Local/global uniform approximation of real-valued continuous functions

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Abstract. For a Tychonoff space X, $C(X)$ is the lattice-ordered group (l-group) of real-valued continuous functions on X, and $C^*(X)$ is the sub-l-group of bounded functions. A property that X might have is (AP) whenever G is a divisible sub-l-group of $C^*(X)$, containing the constant function 1, and separating points from closed sets in X, then any function in $C(X)$ can be approximated uniformly over X by functions which are locally in G. The vector lattice version of the Stone-Weierstrass Theorem is more-or-less equivalent to: Every compact space has AP. It is shown here that the class of spaces with AP contains all Lindelöf spaces and is closed under formation of topological sums. Thus, any locally compact paracompact space has AP. A paracompact space failing AP is Roy's completely metrizable space Δ.

Keywords: real-valued function, Stone-Weierstrass, uniform approximation, Lindelöf space, locally in

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1. Introduction

All spaces will be Tychonoff. The basic theory of $C(X)$ is recorded in [GJ60]. With pointwise $+$, $\cdot$, and partial order $\leq$, and resulting lattice operations $\vee$ and $\wedge$, $C(X)$ is a commutative lattice-ordered ring with identity the constant function 1. We shall barely use the multiplication here, mostly viewing $C(X)$ as an l-group with distinguished element 1.

(The basic theory of l-groups is recorded in [D95], and of archimedean l-groups with "distinguished unit" in [HR77]. We will only barely need to refer to these.)

The notation $G \leq C^*(X)$ (or $G \leq C(X)$) means G is a sub-l-group of $C^*(X)$ (or $C(X)$) containing 1.

Given $G \leq C^*(X)$, define $\operatorname{loc} G \leq C(X)$: $f \in \operatorname{loc} G$ means $f \in \mathbb{R}^X$ and for each $p \in X$ there are $g \in G$ and an open neighborhood of U of p with $[g(x) = f(x)$ for each $x \in U]$.

(Since "locally continuous" implies "continuous", $\operatorname{loc} G \subseteq C(X)$. Let $\otimes$ denote $+, \vee, \wedge$. If $f_1, f_2 \in \operatorname{loc} G$ and $p \in X$, we have g_i and U_i for f_i respectively, then $g_1 \otimes g_2$ and $U_1 \cap U_2$ for $f_1 \otimes f_2$ and p, showing $\operatorname{loc} G$ is a sub-l-group of $C(X)$.)

Let S be a set, and let $L \subseteq M \subseteq \mathbb{R}^S$. If $[\forall m \in M\ \forall \epsilon > 0\, \exists\, l \in L$ with $|m(x) - l(x)| \leq \epsilon\ \forall x \in S]$, we say that L is uniformly dense in M, and write $L \overset{\text{ud}}{\subseteq} M$. (Note that only rational ϵ need be considered.)

For $L \subseteq \mathbb{R}^S$, let $L^+ = \{l \in L \mid 0 \leq l\}$. Note that, if $L \subseteq M \subseteq \mathbb{R}^S$, $L \leq \mathbb{R}^S$ and $M \leq \mathbb{R}^S$, $L^+ \overset{\text{ud}}{\subseteq} M^+$ implies $L \overset{\text{ud}}{\subseteq} M$.

A group $(G,+)$ is divisible if $\forall\, g \in G\ \forall n \in \mathbb{N}\ \exists\ h \in G$ with $nh = g$. For $G \leq C(X)$, this just means $\forall\, g \in G\ \forall\, r \in \mathbb{Q}$, $rg \in G$. (Throughout the paper, all assumptions "G is divisible" could be replaced by "G is a vector lattice" with no real effect.)

Say that $G \leq C^*(X)$ is *full* if G is divisible and separates points and closed sets of X (defined below).

Definition 1.1. The space X has the Approximation Property, AP, if, whenever $G \leq C^*(X)$ is full, then $\operatorname{loc} G \overset{\text{ud}}{\leq} C(X)$.

