Some fixed point theorems and existence of weak solutions of Volterra integral equation under Henstock-Kurzweil-Pettis integrability

Afif Ben Amar

Abstract. In this paper we examine the set of weakly continuous solutions for a Volterra integral equation in Henstock-Kurzweil-Pettis integrability settings. Our result extends those obtained in several kinds of integrability settings. Besides, we prove some new fixed point theorems for function spaces relative to the weak topology which are basic in our considerations and comprise the theory of differential and integral equations in Banach spaces.

Keywords: fixed point theorems, Henstock-Kurzweil-Pettis integral, Volterra equation, measure of weak noncompactness

Classification: 47H10, 28B05, 45D05, 45N05, 26A39

1. Introduction

The resolution of differential and integral problems in a Banach space relative to the strong topology, has been the subject of many papers (see [23], [25], [45]). Besides, some results have been obtained for equations in Banach spaces relative to the weak topology (see [4], [7], [8], [9], [20], [21], [24], [28], [35], [43]). Some examinations of these problems were given under hypotheses of Lebesgue integrability on the real line, respectively Bochner, weak Riemann integral and Pettis integral, in the vector case. Recently, for problems involving highly oscillating functions, many authors have examined the existence of solutions under Henstock-Kurzweil [5], [14], [15], [32], [33], [36], [37], [38], [40], [41] and Henstock-Kurzweil-Pettis integrability [1], [6], [30], [31], [39], [42]. Motivated by those examinations, we first prove some Sadovskii fixed point type results for function spaces which guarantee an existence result for the general operator equation

$$x(t) = Fx(t), \quad t \in [0,T],\ T > 0$$

relative to the weak uniform convergence topology which is not metrizable. These results improve and extend those in [28]. Then by using those results, we give existence criteria of weak solutions for the Volterra integral equation

$$x(t) = h(t) + \int_0^t K(t,s) f(s,x(s))\, ds \quad \text{on} \ \ [0,T],\ T > 0$$

involving the Henstock-Kurzweil-Pettis integral and we prove the existence of a non-empty and compact set of weak solutions on a closed subinterval of $[0, T]$. The main tools used in our study are associated with the techniques of measure of weak noncompactness, properties of the weak uniform convergence topology real bounded variation functions and Henstock-Kurzweil-Pettis integrals. This result generalizes and improves the corresponding results in [4], [28]. We notice in our study that the techniques developed in [42] which are based on fixed point theory for weakly sequentially continuous mappings defined on domains of a metrizable locally convex topological vector space are not useful in establishing existence principles for the problem we are interested in. The major problem encountered is that we are working in function spaces under weak uniform convergence topology features. However, we know that weak uniform convergence topology is not metrizable. Also our theory provides an unified line to the theory of differential and integral equations in Banach spaces relative to the weak topology and under several well known kinds of integrability settings.

2. Preliminaries and fixed point results

The purpose of this section is to give some notations and preliminaries and state some fixed point results for function spaces which will be used throughout this paper.

Let $I = [0, T]$ be an interval of the real line equipped with the usual topology. Let E be a Banach space with norm $\|\cdot\|$. E^* will denote the dual of E and E_w will denote the space E when endowed with its weak topology. On the space $C(I, E_w)$ of continuous functions from I to E_w we define a topology as follows. Let $\mathrm{Fin}(E^*)$ be the class of all non-empty and finite subsets in E^*, Let $\mathcal{O} \in \mathrm{Fin}(E^*)$ and let us define $\|\cdot\|_{\mathcal{O}} : C(I, E_w) \longrightarrow \mathbb{R}_+$ by

$$\|f\|_{\mathcal{O}} := \sup_{t \in I} \sup_{x^* \in \mathcal{O}} |x^*(f(t))|$$

for each $f \in C(I, E_w)$. One may see that $\{\|\cdot\|_{\mathcal{O}}; \mathcal{O} \in \mathrm{Fin}(E^*)\}$ is a family of seminorms on $C(I, E_w)$ which defines a topology of a locally convex, separated space, called the uniform weak convergence topology. We emphasize that this topology (except for the case in which E is finite dimensional) is not metrizable. We will denote by $C_w(I, E)$ the space of weakly continuous functions on I with the topology of weak uniform convergence. For more details see [29]. Moreover, we will denote by β the De Blasi measure of weak noncompactness [10]. Recall that for any nonvoid, bounded subset X of E,

$$\begin{aligned}\beta(X) = \inf\{\varepsilon > 0 : &\text{ there exists a weakly compact set } Y \\ &\text{ such that } X \subset Y + \varepsilon B_E\},\end{aligned}$$

where B_E is the closed unit ball of E. For convenience, we bring back some properties of β:

(i) $X_1 \subset X_2$ implies $\beta(X_1) \leq \beta(X_2)$;

