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Comparison game on Borel ideals

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Abstract. We propose and study a "classification" of Borel ideals based on a natural infinite game involving a pair of ideals. The game induces a pre-order $\sqsubseteq$ and the corresponding equivalence relation. The pre-order is well founded and "almost linear". We concentrate on F_σ and $F_{\sigma\delta}$ ideals. In particular, we show that all F_σ-ideals are $\sqsubseteq$-equivalent and form the least equivalence class. There is also a least class of non-F_σ Borel ideals, and there are at least two distinct classes of $F_{\sigma\delta}$ non-F_σ ideals.

Keywords: ideals on countable sets, comparison game, Tukey order, games on integers

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Introduction

We propose and study a natural Wadge-like two-player game, called the comparison game, associated to a pair of ideals. The game introduces a pre-order $\sqsubseteq$ and the corresponding equivalence relation. On Borel ideals, this pre-order is well-founded and almost-linear (all antichains have size at most 2).

We show that all F_σ-ideals are $\sqsubseteq$-equivalent, and form the least equivalence class. In order to do this, we prove a combinatorial characterization of F_σ-ideals, identifying F_σ-ideals as exactly those Borel ideals which have the P^+(tree)-property considered by Laflamme and Leary [4]. There is also a "second least" equivalence class, the equivalence class of the ideal I_0 defined below. We show that there are at least two distinct classes of $F_{\sigma\delta}$ non-F_σ ideals, and exactly two distinct classes of analytic P-ideals.

We also study a problem of I. Farah concerning inner structure of $F_{\sigma\delta}$-ideals, closely related to the comparison game.

By an *ideal on* ω we mean an ideal I on a countable set X (typically $X = \omega$ the first infinite ordinal) which contains all finite subsets of X and does not contain X. By considering I as a subspace of $\mathcal{P}(X)$, endowed with the product topology of the Cantor space 2^X through the bijection $A \mapsto \chi_A$, we can calculate the Borel complexity of I.

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1. Comparison Game Order

Definition 1.1. Let I and J be ideals on ω. The *Comparison Game* for I and J denoted by $G(\mathsf{I},\mathsf{J})$ is defined as follows: In step n, Player I chooses an element I_n of I and Player II chooses an element J_n of J. Player II wins if $\bigcup_n I_n \in \mathsf{I}$ if and only if $\bigcup_n J_n \in \mathsf{J}$; otherwise, Player I wins.

Comparison game induces an order between ideals on ω.

Definition 1.2. Let I and J be ideals on ω. We say $\mathsf{I} \sqsubseteq \mathsf{J}$ if Player II has a winning strategy in the comparison game $G(\mathsf{I},\mathsf{J})$. We say that $\mathsf{I} \simeq \mathsf{J}$ if $\mathsf{I} \sqsubseteq \mathsf{J}$ and $\mathsf{J} \sqsubseteq \mathsf{I}$.

Let us note that the relation $\sqsubseteq$ is reflexive and transitive, but not antisymmetric; and the relation $\simeq$ is an equivalence relation.

First, we will prove that the comparison game among Borel ideals is determined. To that end we define the following game

Definition 1.3. The game $G'(\mathsf{I},\mathsf{J})$ is defined for ideals I and J on ω as follows: In step n Player I chooses a natural number k_n and Player II chooses a natural number l_n. Player II wins if $\{k_n : n < \omega\} \in \mathsf{I}$ if and only if $\{l_n : n < \omega\} \in \mathsf{J}$.

Let us note that by defining a set $\tilde{\mathcal{X}} = \{x \in \omega^\omega : \operatorname{rng}(x) \in \mathcal{X}\}$ for a subset $\mathcal{X}$ of $\mathcal{P}(\omega)$, we have that game $G'(\mathsf{I},\mathsf{J})$ is equivalent to the *Wadge game* $W(\tilde{\mathsf{I}},\tilde{\mathsf{J}})$ (see [3]).

Theorem 1.4. *Player I has a winning strategy in $G(\mathsf{I},\mathsf{J})$ if and only if Player I has a winning strategy in $G'(\mathsf{I},\mathsf{J})$, and the same for Player II.*

Proof: First, let us assume that Player I has a winning strategy σ on the game $G(\mathsf{I},\mathsf{J})$, and take a bijective function f from ω onto $\omega\times\omega$ such that if $f(n) = \langle k,l\rangle$ then $\max\{k,l\} \leq n$. A winning strategy for Player I in $G'(\mathsf{I},\mathsf{J})$ can be described by playing in parallel the game $G(\mathsf{I},\mathsf{J})$. In step 0, Player I plays the first element k_0 of I_0, where $I_0 = \sigma(\emptyset)$. If in the first n-many steps the players played a sequence $\langle k_0, l_0, \dots, k_n, l_n\rangle$ in the game $G'(\mathsf{I},\mathsf{J})$, and attached to this sequence, we consider the corresponding sequence $\langle I_0, \{l_0\}, I_1, \{l_1\}, \dots, I_n, \{l_n\}\rangle$ in the game $G(\mathsf{I},\mathsf{J})$ according to σ, then, by taking k_{n+1} as the k-th element of I_l, where $f(n+1) = \langle k,l\rangle$, (if it exists, and $k_{n+1} = 0$ if not), we have defined the winning strategy for Player I. This is true since $\bigcup_{n<\omega} I_n \subseteq \{k_n : n < \omega\} = \{0\} \cup \bigcup_n I_n$ and the sequence $\langle I_0, \{l_0\}, I_1, \{l_1\}, \dots\rangle$ follows a winning strategy for Player I in $G(\mathsf{I},\mathsf{J})$, that is $\{k_n : n < \omega\} \in \mathsf{I}$ if and only if $\{l_n : n < \omega\} \notin \mathsf{J}$.

On the other hand, let us assume that Player I has a winning strategy τ in $G'(\mathsf{I},\mathsf{J})$. In step 0, Player I plays $\{k_0\}$, where $k_0 = \tau(\emptyset)$, and in step $n+1$ Player I plays $\{k_{n+1}\}$ where k_{n+1} is the answer given by Player I in $G'(\mathsf{I},\mathsf{J})$ following τ when Player II has played the l-th element l_{n+1} of J_k where $f(n+1) = \langle k,l\rangle$, if J_k has at least l elements, and 0 if not. Then, $\bigcup_n \{k_n\} \in \mathsf{I}$ if and only if $\{k_n : n < \omega\} \in \mathsf{I}$ if and only if $\bigcup_n J_n = \{0\} \cup \{l_n : n < \omega\} \notin \mathsf{J}$.