We shall prove successively that spaces in these progressively larger classes have AP: Compact, in fact, almost compact; both locally compact and σ-compact; Lindelöf. And, a sum of spaces has AP iff each summand has AP.

Indeed, it is not so easy to locate spaces failing AP; Roy's space Δ is essentially the only example we know, this fact due to Sola's proof that Δ fails a property weaker (seemingly) than AP. See §6 here.

We record some technical preliminaries.

Suppose $G \leq C(X)$, and $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of X (e.g., $\mathcal{A}$ = points, $\mathcal{B}$ = closed sets). With some inconsistency in language we say G separates $\mathcal{A}$ and $\mathcal{B}$ if

(*) for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $A \cap B = \emptyset$, there is a $g \in G$ with $g(A) = \{0\}$ and $g(B) = \{1\}$.

If $\mathcal{A} = \mathcal{B}$, we just say G separates $\mathcal{A}$ (e.g., G separates points). Since G is a sub-l-group and $1 \in G$, in (*) we can change $((A,0)$ and $(B,1))$ to $((A,1)$ and $(B,0))$ by replacing g by $1 - g$, and we can suppose $0 \leq g \leq 1$ by replacing g by $(g \wedge 0) \vee 1$, and we can change $(B,1)$ to (B,z) for any integer z by replacing g by zg, and if G is divisible, we can change $(B,1)$ to (B,r) for any rational r.

Let $f \in C(X)$. The set $Z(f) = \{x \in X \mid f(X) = 0\}$ is called a zero-set, and $\operatorname{coz} f = X - Z(f)$ a cozero-set. Zero-sets are closed, any $f^{-1}[a,b]$ is a zero-set, and $f^{-1}(a,b)$ a cozero-set. See [GJ60].

Lemma 1.2. *Let $G \leq C^*(X)$ and let $\mathcal{B}$ be a family of subsets of X.*

(a) *For any X, if G separates points and $\mathcal{B}$, then G separates compact sets and $\mathcal{B}$.*
(b) *For compact X, if G separates points, then G separates zero-sets.*

Proof: (a) Suppose E is compact and $B \in \mathcal{B}$, with $E \cap B = \emptyset$. For each $p \in E$ there is a $0 \leq g_p \in G$ with $[g_p(p) = 2; g_p(B) = \{0\}]$. Then $\{\{x \mid g_p(x) > 1\} \mid$

$p \in E\}$ covers E, so there is a finite $I \subseteq E$ for which $\{\{x \mid g_p(x) > 1\} \mid p \in I\}$ covers E. Then, $g = 1 \wedge (\bigvee_I g_p)$ has $g(E) = \{1\}$ and $g(B) = \{0\}$.

(b) G separates compact sets by (a). Zero-sets are closed, and thus compact when X is compact. □

Each X has its Čech-Stone compactification βX: Each $f \in C^*(X)$ has the unique extension $\beta f \in C(\beta X)$, and extension provides an l-group isomorphism $C^*(X) \approx C(\beta X)$ preserving 1; each $f \in C(X)$ has the unique extension $\beta f \in C(\beta X, [-\infty, \infty])$. See [E89] and [GJ60].

In fact, for any $G \leq C(X)$ which separates points and closed sets of X, there is a similarly associated unique compactification of YG of X, called the Yosida space of G: Each $g \in G$ has the unique extension $\tilde{g} \in C(YG, [-\infty, \infty])$; with $G^* \equiv \{g \in G \mid g \text{ is bounded}\}$, extension provides an isomorphism $G^* \approx \{\tilde{g} \mid g \in G^*\} \leq C(YG)$, and $\{\tilde{g} \mid g \in G^*\}$ separates points in YG, and thus, if divisible, is full in $C(YG)$, by 1.2. See [HR77]. Of course, YG also can be constructed by embedding X in a cube.

Conversely, if K is a compactification of X, then of course, $C(K)$ separates points of K, thus points from closed sets in K (1.2). Thus also in X, so that the set of restrictions $C(K) \mid X$ is full in $C^*(X)$.