(ii) $\beta(X_1) = 0$ iff $\overline{X_1^w}$ is weakly compact, here $\overline{X_1^w}$ is the weak closure of X_1 in E;
(iii) $\beta(X_1) = \beta(\overline{X_1^w})$;
(iv) $\beta(X_1 \cup X_2) = \max\{\beta(X_1), \beta(X_2)\}$;
(v) $\beta(\lambda X_1) = \lambda\beta(X_1)$ for all $\lambda > 0$;
(vi) $\beta(\mathrm{conv}(X_1)) = \beta(X_1)$;
(vii) $\beta(X_1 + X_2) \leq \beta(X_1) + \beta(X_2)$;
(viii) $\beta t(\bigcup_{|\lambda|\leq h} \lambda X_1) = h\beta(X_1)$.

Definition 2.1. A function $f : I \times E \longrightarrow E$ is said to be weakly-weakly continuous at (t_0, x_0) if given $\varepsilon > 0$ and $x^* \in E^*$, there exists $\delta > 0$ and a weakly open set U containing x_0 such that $|x^*(f(t,x) - f(t_0,x_0))| < \varepsilon$ whenever $|t - t_0| < \delta$ and $x \in U$. ♢

Definition 2.2. A family $\mathcal{F} = \{f_i, i \in \mathcal{I}\}$ (where $\mathcal{I}$ is some index set) of E^I is said to be weakly equicontinuous if given $\varepsilon > 0$, $x^* \in E^*$ there exists $\delta > 0$ such that, for $t, s \in I$, if $|t - s| < \delta$, then $|x^*(f_i(t) - f_i(s))| < \varepsilon$ for all $i \in \mathcal{I}$. ♢

Next we recall the Brouwer-Schauder-Tychonoff fixed point theorem.

Theorem 2.1 ([2]). *Let K be a non-empty compact convex subset of a locally convex Hausdorff space and let $f : K \longrightarrow K$ be a continuous function. Then the set of fixed points of f is compact and non-empty.* ♢

Our next fixed point result is motivated by the weak sequential compactness of weakly compact subsets of a Banach space.

Theorem 2.2 ([3]). *Let Q be a non-empty, convex closed set in a Banach space E. Assume $F : Q \longrightarrow Q$ is a weakly sequentially continuous map which is also β-condensing (i.e., $\beta(F(X)) < \beta(X)$ for all bounded subsets $X \subset Q$ such that $\beta(X) \neq 0$). In addition, suppose that $F(Q)$ is bounded. Then F has a fixed point.* ♢

The next lemma is basic for our study.

Lemma 2.1. (a) *Let V be a bounded subset of $C(I,E)$. Then*

$$\sup_{t\in I} \beta(V(t)) \leq \beta(V)$$

where $V(t) = \{x(t) \ : \ x \in V\}$.

(b) *Let $V \subseteq C(I,E)$ be a family of strongly equicontinuous functions. Then*

$$\beta(V) = \sup_{t\in I} \beta(V(t)) = \beta(V(I))$$

where $V(I) = \bigcup_{t\in I}\{x(t) : x \in V\}$, and the function $t \longmapsto \beta(V(t))$ is continuous.

♢

Proof: For a proof of claim (a), see [28]. For a proof of claim (b), we refer to [20], [24]. □

We are now ready to state our fixed point results.

Theorem 2.3. *Let E be a Banach space with Q a non-empty subset of $C(I,E)$. Assume also that Q is a closed convex subset of $C_w(I,E)$, $F : Q \longrightarrow Q$ is continuous with respect to the weak uniform convergence topology, $F(Q)$ is bounded and F is β-condensing (i.e., $\beta(F(X)) < \beta(X)$ for all bounded subsets $X \subset Q$ such that $\beta(X) \neq 0$). In addition, suppose the family $F(Q)$ is weakly equicontinuous. Then the set of fixed points of F is non-empty and compact in $C_w(I,E)$.* ♢

Proof: Let $\mathcal{F}$ the fixed points set of F in Q. We claim that $\mathcal{F}$ is non-empty. Indeed, let $x_0 \in F(Q)$ and $\mathcal{G}$ be the family of all closed bounded convex subsets D of $C(I,E)$ such that $x_0 \in D$ and $F(D) \subset D$. Obviously $\mathcal{G}$ is non-empty, since $\overline{\mathrm{conv}}(F(Q)) \in \mathcal{G}$ (the closed convex hull of $F(Q)$ in $C(I,E)$). We denote $K = \bigcap_{D\in\mathcal{G}} D$. We have that K is closed convex and $x_0 \in K$. If $x \in K$, then $F(x) \in D$ for all $D \in \mathcal{G}$ and hence $F(K) \subset K$. Therefore we have that $K \in \mathcal{G}$. We claim that K is a compact subset of $C_w(I,E)$. Denoting by $K_* = \overline{\mathrm{conv}}(F(K) \cup \{x_0\})$ (the closed convex hull of $F(K)$ in $C(I,E)$), we have $K_* \subset K$, which implies that $F(K_*) \subset F(K) \subset K_*$. Therefore $K_* \in \mathcal{G}$, $K \subset K_*$. Hence $K = K_*$. Clearly K is bounded and if $\beta(K) \neq 0$, we obtain

