Near heaps

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Abstract. On any involuted semigroup $(S,\cdot,')$, define the ternary operation $[abc] := a\cdot b'\cdot c$ for all $a,b,c\in S$. The resulting ternary algebra $(S,[\,])$ satisfies the para-associativity law $[[abc]de]=[a[dcb]e]=[ab[cde]]$, which defines the variety of semiheaps. Important subvarieties include generalised heaps, which arise from inverse semigroups, and heaps, which arise from groups. We consider the intermediate variety of near heaps, defined by the additional laws $[aaa]=a$ and $[aab]=[baa]$. Every Clifford semigroup is a near heap when viewed as a semiheap, and we show that the Clifford semigroup operations are determined by the semiheap operation. We show that near heaps are exactly strong semilattices of heaps, parallelling a known result for Clifford semigroups. We characterise those near heaps which arise directly from Clifford semigroups, and show that all near heaps are embeddable in such examples, extending known results of this kind relating heaps to groups, generalised heaps to inverse semigroups, and general semiheaps to involuted semigroups.

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1. Background on semiheaps

We begin with a review of some established definitions and results.

A *heap* H is a non-empty set with ternary operation $[\,]$ satisfying the following laws.

- $[[abc]de]=[a[dcb]e]=[ab[cde]]$ (para-associative law)
- $[aab]=[baa]=b$

We call the one-element heap the *trivial heap*. Every group gives a heap under the ternary operation $[abc]:=ab^{-1}c$, a construction first considered in the setting of abelian groups by Prüfer in [4]. Conversely, a group arises from a heap H by choosing any element $e\in H$ and defining a binary operation $x*y:=[xey]$; the element e becomes the identity of the constructed group and $[exe]$ the inverse of x. These constructions are mutually inverse up to isomorphism. Hence the varieties of groups and pointed heaps are term equivalent, as shown by Baer in [1].

A *semiheap* H is a non-empty set with a ternary operation $[\,]$ satisfying only the para-associative law above. Semiheaps were first considered by Wagner in [5]. A similar construction gives a semiheap when S is an *involuted semigroup*, that is a semigroup equipped with a unary operation $'$ for which the following laws are satisfied:

- $a'' = a$
- $(ab)' = b'a'$.

If S is an involuted semigroup, setting $[abc] := ab'c$ for all $a, b, c \in S$ gives a semiheap operation on S. Denote by $[S]$ the semiheap obtained from S in this way. Every semiheap can be *embedded in* $[S]$ for some involuted semigroup S: see Section 2 of [5].

An *idempotent semiheap* is a semiheap satisfying

- $[aaa] = a$ (idempotency law).

These were studied in [2]. If S is an involuted semigroup, the semiheap $[S]$ is idempotent if and only if $aa'a = a$ (see [3]). An involuted semigroup with this property is called an *involuted I-semigroup*. Every idempotent semiheap can be *embedded in* $[S]$ for some involuted I-semigroup S. (This specific result does not appear in the literature, but the proof of the analogous result for generalised heaps in [5] is easily modified to show it.)

A *generalised heap* is an idempotent semiheap satisfying

- $[aa[bbc]] = [bb[aac]]$ and $[[abb]cc] = [[acc]bb]$ (generalised heap axiom).

These were considered by Wagner in [5]. They arise naturally in the setting of atlases in differential geometry; see [6]. An *inverse semigroup* is an involuted semigroup satisfying

- $aa'a = a$ (idempotency)
- $aa'bb' = bb'aa'$

Omitting the law $(ab)' = b'a'$, which follows from the others, gives Howie's definition as on page 145 of [3]. It can be shown that the set of idempotents of an inverse semigroup S is $E(S) = \{a'a \mid a \in S\}$ so that idempotents commute in an inverse semigroup. If S is an involuted semigroup, the semiheap $[S]$ is a generalised heap if an only if S is an inverse semigroup, and all generalised heaps can be embedded in a generalised heap constructed in this way (see Section 3 of [5]).

We now summarize these results.

Proposition 1. *Let S be an involuted semigroup.*

1. *$[S]$ is a semiheap. Every semiheap can be embedded in a semiheap of this type.*
2. *$[S]$ is an idempotent semiheap if and only if S is an I-semigroup. Every idempotent semiheap can be embedded in an idempotent semiheap of this type.*
3. *$[S]$ is a generalised heap if and only if S is an inverse semigroup. Every generalised heap can be embedded in a generalised heap of this type.*
4. *$[S]$ is a heap if and only if S is a group. All heaps are of this type.*

The correspondences can be tightened further, to resemble the situation for heaps and groups, if a further assumption is made. If H is a semiheap, we say $e \in H$ is *bi-unitary* if $[aee] = [eea] = a$ for all $a \in H$. A semiheap is a heap if and only if every element is bi-unitary. Note that if S is an involuted monoid with identity 1, then $1 \in [S]$ is bi-unitary.