2. Compact and almost compact

This section is an account, for our purposes, of the Stone-Weierstrass Theorem. What we will say is known, but it is simpler to sketch proofs than do a tedious confusing translation to precise statements of what we need.

Theorem 2.1 ([H47, Theorem 1]). *Let X be any Tychonoff space, let $G \leq C^*(X)$ and suppose G is divisible. Then, $G \overset{\text{ud}}{\leq} C^*(X)$ if and only if G separates zero-sets.*

Proof: Suppose $G \leq C^*(X)$, and G is divisible.

Suppose $G \overset{\text{ud}}{\leq} C^*(X)$ and Z_0, Z_1 are disjoint zero-sets. Take $f_i \in C(X)$ with $Z(f_i) = Z_i$. Then $f = |f_0|/(|f_0| + |f_1|)$ has value i on Z_i. Take $g \in G$ with $|f - g| \leq 1/3$. Then $g' = 3((g \vee 1/3) \wedge 2/3 - 1/3)$ has value i on Z_i.

Suppose $G \leq C^*(X)$ separates zero-sets, $f \in C^*(X)^+$ and $\epsilon > 0$ is rational. For $n \in \mathbb{N}$, let $U_n = f^{-1}(\epsilon(n-2), \epsilon(n+2))$ and $E_n = f^{-1}[\epsilon(n-1), \epsilon(n+1)]$. Note that $X - U_n$ and E_n are disjoint zero-sets. Since f is bounded, there is M with $f < M + 2$, so that $\{U_n \mid n \leq M\}$ covers X. For each $n \leq M$, choose $g_n \in G$ with values 0 on $X - U_n$ and $\epsilon(n-1)$ on E_n, and $0 \leq g_n \leq \epsilon(n-1)$. Then $g \equiv \bigvee_{n \leq M} g_n$ has $|f - g| \leq 3\epsilon$. □

Corollary 2.2 (Stone-Weierstrass: [S48, Corollary 3, p. 174]). *Let X be compact, let $G \leq C(X)$ and suppose G is divisible. If G separates points then $G \overset{\text{ud}}{\leq} C(X)$.*

Every compact space has AP.

Proof: Here $C^*(X) = C(X)$. Apply 1.2 and 2.1. □

The statement "Every compact space has AP," implies the rest of 2.2, though this requires an argument: Suppose divisible $G \leq C^*(X)$, separating points. Then, as in §1, the set of extensions $\tilde{G} \leq C(YG)$ is full. If X is compact, $C^*(X) = C(X)$ and $YG = X$. Assuming X has AP, $\operatorname{loc} G \overset{\text{ud}}{\leq} C(X)$. But, since $YG = X$, $\operatorname{loc} G = G$ by [HR78, 5.2 and 5.5(a)].

X is called almost compact if $|\beta X - X| \leq 1$, equivalently, the only compactification of X is βX. See [GJ60].

Corollary 2.3 ([H47, Theorem 4]). *X is almost compact if and only if for every full $G \leq C^*(X)$, $G \overset{\text{ud}}{\leq} C^*(X)$.*

Every almost compact space has AP.

Proof: We use the discussion after 1.2. For $G \leq C^*(X)$ full, $\tilde{G} \overset{\text{ud}}{\leq} C(YG)$ by 2.2. If X is almost compact, $YG = \beta X$, so $\tilde{G} \overset{\text{ud}}{\leq} C(\beta X)$, so $G = \tilde{G} \mid X \overset{\text{ud}}{\leq} C(\beta X) \mid X = C^*(X)$. Conversely, if there is a compactification $K \neq \beta X$, then $G = C(K) \mid X \leq C^*(X)$ is full, not uniformly dense in $C^*(X)$. □

3. Locally compact σ-compact

This is the most novel step in analyzing AP.

Theorem 3.1. *Every locally compact and σ-compact space has AP.*

The proof will use some known lemmas, and an additional construction using the Stone-Weierstrass Theorem on pieces of the space.