$$\beta(K) = \beta(\overline{\mathrm{conv}}(F(K) \cup \{x_0\}) \leq \beta(\mathrm{conv}(F(K) \cup \{x_0\})) \leq \beta(F(K)) < \beta(K),$$

which is a contradiction, so $\beta(K) = 0$. Since K is a weakly closed subset of $C(I,E)$ (notice that a convex subset of a Banach space is closed iff it is weakly closed), K is a weakly compact subset of $C(I,E)$. We claim that K is closed in $C_w(I,E)$. To see this, let $\mathcal{S} = E^I$ be endowed with the product topology. We consider $C(I,E)$ as a vector subspace of $\mathcal{S}$. Hence its weak topology is the topology induced by the weak topology of $\mathcal{S}$. Suppose (x_α) is a net in K with $x_\alpha \longrightarrow z$ in $C_w(I,E)$. Then $x_\alpha(t)$ tends weakly to $z(t)$ for each $t \in I$. For each $t \in I$, let $H_t = \{x_\alpha(t)\}$. Clearly the weak closure of H_t is a weakly compact subset of E. But the weak topology of E^I is the product topology of the weak topology of E. Hence the subset $H = \prod_{t\in I} \overline{H_t^w}$ is a weakly compact subset of $\mathcal{S}$ by the Tychonoff theorem. Obviously the subset $\{x_\alpha, z\} \subset H$. The set $H \cap K$ is weakly compact in K, hence in $C(I,E)$. Using the fact that for each $x^* \in E^*$ and $t \in I$ the point evaluation mapping $y \longmapsto x^*y(t)$ is a continuous linear functional on $C(I,E)$, we get $z \in K$. Now we apply the Arzela-Ascoli Theorem [19, p. 233]. Because the family $F(Q)$ is weakly equicontinuous, we have by [13, Lemma 6.2] that the family $\overline{\mathrm{conv}}(F(Q))$ (the closure is taken in $C_w(I,E)$) is weakly equicontinuous and therefore, K is weakly equicontinuous. Thus, it remains to show that for each $t \in I$, the set $K(t) = \{x(t), x \in K\}$ is weakly relatively compact in E. By Lemma 2.1(a), $\beta(K(t)) \leq \beta(K)$. Then $\beta(K(t)) = 0$ for each $t \in I$. Thus for each $t \in I$, $K(t)$ is weakly relatively compact in E. Now we apply Theorem 2.1 with the locally convex Hausdorff space $C_w(I,E)$ to obtain that $\mathcal{F} \neq \emptyset$. It remains to show that

$\mathcal{F}$ is compact in $C_w(I,E)$. To do this, we consider $\mathcal{H}$ to be the family of all closed bounded convex subsets D of $C(I,E)$ such that $\mathcal{F} \subset D$ and $F(D) \subset D$. Obviously $\mathcal{H}$ is non-empty, since $\overline{\text{conv}}(F(Q)) \in \mathcal{H}$ (the closed convex hull of $F(Q)$ in $C(I,E)$). We denote $R = \bigcap_{D\in\mathcal{H}} D$. Arguing as above, we prove that R is compact in $C_w(I,E)$, $F(R) \subset R$ and $\mathcal{F} \subset R$. Finally, applying Theorem 2.1 again, we deduce that $\mathcal{F}$ is compact. $\square$

Remark 2.1. Theorem 2.3 extends and improves Theorem 2.2 in [28] (note that in [28], Q was a closed bounded subset of $C(I,E)$, whereas here Q is only a subset of $C(I,E)$).

Corollary 2.1. *Let E be a Banach space and Q be a non-empty subset of $C(I,E)$. Also assume that Q is a closed convex subset of $C_w(I,E)$, $F : Q \longrightarrow Q$ is continuous with respect to the weak uniform convergence topology, $F(Q)$ is bounded and F is β-condensing. In addition, suppose the family $F(Q)$ is strongly equicontinuous. Then the set of fixed points of F is non-empty and compact in $C_w(I,E)$.* $\diamondsuit$

PROOF: Thanks to Theorem 2.3, it suffices to prove that the family $F(Q)$ is weakly equicontinuous which is the case. $\square$

Corollary 2.2. *Let E be a Banach space and Q be a non-empty subset of $C(I,E)$. Also assume that Q is a closed convex subset of $C_w(I,E)$, $F : Q \longrightarrow Q$ is continuous with respect to the weak uniform convergence topology and the family $F(Q)$ is bounded and strongly equicontinuous. In addition, suppose that*

for each $t \in I$, $F(Q)(t)$ is relatively weakly compact in E.