Analogously it can be proved that Player II has a winning strategy in $G(\mathsf{I},\mathsf{J})$ if and only if Player II has a winning strategy in $G'(\mathsf{I},\mathsf{J})$. $\square$

By the previous theorem we can conclude that $\mathsf{I} \sqsubseteq \mathsf{J}$ if and only if $\tilde{\mathsf{I}} \leq_W \tilde{\mathsf{J}}$. As the Wadge order is well founded (Theorem 21.15 in [3]), so is the comparison game order, which is also "almost linear".

Lemma 1.5. *If* I, J *and* K *are Borel ideals,* $\mathsf{I} \not\sqsubseteq \mathsf{J}$ *and* $\mathsf{J} \not\sqsubseteq \mathsf{K}$ *then* $\mathsf{K} \sqsubseteq \mathsf{I}$.

Proof: The hypothesis means that Player I has a winning strategy in games $G(\mathsf{I},\mathsf{J})$ and $G(\mathsf{J},\mathsf{K})$. Then Player II is going to follow those strategies. First, in both games $G(\mathsf{I},\mathsf{J})$ and $G(\mathsf{J},\mathsf{K})$, Player I follows her own strategies, producing I_0 and J_0. Given the first choice K_0 of Player I in $G(\mathsf{K},\mathsf{I})$, let us consider K_0 as the answer of Player II in $G(\mathsf{J},\mathsf{K})$, and then let J_1 be the answer of Player I in the same game, given by her winning strategy. Let us consider J_1 as the answer of Player II in $G(\mathsf{I},\mathsf{J})$ and let I_1 be the answer of Player I given by her winning strategy and then I_1 will be the answer of Player II in $G(\mathsf{K},\mathsf{I})$. Let us suppose that in step n, Player I chooses a set K_n. That set can be considered as the answer of Player II in $G(\mathsf{J},\mathsf{K})$ for the sequence $\langle J_0, K_0, J_1, \dots, J_n\rangle$, and then the winning strategy for Player I in this game makes her choose a set J_{n+1}. Such set J_{n+1} can be considered as the answer of Player II in $G(\mathsf{I},\mathsf{J})$ for the sequence $\langle I_0, J_1, I_1, \dots, I_n\rangle$ and then the winning strategy for Player I makes her choose a set I_{n+1}. Such set will be what Player II plays in $G(\mathsf{K},\mathsf{I})$ in step n. Hence, since the sequences $\langle J_0, K_0, J_1, K_1, \dots\rangle$ and $\langle I_0, J_1, I_1, J_2, \dots\rangle$ follow the winning strategies for Player I in $G(\mathsf{J},\mathsf{K})$ and $G(\mathsf{I},\mathsf{J})$ respectively, we have that $\bigcup_n J_n \in \mathsf{J}$ if and only if $\bigcup_n K_n \notin \mathsf{K}$, and $\bigcup_{n\geq 1} J_n \in \mathsf{J}$ if and only if $\bigcup_n I_n \notin \mathsf{I}$ and then we are done. □

An immediate consequence of the previous lemma is that if we have two incomparable ideals then every other ideal has the same order relation with both ideals of the incomparable pair.

Corollary 1.6. *Let* I *and* J *be two* $\sqsubseteq$*-incomparable ideals. Then, for any ideal* K *on* ω *which is not* $\sqsubseteq$*-equivalent to* I *nor* J, ($\mathsf{K} \sqsubseteq \mathsf{I}$ *iff* $\mathsf{K} \sqsubseteq \mathsf{J}$) *or* ($\mathsf{I} \sqsubseteq \mathsf{K}$ *iff* $\mathsf{J} \sqsubseteq \mathsf{K}$). □

The next lemma shows that the order $\sqsubseteq$ "almost" respects Borel complexities.

Proposition 1.7. *If* I *and* J *are Borel ideals,* $\mathsf{I} \sqsubseteq \mathsf{J}$ *and* I *is* Σ_α *then* J *is* $\Sigma_{\alpha+1}$. □

Proof: It suffices to show that if I is a Σ^0_α (respectively Π^0_α) ideal then $\tilde{\mathsf{I}}$ is a $\Sigma^0_{\alpha+1}$ (resp. $\Pi^0_{\alpha+1}$) set. Define a function $\mathrm{rng}_n : \omega^\omega \to \mathcal{P}(\omega)$ by $\mathrm{rng}_n(x) = \{x(k) : k < n\}$ for all $x \in \omega^\omega$. Note that rng_n is a continuous function and $\mathrm{rng}(x) = \lim_{n\to\infty} \mathrm{rng}_n(x)$ for all $x \in \omega^\omega$. Hence, preimages of clopen sets under rng are Δ^0_2 sets, and inductively we can get the result. □

Another consequence is that comparison game order is at least as long as the Borel hierarchy.

Corollary 1.8.
- *The game* $G(\mathsf{I},\mathsf{J})$ *is determined for every pair* I, J *of Borel ideals.*
- *The order* $\sqsubseteq$ *is well-founded.*

- *The equivalence classes of $\simeq$ are unions of "intervals" of Wadge degrees of ideals.*
- *There are uncountably many $\simeq$-classes.*

Question 1.9. *Is the order $\sqsubseteq$ linear (a well order)? Are there two Borel ideals which are $\sqsubseteq$-equivalent, but one is Σ_α while the other is not?*

2. F_σ-ideals in the comparison game order

The ideal **Fin** is below all ideals in the $\sqsubseteq$-order. We will show that the equivalence class of **Fin** consists exactly of F_σ-ideals. In the process we give a combinatorial characterization of F_σ-ideals as exactly those Borel ideals which satisfy the P^+(tree)-property.

Proposition 2.1. *Let J be an ideal on ω. Then* $\mathbf{Fin} \sqsubseteq \mathsf{J}$.