Lemma 3.2. *The following are equivalent about X.*

(a) *X is locally compact and σ-compact.*
(b) *There is a sequence $\{K_n \mid n \in \mathbb{N}\}$ of compact sets, with $K_n \subseteq \operatorname{int} K_{n+1}$ $\forall\, n$, and $X = \bigcup_{n\in\mathbb{N}} K_n$.*
(c) *There is a $v \in C^*(X)$ with $0 < v(x)$ $\forall\, x$, and $[\forall\, \epsilon > 0\ \exists$ compact K with $(x \notin K \Rightarrow v(x) \leq \epsilon)]$.*
(d) *There is $u \in C(X)$ with $0 < u(x)$ $\forall\, x$, and $[\forall\, 0 < M\ \exists$ compact K with $(x \notin K \Rightarrow M \leq u(x))]$.*

Proof: This is a standard, and we just sketch.

(b) $\Rightarrow$ (a). Obvious.

(a) $\Rightarrow$ (b). Write $X = \bigcup_{n\in\mathbb{N}} E_n$, with E_n compact. By induction: Let $K_1 = E_1$, and given compact K_n, cover it by open sets with compact closure, take a finite subcover with union U, and let $K_{n+1} = \overline{U}$.

(b) $\Rightarrow$ (c). X is normal. By Urysohn's Lemma, there is a $v_n \in C(X)$ with $0 \leq v_n \leq 1$, with value 1 on K_n, and 0 on $X - \operatorname{int} K_{n+1}$. Then $v \equiv \sum 2^{-n} v_n \in C(X)$ by the Weierstrass M-test. (See [E89] if needed.)

(c) $\Rightarrow$ (d). Set $u \equiv 1/v$.

(d) $\Rightarrow$ (b). Given the u, for suitable $M_n \uparrow +\infty$, let $K_n = u^{-1}[0, M_n]$. □

In any $C(X)$, a locally finite partition of unity is a family $\{f_\alpha\} \subseteq C(X,[0,1])$, with $\{\operatorname{coz} f_\alpha\}$ locally finite (meaning: Every point has a neighborhood meeting only finitely many of the sets $\operatorname{coz} f_\alpha$), and with $\sum_\alpha f_\alpha(x) = 1\ \forall\, x$ (in which: $\forall\, x$, $\sum_\alpha f_\alpha(x)$ is a finite sum, by the local finiteness).

Lemma 3.3 ([BH74, 2.1])**.** *For any X: if $\{U_n \mid n \in \mathbb{N}\}$ is a (countable) cover of X by cozero-sets, then there is in $C(X)$ a locally finite partition of unity $\{u_n \mid n \in \mathbb{N}\}$ with $\operatorname{coz} f_n \subseteq U_n\ \forall\, n$.*

Proof of 3.1: Let $u \in C(X)$ be as in 3.2(d), let $U_n = u^{-1}(n-1, n+1)$ and $V_n = u^{-1}(n-3/2, n+3/2)$. Apply 3.3 to $\{U_n\}$. For any $f \in C(X)$, we have $f = f \cdot 1 = f \cdot \sum u_n = \sum (f u_n)$.

Let $C \leq C^*(X)$ be full. For any $K \subseteq X$, $G \mid K \leq C^*(K)$ is full, thus separates points of K. For each n, $\overline{U_n}$ is compact, and so $G \mid \overline{U_n} \overset{\text{ud}}{\leq} C(\overline{U_n})$ by 2.2.

Let $f \in C(X)^+$, and let $\epsilon > 0$ be rational. For each n, take $g_n \in G$ with $|f u_n - g_n| \leq \epsilon$ on $\overline{U_n}$. Let $D_n = \overline{U_n} - U_n$, and by continuity of the functions and compactness of D_n there is an open W_n with $D_n \subseteq W_n \subseteq V_n$ such that $f u_n \leq \epsilon$ and $u_n \leq 2\epsilon$ on W_n.

Now, $\overline{U_n}$ is compact, so $\sup\{(f u_n)(x) \mid x \in \overline{U_n}\} \equiv s_n < +\infty$. Using 1.2, take $h_n \in G$ with values $\geq s_n$ on $\overline{U_n}$, and 0 on $X - (U_n \cup W_n)$. Then $k_n \equiv g_n \wedge h_n \in G$. We have $|f u_n - k_n| \leq 2\epsilon$ on all of X, and we have $\operatorname{coz} k_n \subseteq \operatorname{coz} u_n \cup W_n \subseteq U_n \cup W_n \subseteq V_n$.