Then the set of fixed points of F is non-empty and compact in $C_w(I,E)$. $\diamondsuit$

PROOF: We claim that the set $F(Q)$ is relatively weakly compact in $C(I,E)$. Indeed, the family $F(Q)$ of $C(I,E)$ is bounded and strongly equicontinuous, so by Lemma 2.1, we have $\beta(F(Q)) = \sup_{t\in I} \beta(F(Q)(t)) = 0$. Therefore $F(Q)$ is a relatively weakly compact subset of $C(I,E)$. Accordingly, F is β-condensing. The result now follows from Corollary 2.1. $\square$

Remark 2.2. If $F(Q)$ is bounded and E is reflexive, then for each $t \in I$, $F(Q)(t)$ is relatively weakly compact in E since a subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology. $\diamondsuit$

We close this section by stating a fixed point theorem for weakly sequentially continuous mappings.

Theorem 2.4. *Let E a Banach space and Q be a non-empty, convex closed set in E. Assume $F : Q \longrightarrow Q$ is a weakly sequentially continuous map and the family $F(Q)$ is bounded and strongly equicontinuous. In addition, suppose that*

for each $t \in I$, $F(Q)(t)$ is relatively weakly compact in E.

Then F has a fixed point. ♢

Proof: Arguing as in the proof of Corollary 2.2, we obtain that $F(Q)$ is a relatively weakly compact subset of $C(I,E)$. Hence, F is β-condensing. It suffices now to apply Theorem 2.2 to prove the result. □

Remark 2.3. (a) Theorem 2.4 extends and improves Theorem 3.2 in [28].
(b) It can be proved that the set of fixed points of F is weakly compact in $C(I,E)$. ♢

3. Henstock-Kurzweil-Pettis integrals

In this section, we introduce the concept of Henstock-Kurzweil-Pettis integrability and give some related facts which are useful in Section 4. Concerning basic definitions, we refer to [22] or [34]. Throughout this section and Section 4, E will be considered as a real Banach space.

Definition 3.1. A function $f : I \longrightarrow E$ is said to be Henstock-Kurzweil-integrable, or simply HK-integrable on I, if there exists $w \in E$ with the following property: for $\varepsilon > 0$ there exists a gauge δ on I such that $\|\sigma(g,\mathcal{P}) - w\| < \varepsilon$ for each δ-fine Perron partition $\mathcal{P}$ of I. We set $w = (\mathrm{HK}) \int_0^T f(s)\,ds$. ♢

Remark 3.1. This definition includes the generalized Riemann integral (see [17]). In a special case, when δ is a constant function, we get the Riemann integral. ♢

The following result states that the HK-integrability for real functions is preserved under multiplication by functions of bounded variation.

Lemma 3.1 ([18, Theorem 12.21])**.** *Let $f : I \longrightarrow \mathbb{R}$ be an HK-integrable function and let $g : I \longrightarrow \mathbb{R}$ be of bounded variation. Then fg is HK-integrable.* ♢

Let us recall the following integration by parts result inspired from the previous lemma and [18, Theorem 12.8]:

Lemma 3.2. *$f : [a,b] \longrightarrow \mathbb{R}$ be HK-integrable function and let $g : I \longrightarrow \mathbb{R}$ be of bounded variation. Then, for every $t \in [a,b]$*

$$(HK) \int_a^t f(s)g(s)\,ds = g(t)(HK) \int_a^t f(s)\,ds - \int_a^t \left((HK) \int_a^s f(\tau)\,d\tau \right) dg(s),$$

the last integral being of Riemann-Stieltjes type. ♢

The generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Henstock-Kurzweil integrability produces the Henstock-Kurzweil-Pettis integral (for the definition of Pettis integral see [11]).

Definition 3.2 ([6])**.** A function $f : I \longrightarrow E$ is said to be Henstock-Kurzweil-Pettis integrable, or simply HKP-integrable, on I if there exists a function $g : I \longrightarrow E$ with the following properties:

(i) $\forall\, x^* \in E^*$, $x^* f$ is Henstock-Kurzweil integrable on I;

(ii) $\forall\, t \in I, \forall\, x^* \in E^*, \, x^* g(t) = (\mathrm{HK}) \int_0^t x^* f(s)\, ds$.