Proof: A winning strategy for Player II in $G(\mathbf{Fin}, \mathsf{J})$ is the following. Player II answers the initial interval $J_n = [0, \max(\bigcup_{i\leq n} I_i)]$, given that I_i, $(i \leq n)$ are the finite sets played by Player I until step n. Then, $\bigcup_n I_n \in \mathbf{Fin}$ implies $\bigcup_n J_n$ is a finite set and then an element of J. On the other hand, if $\bigcup_n I_n \notin \mathbf{Fin}$ then $\bigcup_n J_n = \omega \in \mathsf{J}^+$. $\square$

Remark 2.2. If I is an ideal on ω then $\mathsf{I} \sqsubseteq \mathbf{Fin}$ if and only if Player II has a winning strategy in the game $G''(\mathsf{I})$ defined as follows: In step n, Player I chooses an element I_n of I and Player II chooses a natural number k_n. Player II wins if $\bigcup_n I_n \in \mathsf{I}$ if and only if the sequence $\{k_n : n < \omega\}$ is bounded.

To see this, note that if Player II has a winning strategy in $G(\mathsf{I}, \mathbf{Fin})$ then in step n, Player II of $G''(\mathsf{I})$ plays $k_n = \max J_n$, where J_n is the finite set played by Player II following a fixed winning strategy for her in $G(\mathsf{I}, \mathbf{Fin})$, keeping the same play by Player I. On the other hand, the winning strategy for Player II in $G(\mathsf{I}, \mathbf{Fin})$ consists in to play $\{k_n\}$ in step n, where k_n is the answer given in step n for a fixed winning strategy for Player II in $G''(\mathsf{I})$.

Dealing with F_σ ideals, the following theorem is useful. A *lower semicontinuous submeasure for ω* (lscsm) is a function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ such that (1) $\varphi(\emptyset) = 0$, (2) $\varphi(A) \leq \varphi(B)$ if $A \subseteq B$, (3) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ and (4) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [0, n])$. If φ is a lscsm then $\mathrm{Fin}(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ is an F_σ-ideal, and moreover:

Theorem 2.3 (Mazur [5]). *For each F_σ-ideal I there is a lscsm φ such that $\mathsf{I} = \mathrm{Fin}(\varphi)$.*

Using Mazur's theorem we can prove that all F_σ-ideals are equivalent.

Lemma 2.4. *If I is an F_σ-ideal then* $\mathsf{I} \sqsubseteq \mathbf{Fin}$.

Proof: Let φ be a lscsm such that $\mathsf{I} = \mathrm{Fin}(\varphi)$. Let us play the game $G''(\mathsf{I})$. In step n Player II plays k_n, the minimal $k \in \omega$ such that $\varphi(\bigcup_{j\leq n} I_j) < k$. Then $\varphi(\bigcup_n I_n) < \infty$ if and only if $\{k_n : n < \omega\}$ is bounded. $\square$

The definition of a P^+(tree)-ideal is taken from [4].

Definition 2.5 (Laflamme and Leary [4]). Let $\mathcal{X}$ be a set of infinite subsets of ω. A tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ is an *$\mathcal{X}$-tree of finite sets* if for each $s \in T$ there is an $X_s \in \mathcal{X}$ such that $s^\frown a \in T$ for each $a \in [X_s]^{<\omega}$.
An ideal I on ω is a P^+*(tree)-ideal* if every I^+-tree of finite sets has a branch whose union is in I^+.

Laflamme and Leary proved that an ideal I is not P^+(tree) if and only if Player I has a winning strategy in the following game $H(\mathsf{I})$: In step n, Player I chooses an I-positive set X_n and Player II chooses a finite set $F_n \subseteq X_n$. Player II wins if $\bigcup_{n<\omega} F_n \in \mathsf{I}^+$.

In fact, this game characterizes F_σ-ideals, as the following theorem shows:

Theorem 2.6. *Let I be a Borel ideal. Then I is a P^+(tree)-ideal if and only if I is an F_σ-ideal.*

Proof: The theorem follows immediately from the following claim and Borel determinacy.

Claim 2.7. *Let I be a Borel ideal. Then, Player II has a winning strategy in $H(\mathsf{I})$ if and only if I is an F_σ-ideal.*

Proof: If I is an F_σ ideal then there is a lscsm φ such that $\mathsf{I} = \mathrm{Fin}(\varphi)$. In step n, II plays a finite subset F_n of X_n with $\varphi(F_n) \geq n$. That is possible since $\varphi(X_n) = \infty$.

On the other hand, we will prove that Player I has a winning strategy in $H(\mathsf{I})$ if I is not an F_σ ideal. Recall the following result (Theorem 21.22 in [3]).

Theorem 2.8 (Kechris-Louveau-Woodin). *Let X be a Polish space, let $A \subseteq X$ be analytic, and let $B \subseteq X$ be arbitrary with $A \cap B = \emptyset$. Then either there is an F_σ set $K \subseteq X$ separating A from B or there is a perfect set $C \subseteq A \cup B$ such that $C \cap B$ is countable dense in C.* $\square$

By 2.8, there is a perfect set $C \subseteq \mathcal{P}(\omega)$ such that $C \cap \mathsf{I}^+$ is countable dense in C. In the Banach-Mazur game played inside C (denoted by G_0)[1] in $C \cap \mathsf{I}^+$, Player I has a winning strategy, since I is comeager in C. Now, we will prove that if Player I has a winning strategy in $G_0(C \cap \mathsf{I}^+)$ then Player I has a winning strategy in $H(\mathsf{I})$. Let σ be a winning strategy for Player I in $G_0(C\cap\mathsf{I}^+)$. In step 0, let $\tau(\emptyset) = X_0 \in V_0 = \sigma(\emptyset)$ be an I-positive set. Such set exists since V_0 is an open non-empty subset of C and $\mathsf{I}^+ \cap C$ is dense in C. Let us assume that we have defined our strategy τ until step n together with a sequence of σ-legal positions. We will define it for step $n+1$. Given an answer $F \subseteq X_n$ of Player II for a τ-legal sequence $\langle X_0, F_0, \dots, X_{n-1}, F_{n-1}, X_n\rangle$, σ considers F as the clopen set U of all

[1] Banach-Mazur game $G_0(C\cap\mathsf{I}^+)$ is defined as follows: In step 0, Player I chooses a nonempty open set V_0 and Player II chooses a nonempty open subset U_0 of V_0. In step $n+1$, Player I chooses a nonempty open set $V_{n+1} \subseteq U_n$ and Player II chooses a nonempty open set $U_{n+1} \subseteq V_{n+1}$. Player II wins if $\bigcap_{n<\omega} \overline{U_n} = \bigcap_{n<\omega} \overline{V_n} \subseteq \mathsf{I}^+$.

subsets A of ω such that $A \cap (\max(F)+1) = F$, and if $\langle V_0, U_0, \dots, V_{n-1}, U_{n-1}, V_n\rangle$ is the σ-legal position associated to $\langle X_0, F_0, \dots, X_{n-1}, F_{n-1}, X_n\rangle$, then $U = U_n$, $V_{n+1} = \sigma(\langle V_0, U_0, \dots, V_{n-1}, U_{n-1}, V_n, U_n\rangle)$ and let (by density of I^+ in C)

$$\tau(\langle X_0, F_0, \dots, X_{n-1}, F_{n-1}, X_n, F\rangle) = X_{n+1} \in V_n$$

be an I-positive set. Finally, note that τ is a winning strategy for I, since for every τ-legal run of $H(\mathsf{I})$ $\langle X_0, F_0, X_1, F_1, \dots\rangle$, $\bigcup_{n<\omega} F_n \subseteq \bigcap_{n<\omega} U_n \in \mathsf{I}$. $\square$

Returning to the comparison game with the ideal **Fin** as Player II we have the following result.