Call a semiheap equipped with a distinguished bi-unitary element e, viewed as a nullary operation, *bi-unital*. The class of bi-unital semiheaps is a variety with one ternary and one nullary operation.

Let H be a bi-unital semiheap. Define $x' = [exe]$ for all $x \in H$. Then $x'' = x$ for all $x \in H$ as is easily checked. Also define a binary operation on H by setting $x \cdot y = [xey]$. Call the resulting algebra $\langle H \rangle$. In fact $\langle H \rangle$ is an involuted monoid with identity e, as is shown in Section 2 of [5].

The constructions $S \mapsto [S]$ (S an involuted monoid) and $H \mapsto \langle H \rangle$ (H a bi-unital semiheap) are mutually inverse, leading to a term equivalence between the varieties of involuted monoids and bi-unital semiheaps. Indeed the term equivalence specialises to subvarieties as summarised in the following.

Proposition 2. *The following subvarieties of involuted monoids and bi-unital semiheaps are term equivalent.*

bi-unital semiheap type	involuted monoid type
arbitrary	arbitrary
idempotent semiheap	I-monoid
generalised heap	inverse monoid
heap	group

Each of these equivalences is established in [5] (or follows easily from arguments given there).

2. Introducing near heaps

Historically, heaps were the first type of semiheaps considered. There are various ways to weaken the heap axioms, and we have reviewed the most important ones already. Another, first introduced in [2], gives rise to the class of near heaps.

A *near heap* is a semiheap satisfying the laws

- $[aaa] = a$ (idempotent law)
- $[aab] = [baa]$ (near heap axiom).

Every heap is a near heap and every near heap is a generalised heap (see Proposition 15 of [2]).

Let $E(S) = \{aa' \mid a \in S\}$ denote the semilattice of idempotents in the inverse semigroup S, partially ordered via $e \leq f$ if and only if $e = ef$. A *Clifford semigroup* S is an inverse semigroup in which for all $a \in S$ and $e \in E(S)$, $ae = ea$. It is well known that this condition is equivalent to the condition that $aa' = a'a$ for all $a \in S$ (see [3] for example). It is clear that if S is a Clifford semigroup then $[S]$ is a near heap. The converse is also true.

Proposition 3. *Let S be an involuted semigroup. Then $[S]$ is a near heap if and only if S is a Clifford semigroup.*

PROOF: If S is a Clifford semigroup then $[S]$ is a semiheap, and if $a, b \in S$ then $[aab] = aa'b = baa' = ba'a = [baa]$ so $[S]$ is a near heap as claimed.

Conversely assume that $[S]$ is a near heap. Then S is an inverse semigroup by Proposition 1. Then for all $a \in S$, letting $aa' = e \in E(S)$ and $a'a = f \in E(S)$ we have

$$\begin{aligned} aa' &= (aa')(aa') = aa'e = [aae] = [eaa] = ea'a = ef = fe \\ a'a &= (a'a)(a'a) = fa'a = [faa] = [aaf] = aa'f = ef = fe \end{aligned}$$

so $aa' = a'a$ and S is a Clifford semigroup. □

So the first half of a new entry in Proposition 1 can now be made. A further entry can also be added, by specialising near heaps in a direction orthogonal to heaps.

Semilattices may be thought of as Clifford semigroups in which the law $a = a'$ holds; by uniqueness of inverses in inverse semigroups, this is the only possible way to define $'$ to yield a Clifford semigroup.

So if S is a semilattice, the semiheap operation on $[S]$ is given by $[abc] = abc$ for all $a, b, c \in S$. The resulting near heap satisfies the law $[abb] = [aab]$. The following easy result was stated in [2] (and may have first appeared earlier).

Proposition 4. *Let S be an involuted semigroup. $[S]$ is a near heap satisfying the law $[abb] = [aab]$ if and only if S is a semilattice. All such near heaps are of this type, and the two varieties are term equivalent.*

From now on, when we use the term *semilattice* in the context of semiheaps, we mean a near heap satisfying the law $[abb] = [aab]$. Note that the only semilattice which is a heap is the trivial heap, since it satisfies $a = [abb] = [aab] = b$ for all a and b.

A new row in the table provided in Proposition 2 can also be given.

Proposition 5. *The variety of Clifford monoids is term equivalent to the variety of bi-unital near heaps.*

PROOF: It suffices to show that if H is a near heap with distinguished bi-unitary element e, then $\langle H\rangle$ is a Clifford monoid (with identity e). But because H is a near heap, it is a generalised heap, so $\langle H\rangle$ is an inverse semigroup by what is shown in Section 3 of [5], and for any x, y, $[xxy] = [yxx]$ so $xx'y = yx'x$, so letting $y = xx'$ gives $(xx')^2 = (xx')(x'x)$, so in the semilattice $E(S)$, $xx' \leq x'x$, and symmetry establishes the equality needed to show that $\langle H\rangle$ is a Clifford semigroup. □

However, in the absence of a bi-unitary element, the correspondence breaks down, as we next show.