This function g will be called a primitive of f and by $g(T) = \int_0^T f(t)\, dt$ we will denote the Henstock-Kurzweil-Pettis integral of f on the interval I. $\diamondsuit$

Remark 3.2. (i) Any HK-integrable function is HKP-integrable. The converse is not true (see an example in [16]). Then the family of all Henstock-Kurzweil-Pettis integrable functions is larger than the family of all Henstock-Kurzweil integrable ones.
(ii) Since each Lebesgue integrable function is HK-integrable, we find that any Pettis integrable function is HKP-integrable. The converse is not true (see also [16]). $\diamondsuit$

In the sequel we will investigate some properties of the HKP integral which are important in the next part of the paper.

Theorem 3.1. *Let $f : [a, b] \longrightarrow E$ be HKP-integrable on $[a, b]$. Then*

(a) *for any compact interval J of $[a, b]$, f is HKP-integrable on J;*
(b) *if $a_1 = a < a_2 < \ldots < a_n = b$, then $\int_a^b f(s)\, ds = \sum_{i=2}^n \int_{a_{i-1}}^{a_i} f(s)\, ds$.* $\diamondsuit$

PROOF: The proof is straightforward. $\square$

Theorem 3.2 (Mean value theorem [6]). *If the function $f : [a, b] \longrightarrow E$ is HKP-integrable, then*

$$\int_J f(t)\, dt \in |J| \, \overline{\mathrm{conv}}(f(J)),$$

where J is an arbitrary subinterval of $[a, b]$ and $|J|$ is the length of J. $\diamondsuit$

4. Main result

We deal with the existence of weak solution of the Volterra integral equation

$$x(t) = h(t) + \int_0^t K(t, s) f(s, x(s))\, ds \quad \text{on } I,$$

here "$\int$" denotes the HKP-integral.

Theorem 4.1. *Let $f : I \times E \longrightarrow E$, $h : I \longrightarrow E$ and $K : I \times I \longrightarrow \mathbb{R}$ satisfy the following conditions:*

(1) *h is weakly continuous on I.*
(2) *For each $t \in I$, $K(t, \cdot)$ continuous, $K(t, \cdot) \in BV(I, \mathbb{R})$ and the mapping $t \longmapsto K(t, \cdot)$ is $\|\cdot\|_{BV}$-continuous. (Here $BV(I, \mathbb{R})$ represents the space of real bounded variation functions with its classical norm $\|\cdot\|_{BV}$.)*
(3) *$f : I \times E \longrightarrow E$ is a weakly-weakly continuous function such that for all $x \in C_w(I, E)$, for all $t \in I$, $f(\cdot, x(\cdot))$ and $K(t, \cdot) f(\cdot, x(\cdot))$ are HKP-integrable on I.*

(4) *For all* $r > 0$ *and* $\varepsilon > 0$, *there exists* $\delta_{\varepsilon,r} > 0$ *such that*

$$\left\| \int_\tau^t f(s, x(s))\, ds \right\| < \varepsilon, \quad \forall\, |t - \tau| < \delta_{\varepsilon,r}, \forall\, x \in C_w(I, E), \|x\| \le r. \tag{4.1}$$

(5) *There exists a nonnegative function* $L(\cdot, \cdot)$ *such that:*
(a) *for each closed subinterval* J *of* I *and bounded subset* X *of* E,

$$\beta(f[J \times X]) \le \sup\{L(t, \beta(X)), t \in J\}; \tag{4.2}$$

(b) *the function* $s \longmapsto L(s, r)$ *is continuous for each* $r \in [0, +\infty[$, *and*

$$\sup_{t \in I} \left\{ (HK) \int_0^t |K(t, s)|\, L(s, r)\, ds \right\} < r \tag{4.3}$$

for all $r > 0$.

Then there exist an interval $J = [0, a]$ *such that the set of weakly continuous solutions of the Volterra-type integral equation*

$$x(t) = h(t) + \int_0^t K(t, s) f(s, x(s))\, ds, \tag{4.4}$$

defined on J *is non-empty and compact in the space* $C_w(J, E)$. ♢

Remark 4.1. (a) If $f(\cdot, x(\cdot))$ is HKP-integrable on I and for all $\tau \in I$ the mapping $T_{t,\tau} : E^* \longrightarrow \mathbb{R}$, defined by $y^* \longmapsto$ (HK) $\int_0^\tau K(t, s) y^* f(s, x(s))\, ds$, is weak*-continuous, then $K(t, \cdot) f(\cdot, x(\cdot))$ is HKP-integrable on I. Indeed, for $\tau \in I$, because $T_{t,\tau}$ is a linear functional on E^* that is weak*-continuous, then by [29, Theorem 3.10] there exists $w_{t,\tau}$ in E such that $T_{t,\tau}(y^*) = y^* w_{t,\tau}$ for all $y^* \in E^*$. So, (HK) $\int_0^\tau K(t, s) y^* f(s, x(s))\, ds =$ (HK) $\int_0^\tau y^* K(t, s) f(s, x(s))\, ds = y^* w_{t,\tau}$ for all $y^* \in E^*$. Therefore $K(t, \cdot) f(\cdot, x(\cdot))$ is HKP-integrable on I.