Lemma 2.9. *If I is not a P^+(tree)-ideal then Player I has a winning strategy in $G''(\mathsf{I})$.*

Proof: Let T be an I^+-tree of finite sets with all branches in I. In her first few steps, Player I plays in the increasing order the elements of $\bigcup \mathrm{succ}_T(\emptyset)$ until Player II increases her answer. If in step n, Player II chooses a number bigger than all of her previous plays then Player I collects the (finite) set F_0 of answers given by her until the current step and then she begins taking elements of $\mathrm{succ}_T(F_0)$ in the increasing order until the Player II increases her choice. Hence, if eventually Player II does not increase her picks then Player I will choose every element of $\mathrm{succ}_T(t)$ for some $t \in T$ and then he will collect an I-positive set. In the other case Player II will collect a set which follows a branch of T and then its union will be in I. $\square$

Theorem 2.10. *For any Borel ideal I, $\mathsf{I} \simeq$ **Fin** if and only if I is F_σ.*

Proof: It follows from two facts: If I is a Borel ideal then $G''(\mathsf{I})$ is determined, and by Theorem 2.6, J is a P^+(*tree*)-ideal if and only if J is an F_σ-ideal, for all Borel ideal J. $\square$

3. $F_{\sigma\delta}$-ideals in the Comparison Game Order

We now define an ideal I_0 which is the minimal ideal I such that there is an I^+-tree of finite sets which does not have an I-positive branch, i.e. which is not a P^+(tree)-ideal. Let us denote $A_f = \{f \restriction n : n < \omega\}$ for a given $f \in 2^\omega$.

Definition 3.1. The ideal I_0 is the ideal on $2^{<\omega}$ generated by the family of sets A_f where $f \in 2^\omega$ is not eventually zero.

Theorem 3.2. *If I is a Borel ideal which is not F_σ then $\mathsf{I}_0 \sqsubseteq \mathsf{I}$.*

Proof: By the Kechris-Louveau-Woodin theorem 2.8 there is a Cantor set $C \subseteq \mathcal{P}(\omega)$ such that $D = C \setminus \mathsf{I}$ is countable dense in C. Let $T \subseteq 2^{<\omega}$ be a perfect tree such that $[T] = C$. Since D is a countable dense subset of 2^ω, there is a homeomorphism $\varphi : 2^\omega \to C$ such that if $F = \{f \in 2^\omega : (\forall^\infty n) f(n) = 0\}$ then $\varphi'' F = D$. Such φ induces an embedding[2] $\Phi : 2^{<\omega} \to [\omega]^{<\omega}$ which is monotone

[2] The embedding Φ is defined so that for each $s \in 2^{<\omega}$, the finite set $\Phi(s)$ determines the clopen subset $\varphi''\langle s\rangle$ of C.

(i.e. $s \subseteq t$ implies $\Phi(s) \subseteq \Phi(t)$) and such that $\bigcup_n \Phi(f \restriction n) \in \mathsf{I}$ if and only if f is not eventually zero.

Now we describe a winning strategy for Player II in $G(\mathsf{I}_0, \mathsf{I})$. In step n, if Player I plays $I_n \in \mathsf{I}_0$ then Player II plays $J_n = [0, k_n] \cup \bigcup\{\Phi(s) : (\exists k \leq n)(\exists t \in I_k)(s \subseteq t)\}$, where k_n is the maximal cardinality of an antichain in $\bigcup_{k \leq n} I_k$.

We argue why this is a winning strategy for Player II. If $I = \bigcup_n I_n \in \mathsf{I}_0$ then there are $m < \omega$ and $f_0, \ldots, f_m \in 2^\omega \setminus F$ such that $I \subseteq \bigcup_{j \leq m} A_{f_j}$. Then m is an upper bound for k_n and $\bigcup\{\Phi(s) : (\exists k < \omega)(\exists t \in I_k)(s \subseteq t)\} \subseteq \bigcup_{j \leq m} \bigcup_n \Phi(f_j \restriction n) \in \mathsf{I}$, and then $\bigcup_n J_n \in \mathsf{I}$. On the other hand, if $I \notin \mathsf{I}_0$ then either $\langle k_n : n < \omega \rangle$ is unbounded, and then $J = \bigcup_n J_n \notin \mathsf{I}$, or there is an eventually zero function f such that $f \restriction n \in I$ for infinitely many $n < \omega$, and in that case,

$$\bigcup_n \{\Phi(s) : (\exists t \in I_n) s \subseteq t\} \supseteq \bigcup_n \{\Phi(f \restriction n) : n < \omega\} \notin \mathsf{I}. \qquad \square$$

The ideal I_0 is $F_{\sigma\delta}$. Consider another $F_{\sigma\delta}$-ideal.

$$\emptyset \times \mathbf{Fin} = \{A \subseteq \omega \times \omega : (\forall n)(\exists m)(\forall k)((n, k) \in A \to k \leq m)\}.$$

Theorem 3.3.

$$\emptyset \times \mathbf{Fin} \not\sqsubseteq \mathsf{I}_0.$$

PROOF: For every $1 \leq n < \omega$ we define a game G_n as follows. In step k, Player I picks a finite subset I_k of $\omega \times \omega$ and Player II picks an antichain J_k of cardinality n in I_0, and such that for all $i < k$ and all t in J_i there is a unique $s \in J_k$ such that $s \supseteq t$. Player II wins if $\bigcup_n I_n \in \emptyset \times \mathbf{Fin}$ if and only if $\bigcup_n J_n \in \mathsf{I}_0$. Inductively, we will prove that Player I has a winning strategy in game G_n, for all n, having done that, we will show how this fact implies that Player I has a winning strategy in $G(\emptyset \times \mathbf{Fin}, \mathsf{I}_0)$.

