : x_k\in U_n\}$ belongs to the filter $\mathcal{F}$. Since $\mathcal{F}$ is a P^+-filter, the sequence $(F_n)_{n\geq m}$ has a pseudointersection $A\in\mathcal{F}^+$. It follows from the choice of the neighborhoods U_n that the set $\{x_n\}_{n\in A}$ is discrete in $\beta\omega$ and the sequence $(x_n)_{n\in A}$ is $\mathcal{F}|A$-convergent to x_∞. By Rudin's Theorem [16], the map $f:A\to\beta\omega$, $f:n\mapsto x_n$, has injective Stone-Čech extension $\beta f:\beta A\to\beta\omega$, which implies that the filter $\mathcal{F}|A$ is an ultrafilter.

2. If $\chi(\mathcal{F})<\min\{\mathfrak{d},\mathfrak{u}\}$, then $\chi(\mathcal{F})<\mathfrak{d}$ and by the Ketonen's Theorem [10] $\mathcal{F}$ is a P^+-filter. By the preceding statement, $\mathcal{F}|A$ is an ultrafilter for some set $A\in\mathcal{F}^+$. Consequently,

$$\mathfrak{u}\leq\chi(\mathcal{F}|A)\leq\chi(\mathcal{F})<\mathfrak{u}$$

and this is a desired contradiction.

3. If $\mathcal{F}$ is an analytic P^+-filter, then by the first statement, $\mathcal{F}|A$ is an ultrafilter for some subset $A\in\mathcal{F}^+$. On the other hand, the filter $\mathcal{F}|A$ is analytic being a continuous image of the analytic filter $\mathcal{F}$. So, $\mathcal{F}|A$ cannot be an ultrafilter.

4. Assume that $\mathcal{F}$ is an F_σ-filter. In order to apply the preceding statement, it suffices to show that $\mathcal{F}$ is a P^+-filter. This is done in the following lemma. □

Lemma 11. *Each F_σ-filter $\mathcal{F}$ on ω is a P^+-filter.*

Proof: According to a result of Mazur [12] (see also [17]), for the F_σ-filter $\mathcal{F}$ there exists a lower semi-continuous submeasure ϕ on $\mathcal{P}(\omega)$ such that $\mathcal{F}=\{A\subset\omega:\phi(\omega\setminus A)<\infty\}$. Since $\mathcal{F}\neq\mathcal{P}(\omega)$, $\phi(\omega)=\infty$ and the subadditivity of ϕ implies that $\phi(F)=\infty$ for all $F\in\mathcal{F}$. It follows from $\mathcal{F}=\{A\subset\omega:\phi(\omega\setminus A)<\infty\}$ that a set $A\subset\omega$ belongs to $\mathcal{F}^+$ if and only if $\phi(A)=\infty$.

To show that $\mathcal{F}$ is a P^+-filter, fix any decreasing sequence of sets $(A_k)_{k\in\omega}$ in $\mathcal{F}$. Let $n_0=0$ and by induction construct an increasing sequence of positive integers $(n_k)_{k\in\omega}$ such that $\phi([n_k,n_{k+1})\cap A_k)>k$ for every $k\in\omega$. Then the set $A=\bigcup_{k\in\omega}[n_k,n_{k+1})\cap A_k$ is a pseudointersection of $(A_k)_{k\in\omega}$ and belongs to the family $\mathcal{F}^+$ as $\phi(A)=\infty$. □

Let us remark that Lemma 11 cannot be generalized to $F_{\sigma\delta}$-filters. The following example was suggested to the authors by Jonathan Verner.

Example 12. The filter

$$\mathfrak{Fr}\otimes\mathfrak{Fr}=\big\{A\subset\omega\times\omega:\big\{n\in\omega:\{m\in\omega:(n,m)\in A\}\in\mathfrak{Fr}\big\}\in\mathfrak{Fr}\big\}$$

on $\omega\times\omega$ is an $F_{\sigma\delta}$ but not P^+.

Looking at Theorem 10, it is natural to ask the following

Question 13. *Does $\beta\omega$ contain an injective $\mathcal{F}$-convergent sequence for some analytic filter $\mathcal{F}$?*

On the other hand, we have the following fact:

Theorem 14. *Each infinite compact Hausdorff space X contains an injective $\mathcal{F}$-convergent sequence for some meager and null filter $\mathcal{F}$.*

Proof: Choose any finite-to-one function $\xi : \omega \to \omega$ such that

$$\lim_{n\to\infty} |\xi^{-1}(n)| = \infty \quad \text{and} \quad \prod_{n\in\omega}(1 - 2^{-|\xi^{-1}(n)|}) = 0.$$