(b) For $\tau \in I$, if we suppose the HK-equi-integrability of the family $\{y^* K(t, \cdot) f(\cdot, x(\cdot)), y^* \in E^*, \|y^*\| \le 1\}$ on $[0, \tau]$, then we guarantee the continuity of $T_{t,\tau}$ with respect to weak*-topology (see [12]). ♢

Remark 4.2. The condition (4.1) is satisfied if we suppose that $f(\cdot, x(\cdot))$ is HKP-integrable on I and for all $r > 0$, there exists a HK-integrable function $M_r : I \longrightarrow \mathbb{R}_+$ such that

$$\|f(t, y)\| \le M_r(t) \ \text{ for all } \ t \in I \ \text{ and } \ y \in E, \ \|y\| \le r.$$

To see this, let $r > 0$ and $x^* \in E^*$ such that $\|x^*\| \le 1$. For $0 \le t_1 < t_2 \le 1$, we have $|x^* \int_{t_1}^{t_2} f(s, x(s), Tx(s))\, ds| \le |(\mathrm{HK}) \int_{t_1}^{t_2} x^* f(s, x(s), Tx(s))\, ds|$. Because $s \longmapsto M_{b_0}(s)$ is Henstock-Kurzweil integrable and $|x^* f(s, x(s), Tx(s))| \le \|x^*\| \|f(s, x(s), Tx(s))\| \le M_{b_0}(s)$ for all $s \in [0, 1]$, then by [22, Corollary 4.62]),

$s \longmapsto x^* f(s, x(s), Tx(s))$ is absolutely Henstock-Kurzweil integrable on $[t_1, t_2]$ and

$$\left| (\mathrm{HK}) \int_{t_1}^{t_2} x^* f(s, x(s), Tx(s))\, ds \right| \le (\mathrm{HK}) \int_{t_1}^{t_2} M_{b_0}(s)\, ds.$$

Thus

$$\begin{aligned} \left\| \int_{t_1}^{t_2} f(s, x(s), Tx(s))\, ds \right\| &= \sup_{\|x^*\| \le 1} \left| x^* \int_{t_1}^{t_2} f(s, x(s), Tx(s))\, ds \right| \\ &\le (\mathrm{HK}) \int_{t_1}^{t_2} M_{b_0}(s)\, ds, \end{aligned}$$

which thanks to the continuity of the primitive in Henstock-Kurzweil integral, becomes less then ε for t_2 sufficiently close to t_1, and this proves the claim. $\diamondsuit$

Remark 4.3. The inequality condition in Remark 4.2 is fulfilled if we suppose that E is reflexive and the function M_r is independent of $t \in I$ (see [28]). $\diamondsuit$

Proof: Let $c = \sup_{t \in I} \|h(t)\|$, $d = \sup_{t \in I} \|K(t, \cdot)\|_{BV}$ and $\mu > 0$. There exists $b > 0$ such that $\mu < \frac{b-c}{d}$. From (4.1), there exists $a \le T$ such that

$$\sup_{t \in [0,a]} \left\| \int_0^t f(s, x(s))\, ds \right\| < \mu,$$

for any $x \in C_w(I, E)$ satisfying $\|x\| \le b$. Put $J = [0, a]$, denote by $C_w(J, E)$ the space of weakly continuous functions $J \longrightarrow E$, endowed with the topology of weak uniform convergence, and by $\widetilde{B}$ the set of all weakly continuous functions $J \longrightarrow B_b$, where $B_b = \{y \in E : \|y\| \le b\}$. We shall consider $\widetilde{B}$ as a topological subspace of $C_w(J, E)$. It is clear that the set $\widetilde{B}$ is convex and closed. Put

$$F_x(t) = h(t) + \int_0^t K(t, s) f(s, x(s))\, ds \quad \text{on} \quad J.$$

We require that $F : \widetilde{B} \longrightarrow \widetilde{B}$ is continuous.