If $V_i \cap V_j \neq \emptyset$, then $|i-j| \leq 2$, so $\{\operatorname{coz} k_n\}$ is locally finite, so $l \equiv \sum k_n \in C(X)$. Since on V_n, we have $l = \sum\{k_i \mid n-2 \leq i \leq n+2\}$, we have $l \in \operatorname{loc} G$.

A little calculation shows $|f - l| \leq 6\epsilon$ on all of X. □

4. Lindelöf

Here is the next enlargement of the class AP.

Theorem 4.1. *Every Lindelöf space has AP.*

The proof of this uses 3.1, the following known items, and some simple further argument.

Lemma 4.2. *Suppose $X \subseteq Y$. Each of the following implies the next.*

(a) *X is Lindelöf.*
(b) *X is z-embedded in Y: For each zero-set Z of X, there is a zero set Z' of Y with $Z = Z' \cap X$.*
(c) *$\forall\, f \in C(X)\ \forall\, \epsilon > 0\ \exists$ a cozero-set V of Y with $X \subseteq V$, and $\exists\, h \in C(V)$ with $|f(x) - h(x)| \leq \epsilon\ \forall\, x \in X$.*

(a) ⇒ (b) first appeared in [HJ61, 5.3], and is attributed to M. Jerison; there is also a proof in [BH74, 4.1].

(b) ⇒ (c), assuming X dense in Y, first appears in [H69, 3.6]; the density is removed, and the proof cleaned up, in [BH74, 2.2]. This latter proof identifies and invokes 3.3.

Proof of 4.1: Suppose X is Lindelöf, $G \leq C^*(X)$ is full, and let $Y = YG$ be the Yosida space of G (discussed in §1), so that $G \approx \tilde{G} \leq C(Y)$, with $\tilde{G} \leq C(Y)$ full (by 1.2).

Let V be a cozero-set of Y with $V \supseteq X$. Then, $\tilde{G} \mid V \leq C^*(V)$ is full, and V is locally compact and σ-compact, so 3.1 shows $\operatorname{loc}(\tilde{G} \mid V) \overset{\mathrm{ud}}{\leq} C(V)$. Note that $k \in \operatorname{loc}(\tilde{G}|V)$ implies $k \mid X \in \operatorname{loc} G$.

Take $f \in C(X)$ and rational $\epsilon > 0$. Apply 4.2 to find V as above and $h \in C(V)$ with $|f - h| \leq \epsilon$ on X. Apply the previous paragraph to find $k \in \operatorname{loc}(\tilde{G} \mid V)$ with $|h - k| \leq \epsilon$ on V. We then have $|f - k| \leq 2\epsilon$ on X, and $k \mid X \in \operatorname{loc} G$.

Thus $\operatorname{loc} G \overset{\mathrm{ud}}{\subseteq} C(X)$. □

5. Sums

Given $\{X_i \mid i \in I\}$ a set of spaces, the sum (or coproduct) $\sum_I X_i$ is the disjoint union, in which U is open if and only if for each i, $U \cap X_i$ is open in X_i. We enlarge further the class AP.

Theorem 5.1. *$\sum_I X_i$ has AP if and only if each X_i has AP.*

Proof: Suppose each X_i has AP, let $X = \sum X_i$ and suppose $G \leq C^*(X)$ is full. Let $G_i = G \mid X_i$ (the set of restrictions). It is easy to see that $G_i \leq C^*(X_i)$ is full, so $\operatorname{loc} G_i \overset{\mathrm{ud}}{\leq} C(X_i)$. Take $f \in C(X)$ and rational $\epsilon > 0$. So $f \mid X_i \in C(X_i)$, and there is $g_i \in \operatorname{loc} G$ with $|f - g_i| \leq \epsilon$ on X_i. Define $g \in C(X)$ as $g(x) = g_i(x)$ when $x \in X_i$. Evidently, $g \in \operatorname{loc} G$, and $|f - g| \leq \epsilon$ on all of X.