3. Which near heaps are $[S]$ for some S?

Let H be a near heap. Let $\sim$ be the binary relation on H given by $a \sim b$ if $[abb] = a$ and $[baa] = b$. The following is shown in [2]: see Proposition 6, Corollary 7, Lemma 18 and Theorem 19.

Proposition 6. *Let H be a near heap. Then $\sim$ is a congruence on H for which $H/\sim$ is a semilattice, and for every $a \in H$, the congruence class containing a is a subsemiheap of H which is a maximal subheap.*

From Proposition 16 of [2], a semilattice can contain no non-trivial subheaps, this congruence can be regarded as defining the 'heap radical' of the near heap; see [2] for further discussion of radicals of semiheaps.

If H is a neap heap we call $H/\sim$ the *associated semilatttice*.

Proposition 7. *If S is a Clifford semigroup, then every element of S is congruent under $\sim$ to a unique element of $E(S)$.*

Proof: Now $a \sim aa' \in E(S)$ since

$$\begin{aligned}
[aa(aa')] &= aa'aa' = aa' \\
[(aa')aa] &= aa'a'a = aa' \\
[(aa')(aa')a] &= aa'aa'a = a \\
[a(aa')(aa')] &= aaa'aa' = a.
\end{aligned}$$

Also $aa' \sim bb'$ implies $aa' = [(aa')(bb')(bb')] = aa'bb' = [(aa')(aa')(bb')] = bb'$, proving uniqueness. □

The congruence $\sim$ maps $E(S)$ bijectively onto the associated semilattice $[S]/\sim$. It follows that in near heaps of the form $[S]$ where S is a Clifford semigroup, the canonical homomorphism $[S] \longrightarrow [S]/\sim$ is split (has a right inverse $[S]/\sim \longrightarrow [S]$). Equivalently in these near heaps there is a subsemilattice, in this case $E(S)$, with an element in every congruence class of $\sim$.

We define a *spine* for a near heap H to be a subsemilattice L of H such that every element $h \in H$ is congruent under $\sim$ to a unique element of L. Hence a necessary condition for a near heap H to be (isomorphic to) $[S]$ for some Clifford semigroup is that it has a spine.

This condition is not only necessary but also sufficient as we now show.

Theorem 8. *A near heap H is equal to $[S]$ for some Clifford semigroup if and only if it has a spine.*

Proof: As $E(S)$ is a spine of $[S]$, the condition is necessary.

To show that it is sufficient let H be a near heap with spine $L \subseteq H$. For each element $h \in H$ there is a unique $e_h \in L$ with $e_h \sim h$. Define $a' = [e_a a e_a]$ and $a * b = [ae_a b] = [ae_b b]$. We claim that $(H, *, ')$ is a Clifford semigroup under these operations and that $[(H, *, ')] = H$.

First note that

$$[ae_a b] = [ae_a[e_b e_b b]] = [a[e_b e_b e_a]b] = [a[e_b e_a e_a]b] = [[ae_a e_a]e_b b] = [ae_b b]$$

so the product as stated above is well defined. Furthermore $(a*b)*c = [[ae_a b]e_c c] = [ae_a[be_c c]] = a * (b * c)$ so the product is associative.

Now let $a \in H$ and let $e = e_a$ to simplify the notation. Then $[eea'] = [ee[eae]] = [eae] = a'$ and $[a'a'e] = [[eae][eae]e] = [[[eae]ea]ee] = [[ea[eea]]ee] = [[eaa]ee] =$

$[eee] = e$. Hence $a' \sim e$, which gives $e_{a'} = e_a = e$ and $a'' = [e[eae]e] = a$. We also have $a * a' * a = [[ae[eae]]ea] = a$. Furthermore $a * a' = [ae[eae]] = [a[aee]e] = [aae] = e$ so $(a * a') * b = e * b = [eeb] = [bee] = b * (a * a')$ proving the Clifford semigroup condition.

At this point we have proved that H is a Clifford semigroup (the condition $(a * b)' = b' * a'$ follows from the others). It remains to show that the semiheap product can be recovered from the Clifford semigroup operations. But $a * b' = [ae_{b'}b'] = [ae_b[e_bbe_b]] = [abe_b]$ and hence

$$\begin{aligned} a * b' * c &= [[abe_b]e_cc] = [a[e_ce_bb]c] = [a[e_c[e_be_be_b]b]c] = [a[[e_ce_be_b]e_bb]c] \\ &= [a[[e_ce_ce_b]e_bb]c] = [a[e_ce_c[e_be_bb]]c] = [a[e_ce_cb]c] = [ab[e_ce_cc]] \\ &= [abc] \end{aligned}$$

giving $[(H, *, ')] = H$, and completing the proof. □

Not all near heaps have spines and therefore not all near heaps are of the form $[S]$ for a Clifford semigroup S. An example of a near heap without a spine is the free near heap generated by two elements. A simpler example is the free fully symmetric near heap on two generators, which we now construct.