Claim 3.4. *Player I has a winning strategy in the game G_n, for all n.*

PROOF OF CLAIM: First we prove that Player I has a winning strategy in the game G_1. In step 0, Player I plays $\{(0, 0)\}$. In step k, define $N(k) = \min\{\sum h(l) : h$ is a maximal sequence in $J_k \wedge l \in \mathrm{dom}(h)\}$, and Player I just plays a doubleton with the form $\{(0, N(k)), (n_k, m_k)\}$, where $n_0 = m_0 = 0$; (1) if $J_k \supsetneq J_{k-1}$ and there is $m \in J_k \setminus J_{k-1}$ such that $J_k(m) = 1$ then $n_k = n_{k-1}$ and $m_k = m_{k-1} + 1$; and (2) $n_k = n_{k-1} + 1$ and $m_k = m_{k+1}$ otherwise.

We show why is this a winning strategy for Player I. If in some step k, Player II plays an infinite set J_k then she will be playing along the branch $\bigcup J_k$ and then Player I know that she has won because she just will fill the column $\{0\} \times \omega$ if $\bigcup J_k$ is not eventually zero, or the raw $\{k\} \times \omega$ otherwise. Without loss of generality, let us assume that Player II plays finite increasing sets. Then if there is K such that $J_k = J_K$ for all $k \geq K$ then $\bigcup_n J_n \in \mathsf{I}_0$ but Player I will fill the column $\{m_K\} \times (\omega \setminus n_K)$ for K minimal; and if Player II increases the length of J_k for infinitely many steps k then, if there is K such that the increasing of J_k is just with 0's then column $\{0\} \times N(k)$ will not increase and choices of Player I will follow a horizontal line; but if Player II increases the length of J_k and she adds a new 1 in infinitely many steps then Player I will make the column $\{0\} \times N(k)$ increase to $\{0\} \times \omega$ and then $\bigcup_n I_n \notin \emptyset \times \mathbf{Fin}$.

Inductively assume that Player I has a winning strategy in G_n and let us prove that she has a winning strategy in G_{n+1}. Fix a partition $\{X_i^j : j \leq n \wedge i < \omega\}$ of $\omega \setminus \{0\}$. In step 0, Player I plays $\emptyset$ and then, assume that Player II has played an antichain J_k of cardinality $n+1$ (we can assume this by identifying J_k with its maximal elements. Let us enumerate this antichain as $\{a_r^0 : r \leq n\}$ and for each $r \leq n$, we enumerate $J_k = \{a_r^k : r \leq n\}$ in such way that $a_r^k \supseteq a_r^0$ for all $r \leq n$. Then, Player I will play simultaneously the game G_n in $X_i^r \times \omega$ for some i (depending of k and r), where answers of Player I are given by the winning strategy for her when Player II plays $J_k \setminus a_r^k$; and following this rule: If $a_r^k \supsetneq a_r^{k-1}$ and Player I is playing in the copy $X_i^r \times \omega$ then she abandons this copy and begins playing G_n in $X_{i+1}^r \times \omega$; and if not, she still playing in the same $X_i^r \times \omega$, i.e., $i(k,r) = i(k-1,r)$. In both cases Player I adds the column $\{0\} \times N(k)$ (recall $N(k)$ was defined two paragraphs above). Now we prove that this is a winning strategy for Player I.

If all the sequences a_r^k are eventually increasing then we have two cases:

(1) For each $k \leq n$ the sequence $\bigcup_r a_r^k$ is not eventually-zero. Then, Player I will increase the column $\{0\} \times N(k)$ to $\{0\} \times \omega$, making $\bigcup_n J_n \notin \emptyset \times \mathbf{Fin}$.

(2) There is $k \leq n$ such that $\bigcup_r a_r^k$ is an eventually-zero branch. Then, the column $\{0\} \times N(k)$ will not increase and in all the pieces of the partition will be played the game G_n and since all increase, all pieces are eventually abandoned and then, $\bigcup_n J_n \in \emptyset \times \mathbf{Fin}$.

If for some k, the sequence a_r^k does not increase then Player I will be playing the game G_n and since she has a winning strategy in this game, we are done, because the column $\{0\} \times N(k)$ will not increase. $\square$

Let $\{X_r : r < \omega\}$ be a partition of $\omega \setminus \{0\}$ in infinite sets. The main idea is based on the following trick: Player I is going to play the game G_n but in $X_n \times \omega$ instead of $\omega \times \omega$. In step 0, Player I plays $\emptyset$ and in step $k > 0$, let $M(k)$ be the maximal cardinality of an antichain in $\bigcup_{i<k} J_i$. If $M(k) = M(k-1)$ then Player I has to play the game $G_{M(k)}$ in $X_{M(k-1)} \times \omega$ instead of $\omega \times \omega$, and if $M(k) > M(k-1)$, then Player I has to abandon what he has played and begin a new game of $G_{M(k)}$ inside the copy $X_{M(k)} \times \omega$, and in both cases, Player I has to add $\{\min X_{M(k)}\} \times N(k)$ to the sets defined above.

If Player II makes $M(k)$ increase in infinitely many steps, then $\bigcup_n J_n \notin \mathsf{I}_0$, but Player I will abandon all pieces where he played, and then $\bigcup_n I_n \in \emptyset \times \mathbf{Fin}$.

If there is K such that $M(k) = M(K)$ for all $k > K$ then the winning strategy for Player I in $G_{M(K)}$ makes Player I win in $G(\emptyset \times \omega, \mathsf{I}_0)$. $\square$

Now we give a criterion for ideals to be $\sqsubseteq$-below $\emptyset \times \mathbf{Fin}$.