1. Let $t \in [0, a]$. For any $x^* \in E^*$ such that $\|x^*\| \le 1$, and for any $x \in \widetilde{B}$, $x^* F_x(t) = x^* h(t) + \int_0^t K(t, s) x^* f(s, x(s))\, ds$. Using Lemma 3.2 and the definition of the Riemann-Stieltjes integral, we obtain

$$\begin{aligned} &\left| \int_0^t K(t, s) x^* f(s, x(s))\, ds \right| \\ &\quad = \left| K(t, t)(\mathrm{HK}) \int_0^t x^* f(s, x(s)) - \int_0^t \left((\mathrm{HK}) \int_0^s x^* f(\tau, x(\tau))\, d\tau \right) dK_t \right| \\ &\quad \le |K(t, t)| \sup_{v \in [0,t]} \left\| \int_0^v f(s, x(s))\, ds \right\| + (V[K_t; 0, t]) \sup_{s \in [0,t]} \left\| \int_0^s f(\tau, x(\tau))\, d\tau \right\| \end{aligned}$$

$$\le |K(t,t)| \sup_{v\in J} \left\| \int_0^v f(s,x(s))\,ds \right\| + (V[K_t;0,t]) \sup_{s\in J} \left\| \int_0^s f(\tau,x(\tau))\,d\tau \right\|$$
$$\le \|K(t,\cdot)\|_{BV} \sup_{s\in J} \left\| \int_0^s f(\tau,x(\tau))\,d\tau \right\|.$$

Here $K_t(\cdot)$ denotes $K(t,\cdot)$ and $V[K_t;0,t]$ denotes the total variation of K_t on the interval $[0,t]$. Hence,

$$|x^* F_x(t)| \le c + d\mu \le b.$$

Then

$$\sup\{|x^* F_x(t)|\,, x^* \in E^*, \|x^*\| \le 1\} \le b.$$

So, $F_x(t) \in B_b$.

2. Now, we will show that $F(\widetilde{B})$ is a strongly equicontinuous subset.

Let $t,\tau \in J$. We suppose without loss of generality that $\tau < t$ and that $F_x(t) \ne F_x(\tau)$. By the Hahn-Banach theorem, there exists $x^* \in E^*$, such that $\|x^*\| = 1$ and

$$\|F_x(t) - F_x(\tau)\| = x^*(F_x(t) - F_x(\tau))$$
$$\le |x^*(h(t)) - x^*(h(\tau))| + \left| (\mathrm{HK}) \int_0^\tau (K(t,s) - K(\tau,s)) x^* f(s,x(s))\,ds \right|$$
$$+ \left| (\mathrm{HK}) \int_\tau^t K(t,s) x^* f(s,x(s))\,ds \right|$$
$$\le |x^*(h(t)) - x^*(h(\tau))| + \|K(t,\cdot) - K(\tau,\cdot)\|_{BV} \sup_{v\in J} \left\| \int_0^v f(s,x(s))\,ds \right\|$$
$$+ d \sup_{\zeta\in[\tau,t]} \left\| \int_\tau^\zeta f(s,x(s))\,ds \right\|.$$

So, the result follows from hypotheses (1), (2) and (4.1).

3. Now we will prove the continuity of F.

Since f is weakly continuous, we have by the Krasnoselskii type Lemma (see [44]) that for any $x^* \in E^*$, $\varepsilon > 0$ and $x \in \widetilde{B}$ there exists a weak neighborhood U of 0 in E such that $|x^*(f(t,x(t)) - f(t,y(t)))| \le \frac{\varepsilon}{ad}$ for $t \in J$ and $y \in \widetilde{B}$ such that $x(s) - y(s) \in U$ for all $s \in J$. Because the function $s \longmapsto x^*(f(s,x(s)) - f(s,y(s)))$ is HK-integrable on J and the function $s \longmapsto \frac{\varepsilon}{a\,d}$ is Riemann integrable on J, then by [22, Corollary 4.62], $s \longmapsto x^*(f(s,x(s)) - f(s,y(s)))$ is absolutely Henstock-Kurzweil-integrable on J and for all $t \in J$ we have:

$$\left| (\mathrm{HK}) \int_0^t K(t,s) x^*(f(s,x(s)) - f(s,y(s))) \right| ds$$
$$\le \sup_{\zeta\in I} \|K(\zeta,\cdot)\|_{BV} \sup_{\tau\in[0,t]} \left(\left| (\mathrm{HK}) \int_0^\tau x^*(f(s,x(s)) - f(s,y(s)))\,ds \right| \right)$$

$$\leq d \sup_{\tau \in J} \left((\mathrm{HK}) \int_0^\tau |x^*(f(s,x(s)) - f(s,y(s)))| \, ds \right) \leq \varepsilon.$$

Thus F is continuous.