Example 9. *The free fully symmetric near heap on two generators.*

Let a, b be two symbols and form the finite set of strings

$$\mathfrak{H}(a, b) = \{a, b, a^2b, ab^2\}$$

(where a^2 denotes aa and so on), and define a ternary operation on $\mathfrak{H}(a, b)$ by setting $[w_1w_2w_3]$ to be the result of first sorting into alphabetical order the letters in the string concatenation $w_1w_2w_3$ to give a^mb^n, where $m + n$ is necessarily odd, and then reducing to an element of $\mathfrak{H}(a, b)$ by reducing powers modulo 2 down to 1 or 2 (or else 0 if that letter did not occur at all).

Thus for example $[aaa] = a$, and

$$[(aab)(b)(abb)] \to a^3b^4 \to ab^2.$$

It is easy to confirm that $\mathfrak{H}(a, b)$ is a fully symmetric semiheap in the sense that

$$[w_1w_2w_3] = [w_{\theta_1}w_{\theta_2}w_{\theta_3}]$$

where $\theta_1, \theta_2, \theta_3$ is any permutation of $1, 2, 3$. Moreover it is clearly idempotent, and hence is a near heap.

We claim that $\mathfrak{H}(a, b)$ has no spine. The congruence $\sim$ has at least three equivalence classes on $\mathfrak{H}(a, b)$, two of which are $\{a\}$ and $\{b\}$. Any spine for $\mathfrak{H}(a, b)$ must therefore include both a and b, although $[aab] = a^2b \neq ab^2 = [abb]$ so a and b cannot coexist in a subsemilattice.

A similar though more complicated argument can be used in the non-fully symmetric case, to establish that the free near heap on two generators is an infinite near heap without a spine.

To completely fit nears heaps into the pattern suggested by Proposition 1, we must consider whether or not all near heaps can be *embedded* in a near heap of the form $[S]$ for some Clifford semigroup S. The answer to that question is "yes" as we shall prove.

4. Near heaps as strong semilattices of heaps

Every Clifford semigroup S is a semilattice of groups, meaning that there is a congruence θ on S for which each θ-class is a group and S/θ is a semilattice. By Proposition 6, every near heap is a "semilattice of heaps" in the obvious sense. But every Clifford semigroup is not only a semilattice of groups but a *strong semilattice of groups* as in [3]; full information about the multiplication in a Clifford semigroup S can be obtained from such a strong semilattice of group decomposition. Thus if $S = \bigcup_{e\in E(S)} S_e$ is the decomposition of S into groups S_e (one for each $e \in E(S)$), then for every $e, f \in L$ with $e \geq f$, there is a group homomorphism $\phi_{e,f} : S_e \to S_f$ for which

- $\phi_{e,e}$ is the identity map on S_e, and
- for all $e, f, g \in L$ for which $e \geq f \geq g$, $\phi_{f,g} \circ \phi_{e,f} = \phi_{e,g}$.

One then finds that for $a_e \in S_e$ and $a_f \in S_f$, $a_e a_f = \phi_{e,ef}(a_e)\phi_{f,ef}(a_f)$ as calculated in S_{ef}, so that information about the multiplications in each of the groups together with all the homomorphisms $\phi_{e,f}$ completely determines the multiplication on S.

One can define an abstract strong semilattice of groups to be any disjoint union of groups $S = \bigcup_{e\in L} S_e$, L a semilattice, equipped with homomorphisms as above, and with multiplication *defined* as follows: for all $a_e \in S_e$ and $a_f \in S_f$,

$$a_e a_f := \phi_{e,ef}(a_e)\phi_{f,ef}(a_f) \text{ as calculated in } S_{ef}.$$

It then follows easily that $S_e S_f \subseteq S_{ef}$ for all $e, f \in L$. S can be shown to be a semigroup; indeed it is always a Clifford semigroup (with a' defined to be the unique $b \in S$ such that $aba = a, bab = b$). Hence every Clifford semigroup is a strong semilattice of groups. For the details, consult [3].

Of course, in a Clifford semigroup S, the semilattice $L = E(S)$ is embedded in the semigroup: it is both a subsemigroup and a quotient semigroup. Indeed L is the spine of the near heap S. However, as we have seen, not all near heaps have spines. In the cases that do, there will be some sort of strong semilattice of heaps representation. The interest is in the general case.

A *strong semilattice of heaps* is defined to be a disjoint union of heaps $S = \bigcup_{e\in L} S_e$, where L is a semilattice, such that there are heap homomorphisms $\phi_{e,f} : S_e \to S_f$ for each $e, f \in L$ for which $e \leq f$, and for which

- $\phi_{e,e}$ is the identity map on S_e, and
- for all $e, f, g \in L$ for which $e \geq f \geq g$, $\phi_{f,g} \circ \phi_{e,f} = \phi_{e,g}$.