Proposition 3.5. *Let I be an ideal on ω. Then $\mathsf{I} \sqsubseteq \emptyset \times \mathbf{Fin}$ if and only if Player II has a winning strategy in the following game $G'''(\mathsf{I})$: In step n, Player I chooses an element I_n of I and then Player II chooses an increasing function $f_n \in \omega^\omega$. Player II wins if $\bigcup_n I_n \in \mathsf{I}$ if and only if the sequence $\{f_n : n < \omega\}$ is bounded.*

PROOF: Let us assume that Player II has a winning strategy σ in $G(\mathsf{I}, \emptyset \times \mathbf{Fin})$. For every element $J \in \emptyset \times \mathbf{Fin}$, let $f_J : \omega \to \omega$ given by $f_J(n) = \min\{k > f_J(n-1) : (\forall m > k)\ (n,m) \notin J\}$. Then we describe a winning strategy for Player II in $G'''(\mathsf{I})$ as follows: Given $I_0 \in \mathsf{I}$, let f_0 be the function $f_{\sigma(I_0)}$. Assume that the legal position $\langle I_0, f_0, \ldots, I_n, f_n\rangle$ follows the strategy which we are defining. Then in parallel we have a legal position $\langle I_0, J_0, \ldots, I_n, J_n\rangle$ of $G(\mathsf{I}, \emptyset \times \mathbf{Fin})$ following σ. Then, given I_{n+1}, define $J_{n+1} = \sigma(\langle I_0, J_0, \ldots, I_n, J_n, I_{n+1}\rangle)$ and the function $f_{n+1} = f_{J_{n+1}}$. It is easy to check that this is a winning strategy for Player II in $G'''(\mathsf{I})$. On the other hand, for any function $f \in \omega^\omega$ define $J_f = \{(n,m) \in \omega \times \omega : m \le f(n)\}$. Analogous to first part, Player II in $G(\mathsf{I}, \emptyset \times \mathbf{Fin})$ has plays J_f where f is the answer given by Player II in $G'''(\mathsf{I})$. □

Ilijas Farah asked in [2] if for every $F_{\sigma\delta}$-ideal I there is a family of compact hereditary sets $\{C_n : n < \omega\}$ such that

$$\mathsf{I} = \{A \subseteq \omega : (\forall n < \omega)(\exists m < \omega)(A \setminus [0,m) \in C_n)\}.$$

We will say I is a *Farah ideal* if I fulfils that property. Note that every Farah ideal I is an $F_{\sigma\delta}$ ideal. The following is a simple observation.

Proposition 3.6. *Let I be an ideal on ω. Then, I is Farah if and only if there is a sequence $\{F_n : n < \omega\}$ of hereditary F_σ-sets closed under finite changes such that $\mathsf{I} = \bigcap_n F_n$.*

PROOF: Let $\langle C_n : n < \omega\rangle$ be a family of compact hereditary sets such that $\mathsf{I} = \{A \subseteq \omega : (\forall n)(\exists k)(A \setminus k \in C_n)\}$. For any n, define F_n as the closure of C_n under finite changes. It is clear that F_n is hereditary, F_σ, closed under finite changes, and contains I. If $A \in F_n$ then there is a finite set F such that $A \triangle F \in C_n$ and by taking an adequate $k > \max(F)$ we have that $A \setminus k \in C_n$.

Now, let $\{F_n : n < \omega\}$ be an increasing sequence of hereditary F_σ-sets closed under finite changes such that $\mathsf{I} = \bigcap_n F_n$. Let us write $F_n = \bigcup_k E^n_k$ where $\langle E^n_k : k < \omega\rangle$ is an increasing sequence of closed sets. We can assume that each E^n_k is a hereditary set, and we can define

$$\tilde{E}^n_k = \{A \setminus (k+1) \cup \{k\} : A \in E^n_k\}$$

and $C_n = \{\emptyset\} \cup \bigcup_k \tilde{E}^n_k$. Note that each C_n is a closed hereditary set, and if $A \setminus k \in C_n$ we can assume $k \in A$ and then $A \in \tilde{E}^n_k \subseteq F_n$, for all n. Finally, if A is an infinite set in I (the finite case is trivial) then for each n take k such that $A \setminus k \in E^n_k$ and $k \in A$ (this is possible since the E^n_k is an increasing family). Hence $A \setminus k \in C_n$. □

We denote by **nwd** the ideal of all nowhere dense subsets of the set of rational numbers $\mathbb{Q}$.

Example 3.7. *The ideal* **nwd** *is Farah.*

Proof: Let $\{U_n : n < \omega\}$ be a base of the topology of $\mathbb{Q}$, and define $F_n = \{A \subseteq \mathbb{Q} : (\exists m)(U_m \subseteq U_n \wedge A \cap U_m = \emptyset)\}$. Note that $\mathbf{nwd} = \bigcap_n F_n$ and each F_n is F_σ hereditary and closed under finite changes. $\square$

We refine Proposition 3.6 as follows.

Theorem 3.8. *Let* I *be an ideal on* ω*. Then,* I *is Farah if and only if there is a sequence* $\{F_n : n < \omega\}$ *of* F_σ *sets closed under finite changes such that* $\mathsf{I} = \bigcap_n F_n$*.*

Proof: Without loss of generality, we can assume that every F_n is meager, because if F_n is non-meager then there is a non-empty clopen set contained in F_n and by closedness under finite changes, $F_n = 2^\omega$.

Sufficiency is a consequence of Proposition 3.6, and by the same result, it will be enough to prove that if F is a meager F_σ-set closed under finite changes and containing I, then there is a hereditary F_σ-set E such that $\mathsf{I} \subseteq E \subseteq F$, since the closure of E under finite changes would be the hereditary closed under finite changes wanted. Let us consider the game H defined so that in step k, Player I chooses a set $B_k \notin F$ and Player II picks a finite subset a_k of B_k. Player I wins if $\bigcup_k a_k \in \mathsf{I}$. Note that H is determined since I is Borel.

Claim 3.9. *Player II has a winning strategy in* H*.*

Proof of claim: Let $\{E_n : n < \omega\}$ be an increasing sequence of closed sets such that $F = \bigcup_n E_n$ and for each n, let T_n be a pruned tree such that $E_n = [T_n]$. Since each E_n is a nowhere dense set, in step k, if Player I plays B_k then there is $m_k < \omega$ such that $m_{k-1} < m_k$ $(m_{-1} = 0)$ and $\chi_{B_k} \restriction m_k \notin T_k$. Then, Player II plays $a_k = B_k \cap m_k$. It is clear that $\bigcup_k a_k \notin F$ and then $\bigcup_k a_k \notin \mathsf{I}$. $\square$

It is very easy to see that

Claim 3.10. *Player II has a winning strategy in* H *if there is a tree* $T \subseteq ([\omega]^{<\omega})^{<\omega}$ *such that* (a) *for all* $A \notin F$ *and all* $t \in T$ *there is* $a \in \operatorname{succ}_T(t)$ *such that* $a \subseteq A$ *and* (b) $\bigcup_n f(n) \in \mathsf{I}^+$ *for all* $f \in [T]$. $\square$

Hence, by defining $C_t = \{A \subseteq \omega : (\forall a \in \operatorname{succ}_T(t))(a \not\subseteq A)\}$, for all $t \in T$, we have immediately that C_t is closed and hereditary and $\mathsf{I} \subseteq \bigcup_{t\in T} C_t$. Finally, (a) is equivalent to $\bigcup_{t\in T} C_t \subseteq F$. Hence, $\bigcup_t C_t$ is the hereditary F_σ-set required. $\square$

By Theorem 3.6 it is clear that any Farah ideal satisfies the following.