For the converse, it suffices to show that if X has AP and U is clopen in X, then U has AP. So, suppose we have such X and U, and $G \leq C^*(U)$ is full. Let $H = \{h \in C^*(X) | h|U \in G\}$. Because U is clopen, $H \leq C^*(X)$ full and $\operatorname{loc} H = \{f \in C(X) | f|U \in \operatorname{loc} G\}$. Thus, $\operatorname{loc} H \overset{\mathrm{ud}}{\leq} C(X)$, and this implies $\operatorname{loc} G \overset{\mathrm{ud}}{\leq} C(U)$. □

Corollary 5.2. *Any locally compact paracompact space has AP.*

Proof: Such a space is the sum of locally compact σ-compact spaces, by [E89, p. 308]. Apply 3.1 and 5.1. □

In 5.2, "locally compact" cannot be dropped, because of Example 6.3. I do not know if "paracompact" can be dropped, but strongly doubt it; see comments in §7.

6. One example

We explain why Roy's space Δ [R68] fails AP, courtesy of M. Sola [S87]. This is essentially the only example we know of a space failing AP. (Of course, any space with Δ as a summand will fail AP, by 5.1.)

Let K be a compactification of X. Set $C[K,X] = \bigcup\{C(V)|X|V$ is open in K and $V \supseteq X\}$. Note that $C[\beta X, X] = C(X)$ (because $f \in C(X)$ extends to $\beta f \in C(\beta X, [-\infty, +\infty])$, so $f \in C(V) \mid X$ for $V = \beta f^{-1}(-\infty, +\infty)$).

We say that $H \leq C(X)$ is uniformly complete if H is closed in $C(X)$ under uniform convergence of sequences. Of course, $C(X)$ and $C^*(X)$ are uniformly complete.

Proposition 6.1. (a) *Suppose K is a compactification of X. Then $C[K,X] \leq C(X)$, separates points and closed sets and is divisible, and $C[K,X] = \operatorname{loc} C[K,X]$.*

(b) *Suppose $G \leq C^*(X)$ separating points and closed sets. Then $G \leq \operatorname{loc} G \leq C[YG, X] \leq C(X)$; if G is uniformly complete, then $\operatorname{loc} G = C[YG, X]$.*

($C[K,X]$ need not be uniformly complete: Let $K = [0,1], X$ the irrational points. Then $C[K,X] \overset{\text{ud}}{\leq} C(X)$ by 4.2, but there is $f \in C(X)$ with no continuous extension to any $p \in K - X$ [FGL65].)

At the risk of excessive jargon, let us say: "H is almost $C(X)$" if $H \leq C(X)$, H separates points and closed sets, H is divisible and uniformly complete and $H = \operatorname{loc} H$. And "K is almost βX" if K is a compactification of X, and $\beta V = \beta X$ for each V open in K with $V \supseteq X$.

Proposition 6.2. (a) *Suppose K is almost βX. Then, $C[K,X]$ is almost $C(X)$ and $YC[K,X] = K$; and $C[K,X] = C(X)$ if and only if $K = \beta X$.*

(b) *H is almost $C(X)$ if and only if $H = C[YH, X]$ and YH is almost βX.*

It is fairly easy to derive 6.2 from 6.1, and the proof of 6.1 is not difficult. In any event, the proofs appear, more-or-less, in [H76] (*mutatis mutandis*), because 6.1 and 6.2 can be shown equivalent to statements in [H76]. See also remarks below.

If X has AP, then [H almost $C(X)$ implies $H = C(X)$] (because $H = \operatorname{loc} H = \operatorname{loc} H^*$), and so by 6.2 [$K$ almost βX implies $K = \beta X$]. Thus, if X has a compactification K which is almost βX but $K \neq \beta X$, then X will fail AP. In [R68] is constructed a completely metrizable space Δ for which $\operatorname{ind}\Delta = 0 < 1 = \dim \Delta$. Consequently, the maximal zero-dimensional compactification ζX is not $\beta\Delta$. In [S87] it is shown (in response to a question from [H76] — see 6.4 below) $\zeta\Delta$ is almost $\beta\Delta$. (This is not easy.)