Such an S is turned into a ternary algebra by setting, for all $a_e \in S_e$, $a_f \in S_f$ and $a_g \in S_g$,

$$[a_e a_f a_g] = [\phi_{e,efg}(a_e)\phi_{f,efg}(a_f)\phi_{g,efg}(a_g)].$$

Notation: $[S_e, L, \phi_{e,f}]$.

Theorem 10. *A strong semilattice of heaps $[S_e, L, \phi_{e,f}]$ is a near heap which is a semilattice of the heaps S_e, with the semilattice isomorphic to L.*

Proof: It is obvious that the S_e are closed under the ternary operation on S (and of course are heaps). We next show S is a semiheap.

Let $a_\alpha \in S_\alpha$ for each $\alpha \in \{e, f, g, h, i\}$. Then

$$\begin{aligned} & [[a_e a_f a_g] a_h a_i] \\ = \ & [[\phi_{e,efg}(a_e)\phi_{f,efg}(a_f)\phi_{g,efg}(a_g)] a_h a_i] \\ = \ & [\phi_{efg,efghi}([\phi_{e,efg}(a_e)\phi_{f,efg}(a_f)\phi_{g,efg}(a_g)])\phi_{h,efghi}(a_h)\phi_{i,efghi}(a_i)] \\ = \ & [[\phi_{e,efghi}(a_e)\phi_{f,efghi}(a_f)\phi_{g,efg}(a_g)]\phi_{h,efghi}(a_h)\phi_{i,efghi}(a_i)] \\ = \ & [[\phi_{e,efghi}(a_e)[\phi_{h,efghi}(a_h)\phi_{g,efg}(a_g)]\phi_{f,efghi}(a_f)]\phi_{i,efghi}(a_i)], \end{aligned}$$

which a very similar routine calculation shows is equal to $[a_e[a_h a_g a_f]a_i]$, and so also by symmetry to $[a_e a_f [a_g a_h a_i]]$, so S is a semiheap.

We turn to the near heap laws. Idempotence is immediate (since the calculation of $[a_e a_e a_e]$ takes place wholly within S_e, which is a heap). Finally,

$$\begin{aligned} & [a_e a_e a_f] \\ = \ & [\phi_{e,ef}(a_e)\phi_{e,ef}(a_e)\phi_{f,ef}(a_f)] \\ = \ & [\phi_{e,ef}(a_e)\phi_{e,ef}(a_e)\phi_{f,ef}(a_f)] \text{ since the computation is inside the heap } S_{ef} \\ = \ & [\phi_{e,ef}(a_e)\phi_{f,ef}(a_f)\phi_{f,ef}(a_f)] \text{ again working in } S_{ef} \\ = \ & [a_e a_f a_f] \end{aligned}$$

as required. It is obvious that $[S_e S_f S_g] \subseteq S_{efg} = S_{[efg]}$ for all $e, f, g \in L$, so the partition of S into the disjoint S_e is a congruence θ, and that $S/\theta \cong L$. $\square$

This result justifies the term "strong semilattice of heaps". Note that L in this proof is not in general represented as a subset of S, only as a quotient.

The following result extends Theorem 4.2.1 of [3] stating that every Clifford semigroup is a semilattice of groups, to cover "spineless" cases.

Theorem 11. *Let H be a ternary algebra, L a semilattice. The following are equivalent.*

(1) *H is a near heap with $L \cong H/\sim$.*
(2) *H is a semilattice of heaps $\bigcup_{e\in L} H_e$.*
(3) *H is a strong semilattice of heaps $[H_e, L, \phi_{e,f}]$.*

PROOF: (1) $\Rightarrow$ (2) has been shown already.

For (2) $\Rightarrow$ (3), let $H = \bigcup_{e\in L} H_e$ be a semilattice of heaps. Let $e, f \in L$, with $f \leq e$. Then for all $a_e \in S_e$ and $a_f \in S_f$, $[a_e a_f a_f] \in S_{[eff]} = S_f$. So define $\psi_{e,f} : S_e \to S_f$ by setting $\phi_{e,f}(a_e) = [a_e a_f a_f]$ for any $a_f \in S_f$. This is well-defined (independent of the choice of $a_f \in S_f$), because if also $b_f \in S_f$, then, using the heap laws as needed, we have

$$\begin{aligned}
[a_e b_f b_f] &= [a_e[a_f a_f b_f][a_f a_f b_f]]\\
&= [a_e[b_f a_f a_f][b_f a_f a_f]]\\
&= [[a_e a_f a_f] b_f [b_f a_f a_f]]\\
&= [[[a_e a_f a_f] b_f b_f] a_f a_f]\\
&= [[a_e a_f a_f] a_f a_f] \text{ since } [a_e a_f a_f] \in S_f\\
&= [a_e a_f a_f].
\end{aligned}$$

Now $\phi_{e,e}$ is the identity on S_e because for any $a_e \in S_e$, $\phi_{e,e}(a_e) = [a_e b_e b_e] = a_e$ for any $b_e \in S_e$.