Definition 3.11. An ideal I is *weakly Farah* if there is a sequence $\langle F_n : n < \omega\rangle$ of hereditary F_σ-sets such that $\mathsf{I} = \bigcap_n F_n$.

Without loss of generality, the sequence in the previous definition is decreasing, and it is clear that any weakly Farah ideal is $F_{\sigma\delta}$.

Theorem 3.12. *If* I *is a weakly Farah ideal then* $\mathsf{I} \sqsubseteq \emptyset \times \mathbf{Fin}$.

Proof: Let $\{F_n : n < \omega\}$ be a family of hereditary F_σ-sets such that $\mathsf{I} = \bigcap_n F_n$. Without loss of generality, we can assume that for any n, $F_n = \bigcup_k E_k^n$ where $(E_k^n)_k$ is an increasing sequence of closed hereditary sets. Then, for any $A \subseteq \omega$

$$A \in \mathsf{I} \quad \text{iff} \quad (\exists f_A \in \omega^\omega)(\forall k, n < \omega)(A \notin E_k^n \leftrightarrow k < f_A(n)).$$

Hence, playing the game $G'''(\mathsf{I})$, for any step n, Player II plays $f_{\bigcup_{j<n} I_j}$. So, if $I = \bigcup_{n<\omega} I_n \in \mathsf{I}$ then f_I bounds all the f_{I_n} functions; and if $I \notin \mathsf{I}$ then there is j such that $I \notin E_k^j$ for all $k < \omega$ and then, $\langle f_{I_n}(j) : n < \omega\rangle$ increases to infinity, because in other case, there were k such that $I_n \in E_k^j$ for all n and $I \notin E_k^j$, contradicting the closedness of E_k^j. ☐

A positive answer to Farah's question would imply that every $F_{\sigma\delta}$-ideal is $\sqsubseteq$-below $\emptyset \times \mathbf{Fin}$.

Recall the following characterization of analytic P-ideals.

Theorem 3.13 (Solecki [7]). *If I is an analytic P-ideal then there is a lscsm φ such that $\mathsf{I} = \mathrm{Exh}(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus [n, \infty)) = 0\}$.*

Note that by Solecki's theorem, every analytic P-ideal is a Farah ideal, and then, if I is an analytic P-ideal then $\mathsf{I} \sqsubseteq \emptyset \times \mathbf{Fin}$. Concerning analytic P-ideals, every one of them is either equivalent with $\mathbf{Fin}$ (i.e., is F_σ) or equivalent with $\emptyset \times \mathbf{Fin}$, i.e., the class of P-ideals "skips" the intermediate class of I_0.

Theorem 3.14. *Let I be an analytic P-ideal. Then either $\mathsf{I} \simeq \mathbf{Fin}$ or $\mathsf{I} \simeq \emptyset \times \mathbf{Fin}$.*

Proof: Let φ be a lscsm such that $\mathsf{I} = \mathrm{Exh}(\varphi)$. Consider two cases:

Case 1. There is $\varepsilon > 0$ such that for any set X, $\varphi(X) < \varepsilon$ implies $X \in \mathsf{I}$. Note than in such case I is an F_σ ideal, because $C = \{A \subseteq \omega : \varphi(X) \le \varepsilon\}$ is a closed set and $\mathsf{I} = \bigcup_n \{A \subseteq \omega : A \setminus n \in C\}$.

Case 2. For all $\varepsilon > 0$ there is an I-positive set X such that $\varphi(X) < \varepsilon$. We will use the following result, which is a known consequence of Jalali-Naini–Talagrand theorem (see [1]).

Lemma 3.15 (Disjoint Refinement Lemma for Definable Ideals, see [6]). *If I is a hereditarily meager ideal and $\{X_m : m < \omega\}$ is a family of I-positive sets then there is a pairwise disjoint family $\{Y_m : m < \omega\}$ of I-positive sets such that $Y_m \subseteq X_m$ for all $m < \omega$.* ☐

Take a family Y_m of I-positive sets such that $\varphi(Y_m) \le 2^{-m}$ and by the Disjoint Refinement Lemma for hereditary meagre ideals, there is a disjoint family of positive sets $\{X_m : m < \omega\}$ such that $\varphi(X_m) \le 2^{-m}$. Let $\{x_k^m : k < \omega\}$ be an enumeration of X_m. Let us describe a winning strategy for Player II in $G(\emptyset \times \mathbf{Fin}, \mathsf{I})$. In step n, if Player I plays I_n, we consider the function f_n given by $f_n(i) = \max\{0\} \cup \{j : (\exists l \le n)((i,j) \in I_l)\}$ and then Player II plays $J_n = \{x_j^i : j \le f_n(i)\}$. Hence, if $I = \bigcup_n I_n \in \mathsf{I}$ then the family $\langle f_n : n < \omega\rangle$ is bounded by a function f, and then $J = \bigcup_n J_n$ intersects each X_n in a finite set F_n which has submeasure smaller than 2^{-n} and so, J is a φ-exhaustive set. On the other hand, if $I \notin \emptyset \times \mathbf{Fin}$ then there is m such that $f_n(m)$ increases to infinity, and so, $J \cap X_m = X_m \in \mathsf{I}^+$. $\square$

Recall the *asymptotical density zero ideal* $\mathcal{Z}$ is defined by

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap [0,n)|}{n} = 0 \right\}$$

and (by its definition) is an analytic P-ideal.

Remark 3.16. The following ideals on ω are comparison game equivalent:

(1) $\mathcal{Z}$,
(2) **nwd**, and
(3) $\emptyset \times \mathbf{Fin}$.

PROOF: (1) $\simeq$ (3) use $\mathcal{Z}$ is an analytic P-ideal which is not F_σ.

(2) $\sqsubseteq$ (3) use **nwd** is a Farah ideal.

