Cellularity and the index of narrowness in topological groups

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Abstract. We study relations between the cellularity and index of narrowness in topological groups and their G_δ-modifications. We show, in particular, that the inequalities $\mathrm{in}((H)_\tau) \leq 2^{\tau\cdot \mathrm{in}(H)}$ and $c((H)_\tau) \leq 2^{2^{\tau\cdot \mathrm{in}(H)}}$ hold for every topological group H and every cardinal $\tau \geq \omega$, where $(H)_\tau$ denotes the underlying group H endowed with the G_τ-modification of the original topology of H and $\mathrm{in}(H)$ is the index of narrowness of the group H.

Also, we find some bounds for the *complexity* of continuous real-valued functions f on an arbitrary ω-narrow group G understood as the minimum cardinal $\tau \geq \omega$ such that there exists a continuous homomorphism $\pi\colon G \to H$ onto a topological group H with $w(H) \leq \tau$ such that $\pi \prec f$. It is shown that this complexity is not greater than 2^{2^ω} and, if G is weakly Lindelöf (or 2^ω*-steady*), then it does not exceed 2^ω.

Keywords: cellularity, G_δ-modification, index of narrowness, ω-narrow, weakly Lindelöf, $\mathbb{R}$-factorizable, complexity of functions

Classification: 54H11, 54A25, 54C30

1. Introduction

Passing to a subspace of a (compact) space can increase the cellularity of a space. Indeed, for every uncountable cardinal τ, the Tychonoff cube I^τ of weight τ contains a discrete subspace of cardinality τ, while the cellularity of the cube itself is countable. The same happens in (compact) topological groups — it suffices to replace the Tychonoff cube I^τ with $\mathbb{T}^\tau$, where $\mathbb{T}$ is the circle group with the usual multiplication and topology inherited from the complex plane $\mathbb{C}$.

However, the gap between the cellularity $c(G)$ of a topological group G and the cellularity of subgroups of G becomes considerably smaller. According to [1, Theorem 5.4.11], the inequality $c(H) \leq 2^{c(G)}$ holds for every subgroup H of G. In addition, if G is precompact, then every subgroup of G has countable cellularity.

Another important fact for our study was proved by I. Juhász in [3]: If X is a compact space and γ is a disjoint family of G_δ-sets in X, then the cardinality of γ is at most $2^{c(X)}$. This result shows that the cellularity of the G_δ*-modification* of X, say $(X)_\omega$, does not exceed $2^{c(X)}$. As usual, by G_δ-modification of X we mean the underlying set X which carries the topology whose base consists of G_δ-sets in X. Similarly, one defines the G_τ*-modification* of X, for any cardinal $\tau \geq \omega$, which will be denoted by $(X)_\tau$.

Our main concern here is to find a bound for the cellularity of the G_τ-modification of a topological group H in terms of the cellularity of H. This is done in Theorem 3.1 where we show that the inequalities $\mathrm{in}((H)_\tau) \le 2^{\tau\cdot\mathrm{in}(H)}$ and $c((H)_\tau) \le 2^{2^{\tau\cdot\mathrm{in}(H)}}$ hold for every topological group H and every cardinal $\tau \ge \omega$, where $\mathrm{in}(H)$ is the index of narrowness of H (see Section 2 below). This means, in particular, that every τ-narrow topological group H satisfies $c((H)_\tau) \le 2^{2^\tau}$. It turns out that this bound is exact — in Example 3.4 we present an ω-narrow Abelian group H such that $c((H)_\omega) = 2^{2^\omega}$.

A topological group G is called $\mathbb{R}$-*factorizable* if every continuous real-valued function f on G can be represented as a composition of a continuous homomorphism of G to a *second countable* group H and a continuous real-valued function on H (see [7, Section 5] or [1, Chapter 8]). In other words, G is $\mathbb{R}$-factorizable if every continuous real-valued function on G has 'countable complexity'. By [7, Proposition 5.3], every $\mathbb{R}$-factorizable group is ω-narrow, but ω-narrow groups need not be $\mathbb{R}$-factorizable according to [7, Example 5.14]. These facts give rise to the problem of finding bounds for the complexity of continuous real-valued functions on ω-narrow groups (see [6, Problem 3.3] or Problem 4.1 below).

We show in Theorem 4.2 that 2^{2^ω} is such a bound. However, we do not know whether this bound is exact. However, it is shown in Proposition 4.3 that 2^ω is a bound for the complexity of continuous real-valued functions on *weakly Lindelöf* topological groups, while Proposition 4.4 extends this fact to 2^ω-*steady* groups (the terms are explained in the next section).