We next show $\phi_{e,f}$ is a homomorphism $S_e \to S_f$. So let $a_e, b_e, c_e \in S_e$, with $d_f \in S_f$. Then repeatedly using the heap laws in S_f,

$$\begin{aligned}
\phi_{e,f}([a_e b_e c_e]) &= [[a_e b_e c_e] d_f d_f]\\
&= [[[a_e b_e c_e] d_f d_f] d_f d_f] \text{ since } S_e \subseteq S_f \text{ and } S_f \text{ is a heap}\\
&= [[a_e[d_f c_e b_e] d_f] d_f d_f]\\
&= [[a_e d_f d_f][d_f c_e b_e] d_f]\\
&= [[[a_e d_f d_f] b_e c_e] d_f d_f]\\
&= [[a_e d_f d_f] b_e c_e].
\end{aligned}$$

However,

$$\begin{aligned}
[\phi_{e,f}(a_e)\phi_{e,f}(b_e)\phi_{e,f}(c_e)] &= [[a_e d_f d_f][b_e d_f d_f][c_e d_f d_f]]\\
&= [[[a_e d_f d_f] d_f d_f] b_e [c_e d_f d_f]]\\
&= [[a_e d_f d_f] b_e [c_e d_f d_f]]\\
&= [a_e [b_e d_f d_f][c_e d_f d_f]]\\
&= [[a_e [b_e d_f d_f] c_e] d_f d_f]\\
&= [[a_e d_f [d_f b_e c_e]] d_f d_f]\\
&= [a_e d_f [[d_b b_e c_e] d_f d_f]]\\
&= [a_e d_b [d_b b_e c_e]]\\
&= [[a_e d_f d_f] b_e c_e]\\
&= \phi_{e,f}([a_e b_e c_e])
\end{aligned}$$

from the above.

Finally we must show that for all $e, f, g \in L$ for which $e \geq f \geq g$, $\phi_{f,g} \circ \phi_{e,f} = \phi_{e,g}$. So suppose $e, f, g \in L$ satisfy $e \geq f \geq g$. Then for any $a_e \in S_e$, $a_f \in S_f$ and $a_g \in S_g$,

$$\begin{aligned}
(\phi_{f,g} \circ \phi_{e,f})(a_e) &= [[a_e a_f a_f] a_g a_g] \\
&= [[[[a_e a_f a_f] a_g a_g] a_g a_g] \\
&= [[[a_e [a_g a_f a_f] a_g] a_g a_g] \\
&= [[a_e a_g a_g][a_g a_f a_f] a_g] \\
&= [[[a_e a_g a_g] a_f a_f] a_g a_g] \\
&= [[a_e a_g a_g] a_f [a_f a_g a_g]] \\
&= [a_e [a_f a_g a_g][a_f a_g a_g]] \\
&= \phi_{e,g}(a_e)
\end{aligned}$$

since $[a_f a_g a_g] = \phi_{f,g}(a_f) \in S_g$. This completes the proof that any semilattice of heaps is a strong semilattice of heaps.

For (3) $\Rightarrow$ (1), the fact that $H = [H_e, L, \phi_{e,f}]$ is a near heap was shown in Theorem 10. For each $a \in H$, let a^* be the $\sim$-class containing a. To show that L is the same as in Proposition 6, it suffices to show that the heaps H_e in $H = [H_e, L, \phi_{e,f}]$ are precisely the subheaps a^* of H. It suffices to show that for all $a \in H$, if $a \in H_e$ then $a^* = H_e$. So suppose $a \in H_e$. Of course $H_e \subseteq a^*$ by maximality of a^*. Conversely, if $b \in a^*$, suppose $b \in H_f$. Then $[abb] = a$, so in particular, $H_e \ni a = [aaa] = [bba] \in H_{ef}$, so $ef = e$, as otherwise $H_{ef} \cap H_e = \emptyset$. By symmetry (since also $a \in b^*$) $ef = f$, so $e = f$ and $b \in H_e$. Hence $a^* \subseteq H_e$. □

Note that the homomorphisms $\phi_{e,f}$ used to define a given strong semilattice of heaps $H = [S_e, L, \phi_{e,f}]$ (that is, a near heap by the above result) are uniquely determined by the near heap. First, the maximal subheap decomposition $\bigcup_{e \in L} H_e$ (including L up to isomorphism) depends only on the structure of H, and for $a_e \in H_e$ and $a_f \in H_f$, we have $[a_e a_f a_f] = [\phi_{e,ef}(a_e)\phi_{f,ef}(a_f)\phi_{f,ef}(a_f)] \in H_{ef}$, a heap, and so $[a_e a_f a_f] = \phi_{e,ef}(a_e)$, so $\phi_{e,ef}$ is wholly determined by the near heap operation. This parallels the situation for Clifford semigroups.