(3) $\sqsubseteq$ (2) Let $\{V_n : n < \omega\}$ be a sequence of pairwise disjoint open subsets of $\mathbb{Q}$ and for each n, let $\{q^n_k : k < \omega\}$ be an enumeration of V_n. Let us play the $G(\emptyset \times \mathbf{Fin}, \mathbf{nwd})$ game. In step n, if Player I has played $I_n \in \emptyset \times \mathbf{Fin}$, take a function $f \in \omega^\omega$ such that for all k, m, $(k,m) \in I_n$ implies $m \leq f(k)$, and then Player II must play $J_n = \{q^k_s : s < f(k) \wedge k < \omega\}$. J_n is a nowhere dense subset of $\mathbb{Q}$ since it intersects each V_n in a finite set, and if $I = \bigcup_n I_n \in \emptyset \times \mathbf{Fin}$ then $J = \bigcup_n J_n$ intersects each V_n in a finite set, and then, $J \in \mathbf{nwd}$; and if for some k, $I \cap (\{k\} \times \omega)$ is infinite, then J will contain V_k, and then $J \in \mathbf{nwd}^+$. $\square$

4. Final remarks

Recall that $\mathbf{Fin} \times \mathbf{Fin}$ is the ideal on $\omega \times \omega$ generated by the columns $\{n\} \times \omega$ and the sets $\{(n,m) : m < f(n)\}$, for $f \in \omega^\omega$. We finally will show that the ideal $\mathbf{Fin} \times \mathbf{Fin}$ belongs to a higher class than $\emptyset \times \mathbf{Fin}$. It is easy to see that $\emptyset \times \mathbf{Fin} \sqsubseteq \mathbf{Fin} \times \mathbf{Fin}$.

Proposition 4.1. $\emptyset \times \mathbf{Fin} \sqsubseteq \mathbf{Fin} \times \mathbf{Fin}$.

PROOF: Let $\{X_n : n < \omega\}$ be an infinite partition of ω in infinite pieces. Given I in $\emptyset \times \mathbf{Fin}$, we define an element J_I of $\emptyset \times \mathbf{Fin}$ by

$$J_I = \{(k,l) : (\exists n < \omega)(k \in X_n \wedge (n,l) \in I)\}.$$

The winning strategy for Player II consists in playing J_{I_n} as an answer to a set I_n played by Player I in step n. If $I = \bigcup_n I_n \in \emptyset \times \mathbf{Fin}$ then $J = \bigcup_n J_{I_n} \in \mathbf{Fin} \times \mathbf{Fin}$, and if for some $k < \omega$, $I \cap (\{k\} \times \omega)$ is infinite then $J \cap (\{l\} \times \omega)$ will be infinite for all $l \in X_k$, and so $J \notin \mathbf{Fin} \times \mathbf{Fin}$. $\square$

Theorem 4.2. $\mathbf{Fin} \times \mathbf{Fin} \not\sqsubseteq \emptyset \times \mathbf{Fin}$.

Proof: We will describe a winning strategy for Player I in $G'''(\mathbf{Fin} \times \mathbf{Fin})$. Without loss of generality, we can assume that Player II plays in such a way that $f_k(n) \geq f_{k-1}(n)$ for all n. First, take an infinite partition $\{X_n : n < \omega\}$ of ω in infinite pieces, and let $\{x_r^n : r < \omega\}$ be an enumeration of X_n. Player I will play just selectors of the family $\{X_n \times \omega : n < \omega\}$. In step 0, Player I plays $\{(x_r^0, 0) : r < \omega\}$. In step k, if $f_k = f_{k-1}$ ($f_{-1} \equiv 0$) and $J_{k-1} = \{(x_r^n, m_r^n) : r < \omega\}$ then $J_{k+1} = \{(x_r^n, m_r^n + 1) : r < \omega\}$, and otherwise, if $l = \min\{n : f_k(n) > f_{k-1}(n)\}$ then $J_{k+1} = \{(x_r^n, m_r^n + 1) : r \leq l\} \cup \{(x_{r+1}^n, m_r^n) : r > l\}$.

If there is N such that $\{f_k(N) : k < \omega\}$ increases infinitely often then $\bigcup_n J_n \in \mathsf{I}$ since all but finitely many pieces X_r are "turning to the right" infinitely often and if $\{f_k : k < \omega\}$ is bounded by a function f then for each r, there are k and N such that Player I will be "filling" the column $\{x_r^k\} \times (\omega \setminus N)$, making $\bigcup_n J_n \notin \mathbf{Fin} \times \mathbf{Fin}$. □

Recall that a function f from I to J is a *Tukey function* if for each $A \in \mathsf{J}$ there is $B \in \mathsf{I}$ such that $I \subseteq B$ if $f(I) \subseteq A$. Tukey order is defined by $\mathsf{I} \leq_T \mathsf{J}$ if there is a Tukey function from I to J; and let us denote by $\mathsf{I} \leq_{MT} \mathsf{J}$ when there is a monotone (with respect to inclusion) Tukey function from I to J. The order $\sqsubseteq$ refines the monotone Tukey order.

Lemma 4.3. *If $\mathsf{I} \leq_{MT} \mathsf{J}$ then $\mathsf{I} \sqsubseteq \mathsf{J}$.*

Proof: Let $f : \mathsf{I} \to \mathsf{J}$ be a monotone Tukey function. Then Player II only has to answer $f(I_n)$ for any I_n given by Player I. If $\bigcup_n I_n \in \mathsf{I}$ then by monotonicity, $\bigcup_n f(I_n) \subseteq f(\bigcup_n I_n) \in \mathsf{J}$. If $\bigcup_n I_n \notin \mathsf{I}$ then by Tukeyness $\bigcup_n f(I_n) \notin \mathsf{J}$. □

Note that the Tukey and monotone Tukey orders are quite different: There is a Tukey-maximal ideal among all ideals, which is F_σ. On the other hand, by Lemma 4.3 and Proposition 1.7, if $\mathsf{I} \leq_{MT} \mathsf{J}$ and I is F_σ then J is $F_{\sigma\delta\sigma}$.

5. Questions

(1) Are there exactly two classes of $F_{\sigma\delta}$ non-F_σ-ideals?
(2) How many classes of $F_{\sigma\delta\sigma}$-ideals are there?
(3) Is every $F_{\sigma\delta}$-ideal weakly Farah? Is every weakly Farah a Farah ideal?

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