Example 6.3. Δ fails AP.

Remark 6.4. My paper [H76] was an inconclusive attempt to give a new order-algebraic characterization of $C(X)$. In 6.1 and 7.3 there, appears what is called a "working conjecture," which is equivalent to:

(†) If H is almost $C(X)$ and $X = R(H)$ (the real ideal space of H), then $H = C(X)$.

(Here, that $X = R(H)$ is equivalent to: $\forall\, p \in YH - X$ there is a G_δ-set U of YH with $p \in U$ and $U \cap X = \emptyset$.) And I said "I suspect (†) false" (p. 18). I pointed out there in 6.4 (a) $\Leftrightarrow$ (b) what is, in effect, the consequence of 6.2 above, that (H almost $C(X) \Rightarrow H = C(X)$) if and only if ($K$ almost $\beta X \Rightarrow K = \beta X$) — this *sans* "$X = R(H)$" — and said that I did not know if either/both is/are true. In the "if and only if" here, "$X = R(H)$" translates to "X is G_δ-closed in K".

Sola counterexampled (K almost $\beta X \Rightarrow K = \beta X$) with $X = \Delta$, $K = \zeta\Delta$. This does not counterexample (†) because the G_δ-closure of X in ζX is what is called $\nu\Delta$, the so-called "$\mathbb{N}$-compactification" of Δ, and the associated "H almost $C(\Delta)$" is $H = \operatorname{loc}(C(\zeta\Delta) \mid \Delta) = C(\nu\Delta)|\Delta$; here $R(H) = \nu\Delta$. (See [S87] and [N73] for some of these details.)

The upshot of this is: (†) still remains an open question. I still suspect (†) false. (A related question is: H almost $C(X) \overset{?}{\Rightarrow} H \approx C(R(H))$? As noted above, Δ fails to counterexample this.)

7. Other AP's

The condition on X discussed in §6 above, that H almost $C(X) \Rightarrow H = C(X)$, can be viewed as another Approximation Property for X, and these are several similar ones, which might or might not be worth further study, and which we list. In the following, G and H are assumed to separate points and closed sets of X.

(AP^1) $\forall\, G \leq C^*(X)$, $\beta(\operatorname{loc} G)^* \leq C(\beta X)$ separates points of βX.

(AP) $\forall$ divisible $G \leq C^*(X)$, $\operatorname{loc} G \overset{\mathrm{ud}}{\leq} C(X)$.

(AP_1) $\forall$ divisible uniformly complete $G \leq C^*(X)$, $\operatorname{loc} G \overset{\mathrm{ud}}{\leq} C(X)$.

(AP_2) H almost $C(X) \Rightarrow H = C(X)$.

It can be shown (and will be, in [H∞]) that $\mathrm{AP}^1 \Rightarrow \mathrm{AP}$, and it is easy to see that $\mathrm{AP} \Rightarrow \mathrm{AP}_1 \Rightarrow \mathrm{AP}_2$. I have no knowledge of any of the converse implications. It is true (and not so easy) that any locally compact σ-compact space has AP^1, and also true that any locally compact space has AP_1, (because in the conclusion of AP_1, $\operatorname{loc} G = C(X)$ by applying the Stone-Weierstrass Theorem on compact neighborhoods).

It does not seem plausible to me that every locally compact space has AP; it seems to me that some version of paracompactness is required for AP.

Here are three more specific questions about the extent of the class AP.

(1) What is the relationship (if any) between the conditions "X has AP" and "υX has AP"? (υX is the Hewitt real compactification [GJ60].)

(2) Does every pseudocompact space have AP?

(3) Does every almost Lindelöf space have AP? (X is almost Lindelöf if υX is Lindelöf and $|\upsilon X - X| \leq 1$. See [HM02].)

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