However, it follows from the main result of the previous section that for any near heap of the form $[S]$ where S is a Clifford semigroup, the structure of $[S]$ completely determines the Clifford semigroup operations on S.

Corollary 12. *Suppose S_1 and S_2 are two Clifford semigroups on the same underlying set for which $[S_1] = [S_2]$. Then $S_1 = S_2$.*

Proof: First, it is a routine exercise to check that, given a representation of the Clifford semigroup S as a strong semilattice of groups, there is an induced representation of $[S]$ as a strong semilattice of heaps, using the same semilattice, the subheaps associated with the subgroups, and the same homomorphisms. Then, if S_1 and S_2 are two Clifford semigroups on the same underlying set for which $[S_1] = [S_2]$, the homomorphisms inherited from S_1 and S_2 (as well as the S_e of course) must be the same, and so S_1 and S_2 are also the same. □

The corresponding fact for arbitrary involuted semigroups fails: the involuted semigroup operations on S are not determined by the structure of $[S]$. For example, the zero semiheap on a set, in which all ternary products are zero, arises from distinct, even non-isomorphic, involuted semigroups on the set. It would be interesting to determine those varieties $\mathcal{V}$ of involuted semigroups for which the operations on $S \in \mathcal{V}$ are completely determined by $[S]$ (at least up to isomorphism).

5. Embedding near heaps in Clifford semigroups

As we have seen, Clifford semigroups give rise to near heaps, and indeed all of the information present in the Clifford semigroup is retained by the near heap. However, not every near heap is $[S]$ where S is a Clifford semigroup. So what can be said? Can we give an embedding theorem for near heaps, thereby providing a completed entry in Proposition 1?

Note that the cases considered in Proposition 1 can all be dealt with by first showing that every semiheap of a given type may be embedded in a bi-unital one of the same type, and then invoking Proposition 2. This is the approach taken in [5]. However, that approach does not readily extend to near heaps.

First some observations about representations in terms of partial mappings. By the Wagner-Preston theorem, any inverse semigroup G is representable as a subsemigroup of the symmetric semigroup of one-to-one partial maps $X \to X$ for some set X. The actual representation used is a left regular one, which maps $a \in G$ to the partial map $\psi_a : G \to G$ given by $\psi_a(x) = ax$ for all x such that $a'ax = x$; when this is done, $\psi_{a'a}$ is the restriction of the identity map to the domain of ψ_a and $\psi_{aa'}$ is the restriction of the identity map to its range.

Representing a Clifford semigroup in this way, the inverse semigroup of partial maps has the property that every partial map has equal domain and range (since $aa' = a'a$), and that the partial maps having a given domain form a group (since $aa' = a'a$ is an identity element). Moreover the bijections associated with $a'a$ and $b'b$ agree on $a'ab'b$: the two heaps of maps restrict down to the same heap of maps on the smaller domain. This is a concrete way to interpret the fact that every Clifford semigroup is a semilattice of heaps: the semilattice is the set of domains (=ranges) determined by $E(G) = \{a'a \mid a \in G\}$, and the heaps are the associated partial maps with domains and ranges given by the aa'.

Likewise, it is well known that every generalised heap may be represented as a semiheap of one-to-one partial maps $X \to Y$ (where without loss of generality every element of x is in the domain of one of the maps and every element of y is mapped to by one of the maps): the operation on such maps is $[fgh] = f \circ g^{-1} \circ h$. Again, interpreting the near heap law shows that the maps can be organised into subheaps according to their domains, and those maps with a given domain also have identical ranges (not equal to their domains this time, since they are in different sets). For a fixed represented near heap, let L_X be the collection of domains and L_Y the collection of ranges: both sets are semilattices under intersection, as for generalised heaps in general. Again, it follows easily that two sets of heaps (corresponding to two possible domains) restrict down to the same heap when the intersection of their domains in L_X is considered. Again, all of this is nothing but a concrete realisation of Theorem 11: every near heap is a semilattice of heaps.

We are now in a position to give the main result of this section.

Theorem 13. *Every near heap is embeddable in the semiheap obtained from a Clifford semigroup.*

PROOF: Without loss of generality, let H be a near heap of partial maps $X \to Y$ as described above. We shall show how to identify X and Y in such a way that the resulting Clifford semigroup embeds the original near heap.

Choosing $S \in L_X$, we have a fixed set (indeed heap) of bijections H_S from S to $S' \in L_Y$. Choose $x \in X$ and for any $S \in L_X$ for which $x \in S$, define $T_x = \{p(x) \mid p \in H_S\}$, a subset of Y independent of the choice of S by the restriction property. This can be extended to arbitrary subsets of X in the obvious way: for $W \subseteq X$, define $T(W) = \bigcup_{x \in W} T_x$.