By Corollary 7, any infinite compact Hausdorff space X contains an injective $\mathcal{F}$-convergent sequence for some ξ-meager filter $\mathcal{F}$. It is clear that $\mathcal{F}$ is meager. It remains to check that $\mathcal{F}$ is null. The filter $\mathcal{F}$, being ξ-meager, lies in the union $\bigcup_{n\in\omega}\mathcal{F}_n$ where $\mathcal{F}_n = \{A \subset \omega : \forall\, k \geq n \; A \cap \xi^{-1}(k) \neq \emptyset\}$. It suffices to prove that each set $\mathcal{F}_n$ has Haar measure zero. Observe that the set $\mathcal{F}_n$ can be identified with the product $\prod_{k\geq n}(\mathcal{P}(\varphi^{-1}(k)) \setminus \{\emptyset\})$, which has Haar measure

$$\prod_{k\geq n} \frac{2^{|\varphi^{-1}(k)|} - 1}{2^{|\varphi^{-1}(k)|}} = \prod_{k\geq n}(1 - 2^{-|\varphi^{-1}(k)|}) = 0.$$

□

Remark 15. After writing this paper the authors learned from V. Tkachuk that the meager property of the function space $C_p(\omega^*, 2)$ was also established by E.G. Pytkeev in his Dissertation [15, 3.24]. Game characterizations of topological spaces X with Baire function space $C_p(X, \mathbb{R})$ were given in [9], [19] and [14].

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References

[1] Aviles Lopez A., Cascales S., Kadets V., Leonov A., *The Schur l_1 theorem for filters*, Zh. Mat. Fiz. Anal. Geom. **3** (2007), no. 4, 383–398.

[2] Bartoszynski T., Goldstern M., Judah H., Shelah S., *All meager filters may be null*, Proc. Amer. Math. Soc. **117** (1993), no. 2, 515–521.

[3] van Douwen E., *The integers and topology*, in: Handbook of Set-Theoretic Topology (K. Kunen, J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 111–167.

[4] Ganichev M., Kadets V., *Filter convergence in Banach spaces and generalized bases*, in General Topology in Banach Spaces (T. Banakh, ed.), Nova Sci. Publ., Huntington, NY, 2001, pp. 61–69.

[5] García-Ferreira S., Malykhin V., A. Tamariz-Mascarúa A., *Solutions and problems on convergence structures to ultrafilters*, Questions Answers Gen. Topology **13** (1995), no. 2, 103–122.

[6] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[7] Hart K.P., *Efimov's problem*, in: Open Problems in Topology II (E. Pearl, ed.), Elsevier, Amsterdam, 2007, 171–177.

[8] Lašnev N., *On continuous decompositions and closed mappings of metric spaces*, Dokl. Akad. Nauk SSSR **165** (1965), 756–758 (in Russian).

[9] Lutzer D.J., McCoy R.A., *Category in function spaces. I.* Pacific J. Math. **90** (1980), no. 1, 145–168.

[10] Ketonen J., *On the existence of P-points in the Stone-Cech compactification of integers*, Fund. Math. **92** (1976), no. 2, 91–94.

[11] Malykhin V.I., Tironi G., *Weakly Fréchet-Urysohn and Pytkeev spaces*, Topology Appl. **104** (2000), 181–190.

[12] Mazur K., *F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$*, Fund. Math. **138** (1991), no. 2, 103–111.

[13] Mykhaylyuk V., *On questions connected with Talagrand's problem*, Mat. Stud. **29** (2008), 81–88.

[14] Pytkeev E.G., *The Baire property of spaces of continuous functions*, Mat. Zametki **38** (1985), no. 5, 726–740.

[15] Pytkeev E.G., *Spaces of continuous and Baire functions in weak topologies*, Doktor Sci. Dissertation, Ekaterinburg, 1993 (in Russian).

[16] Rudin M.E., *Types of ultrafilters*, 1966 Topology Seminar (Wisconsin, 1965), pp. 147–151, Ann. of Math. Studies, No. 60, Princeton Univ. Press, Princeton, N.J.

[17] Solecki S., *Analytic ideals*, Bull. Symbolic Logic **2** (1996), no. 3, 339–348.

[18] Talagrand M., *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67** (1980), no. 1, 13–43.

[19] Tkachuk V., *Characterization of the Baire property of $C_p(X)$ by the properties of the space X*, Cardinal Invariants and Mappings of Topological Spaces, Izhevsk, 1984, pp. 76–77 (in Russian).

[20] Vaughan J., *Small uncountable cardinals and topology*, in: Open Problems in Topology (J. van Mill, G. Reed, eds.), North-Holland, Amsterdam, 1990, pp. 195–218.

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