We have already shown that $F(\widetilde{B})$ is bounded and strongly equicontinuous, thus by Lemma 2.1 in [27], $Q = \overline{\mathrm{conv}} F(\widetilde{B})$ (the closed convex hull of $F(\widetilde{B})$ in $C(J,E)$) is also bounded and strongly equicontinuous. Clearly $F(Q) \subset Q \subset \widetilde{B}$. We claim that F is β-condensing on Q. Indeed, let V be a subset of Q such that $\beta(V) \neq 0$, $V(t) = \{x(t), x \in V\}$ and $F(V)(t) = \{F_x(t), x \in V\}$. Because V is bounded and strongly equicontinuous, we have by Lemma 2.1(b) that $\sup_{t \in J} \beta(V(t)) = \beta(V) = \beta(V(J))$. Fix $t \in J$ and $\varepsilon > 0$. From the continuity of the functions $s \longmapsto K(t,s)$ and $s \longmapsto L(s, \beta(V))$ on I, it follows that there exists $\delta > 0$ such that

$$|K(t,\tau)L(q,\beta(V)) - K(t,s)L(s,\beta(V))| < \varepsilon, \tag{4.5}$$

if $|\tau - s| < \delta$, $|q - s| < \delta$, $q, s, \tau \in I$. Divide the interval $[0,t]$ into n subintervals $0 = t_0 < t_1 \ldots < t_n = t$ so that $t_i - t_{i-1} < \delta$ $(i = 1, \ldots, n)$ and put $T_i = [t_{i-1}, t_i]$. For each i, there exists $s_i \in T_i$ such that $L(s_i, \beta(V)) = \sup_{s \in T_i} L(s, \beta(V))$. By Theorem 3.1(b) and Theorem 3.2, we have

$$\begin{aligned} F_x(t) &= h(t) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} K(t,s) f(s,x(s)) \, ds \\ &\in h(t) + \sum_{i=1}^n (t_i - t_{i-1}) \overline{\mathrm{conv}} \{K(t,s) f(s,x(s)), s \in T_i, x \in V\}. \end{aligned}$$

Using (4.2), (4.3) and the proprieties of the measure of weak non-compactness, we have

$$\begin{aligned} \beta(F(V)(t)) &\leq \sum_{i=1}^n (t_i - t_{i-1}) \beta(\overline{\mathrm{conv}} \{K(t,s) f(s,x(s)), s \in T_i, x \in V\}) \\ &\leq \sum_{i=1}^n (t_i - t_{i-1}) \beta(\{K(t,s) f(s,x(s)), s \in T_i, x \in V\}) \\ &\leq \sum_{i=1}^n (t_i - t_{i-1}) \sup_{s \in T_i} |K(t,s)| \, \beta(f(T_i \times V(T_i))) \\ &\leq \sum_{i=1}^n (t_i - t_{i-1}) \, |K(t,\tau_i)| \, L(s_i, \beta(V)), \end{aligned}$$

here for each i, $\tau_i \in T_i$ is a number such that $|K(t,\tau_i)| = \sup_{s\in T_i} |K(t,s)|$. Hence, using (4.5), we have

$$\begin{aligned}\beta(F(V)(t)) &\leq \sum_{i=1}^{n}\left((\mathrm{HK})\int_{t_{i-1}}^{t_i} |K(t,\tau_i)L(s_i,\beta(V)) - K(t,s)L(s,\beta(V))|\, ds\right)\\ &\quad + \sum_{i=1}^{n}\left((\mathrm{HK})\int_{t_{i-1}}^{t_i} |K(t,s)|\, L(s,\beta(V))\, ds\right)\\ &\leq \varepsilon t + (\mathrm{HK})\int_0^t |K(t,s)|\, L(s,\beta(V))\, ds\\ &\leq \varepsilon t + \sup\left\{(\mathrm{HK})\int_0^{t'} |K(t,s)|\, L(s,\beta(V)), t' \in J\right\}.\end{aligned}$$

As the last inequality is satisfied for every $\varepsilon > 0$, we get

$$\beta(F(V)(t)) \leq \sup\left\{(\mathrm{HK})\int_0^{t'} |K(t,s)|\, L(s,\beta(V))ds, t' \in J\right\}.$$

Applying Lemma 2.1(b) again for the bounded strongly equicontinuous subset $F(V)$, we obtain $\beta(F(V)) = \sup_{t\in J}\{F(V)(t)\}$. Accordingly

$$\beta(F(V)) \leq \sup\left\{(\mathrm{HK})\int_0^{t'} |K(t,s)|\, L(s,\beta(V))\, ds, t' \in J\right\} < \beta(V),$$

so, F is β-condensing on Q. Since Q is a closed convex subset of $C(J,E)$, the set Q is weakly closed, and using similar arguments as in the proof of Theorem 2.2, we can suppose that Q is a closed convex subset of $C_w(J,E)$ and so by Corollary 2.1 the set of the fixed points of F in $\widetilde{B}$ is non-empty and compact. This means that there exists a set of weakly continuous solutions of the problem (4.4) on J which is non-empty and compact in $C_w(J,E)$. □

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Département de Mathématiques, Université de Gafsa, Faculté des sciences de Gafsa, Cité Universitaire Zarrouk 2112, Gafsa, Tunisie

E-mail: Afif.BenAmar@ipeis.rnu.tn

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