Likewise for $y \in Y$, define $T'_y = \{q(y) \mid q^{-1} \in H_S\}$, where $S \subseteq X$ is such that $y \in f(S)$, and extend to subsets of Y as for T above to give $T'(S')$. Now if $a = q^{-1}(p(x)) \in T'(T_x)$, then $q(a) = p(x) \in T_x$, so $T(T'(T_x)) \subseteq T_x$, and because the opposite inclusion obviously holds, we have $T(T'(T_x)) = T_x$. It now follows easily that there is a one-to-one correspondence between subsets of the form T_x in Y and $T'(T_x)$ in X.

Now suppose $x' \notin T'(T_x)$. Suppose $b \in T'_x \cap T_x$. So $b = p_1(x') = p_2(x)$ for some bijections $p_1 \in H_{S_1}$ (where $S_1 \in L_X$ contains x') and $p_2 \in H_{S_2}$ (where $S_2 \in L_X$ contains x). Hence $y = p_1^{-1} \circ p_2(x) \in T'(T_x)$, a contradiction. Hence $T_x \cap T_{x'} = \emptyset$. Similarly then, $S(T_x) \cap S(T_{x'}) = S(T_x \cap T_{x'}) = S(\emptyset) = \emptyset$. Thus the T_x form a partition of Y and the $S(T_x)$ form a partition of X.

Note that for any $S \in L_X$ for which $x \in S$, if $a \in T'(T_x)$, then $a = q^{-1}(p(x))$ for some $p, q \in H_S$, so $a \in S$; hence $T'(T_x) \subseteq S$ for every $S \in L_X$ containing x. Pick $p \in H_S$ and define $\psi_x : T'(T_x) \to T_x$ by setting $\psi_x(a) = p(a)$ for all $a \in T'(T_x)$, a one-to-one function (being a restriction of the bijective function $p : S \to f(S)$). It is also surjective, as if $b \in T_x$, then $a = p^{-1}(b) \in T'(T_x)$ satisfies $p(a) = b$. (Hence only one choice of bijection was really needed in defining T_x and so on.)

We build a bijection $\psi : X \to Y$ out of the bijections ψ_x in the expected way: $\psi(x) = \psi_x(x)$ for all $x \in X$. This works because the $T'(T_x)$ are a partition of X (and likewise for the T_x in Y). For convenience we make direct use of the inverse bijection $\phi = \psi^{-1}$, mapping $Y \to X$.

We now map H into the inverse semigroup $I(X)$ of one-to-one partial mappings on X. Thus let θ be the mapping taking H into $I(X)$ such that for each $f \in H$, $\theta(f) = \phi \circ f$; clearly $\theta(f) \in I(X)$. We show θ is an embedding of H into the generalised heap $I(X)$ (equipped with its usual semiheap operation).

For $f, g, h \in H$,

$$\begin{aligned}
[\theta(f)\theta(g)\theta(h)] &= \theta(f) \circ \theta(g)^{-1} \circ \theta(h) \\
&= \theta(f) \circ (\phi \circ g)^{-1} \circ \phi \circ h \\
&= \theta(f) \circ g^{-1} \circ \phi^{-1} \circ \phi \circ h \\
&= \phi \circ f \circ g^{-1} \circ h \\
&= \phi \circ [fgh] \\
&= \theta([fgh]).
\end{aligned}$$

So θ is a homomorphism which is obviously injective (since ϕ is a bijection).

Now let M be the inverse subsemigroup of $I(X)$ generated by $H_1 = \{\theta(f) \mid f \in H\}$ under the operations of inversion and composition.

Note that each $\theta(f) \in M$ (where $f \in H$) has equal domain and range, so $\theta(f) \circ \theta(f)^{-1} = \theta(f)^{-1} \circ \theta(f)$, and if also $g \in H$, then $\theta(f) \circ \theta(f)^{-1} \circ \theta(g) = \theta([ffg]) = \theta([gff]) = \theta(g) \circ \theta(f)^{-1} \circ \theta(f)$. A typical element of M is a composite $w = a_1 a_2 \cdots a_n$ of elements of $I(X)$ of the form $\theta(f)$ or $\theta(f)^{-1}$ for some $f \in H$, and for such elements we have just shown that $xx' = x'x$ and $xy'y = yy'x$. It therefore follows easily that $ww^{-1} = (a_1 a_2 \cdots a_n)(a_n^{-1} \cdots a_2^{-1} a_1^{-1})$ which easily rearranges to $(a_1 a_1^{-1})(a_2 a_2^{-1}) \cdots (a_n a_n^{-1})$, which by symmetry also equals $w^{-1}w$. Hence M is a Clifford semigroup, embedding H. □

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