2. Notation and terminology

Given a topological group G, we define the *index of narrowness* of G, $\mathrm{in}(G)$, as the minimum infinite cardinal τ such that G can be covered by at most τ translates of every neighborhood of the identity. It is easy to verify that $\mathrm{in}(G) \le c(G)$ for every topological group G, where $c(G)$ is the cellularity of G (see [1, Proposition 5.2.1]). We say that G is τ-*narrow* if it satisfies $\mathrm{in}(G) \le \tau$.

Suppose that $p\colon G \to H$ is a continuous homomorphism. Given a continuous mapping $f\colon G \to X$ of the group G to a space X, we write $p \prec f$ if there exists a continuous mapping $h\colon H \to X$ such that $f = h \circ p$.

A space X is *weakly Lindelöf* if every open covering of X contains a countable subfamily whose union is dense in X. All Lindelöf spaces as well as all spaces of countable cellularity are weakly Lindelöf. By virtue of [1, Proposition 5.2.8], every weakly Lindelöf topological group is ω-narrow.

A topological group G is called τ-*steady* (see [1, Section 5.6]) if every continuous homomorphic image H of G with $\psi(H) \le \tau$ satisfies $nw(H) \le \tau$. By [1, Corollary 5.6.11], every τ-steady topological group is τ-narrow.

The *Nagami number* of a Tychonoff space X is $\mathrm{Nag}(X)$ (see [1, Section 5.3]). Every topological group G with $\mathrm{Nag}(G) \le \tau$ is τ-steady and the class of τ-steady groups is productive according to [1, Theorem 5.6.4]. It is also clear that a continuous homomorphic image of a τ-steady group is τ-steady.

3. Cellularity and index of narrowness

Let us consider the behavior of the cellularity in topological groups when passing from a group H to $(H)_\omega$ or $(H)_\tau$, for an infinite cardinal τ. It is known that if H is σ-compact or, more generally, a Lindelöf Σ-group, then every family γ of G_δ-sets in H contains a countable subfamily λ such that $\bigcup\lambda$ is dense in $\bigcup\gamma$ (see [9, Theorem 2] or [7, Theorem 4.14]). Further, the cellularity of an ω-bounded group G cannot be greater than 2^ω [7, Theorem 4.29], and this bound is attained even if G is Lindelöf [2, Example 8]. An interesting complement to the former fact was found in [4]: If H is a Lindelöf topological group, then every family γ of G_δ-sets in H contains a subfamily λ with $|\lambda|\le 2^\omega$ such that $\bigcup\lambda$ is dense in $\bigcup\gamma$. It is an open problem whether this result remains valid for the class of ω-narrow groups [4]. We also recall that if X is a compact space of countable cellularity, then the cellularity of the space $(X)_\omega$ does not exceed 2^ω [3]. It turns out that if H is a τ-narrow topological group, then the cellularity of $(G)_\tau$ does not exceed the second exponent of τ:

Theorem 3.1. *The inequalities* $\mathrm{in}((G)_\tau)\le 2^{\tau\cdot \mathrm{in}(G)}$ *and* $c((G)_\tau)\le 2^{2^{\tau\cdot\mathrm{in}(G)}}$ *hold for every topological group G and every cardinal $\tau\ge\omega$. In particular, if G is τ-narrow, then $c((G)_\tau)\le 2^{2^\tau}$.*

Proof: First we show that $\mathrm{in}((G)_\tau)\le 2^\lambda$, where $\lambda=\tau\cdot\mathrm{in}(G)$. Let O be a neighbourhood of the identity e in $(G)_\tau$. Then there exists a family $\gamma=\{U_\alpha:\alpha<\tau\}$ of open neighbourhoods of e in G such that $\bigcap\gamma\subseteq O$. By [7, Lemma 3.7], for every $\alpha<\tau$, one can find a continuous homomorphism $p_\alpha\colon G\to H_\alpha$ onto a topological group H_α with $w(H_\alpha)\le\lambda$ and an open neighbourhood V_α of the identity in H_α such that $p_\alpha^{-1}(V_\alpha)\subseteq U_\alpha$. Denote by p the diagonal product of the homomorphisms p_α, $\alpha<\tau$. Then the homomorphism $p\colon G\to\prod_{\alpha<\tau}H_\alpha$ is continuous and the group $H=p(G)\subseteq\prod_{\alpha<\tau}H_\alpha$ satisfies $w(H)\le\lambda$. Therefore, $|H|\le 2^\lambda$. For every $\alpha<\tau$, there exists a continuous homomorphism $\pi_\alpha\colon H\to H_\alpha$ such that $p_\alpha=\pi_\alpha\circ p$. Then $W_\alpha=\pi_\alpha^{-1}(V_\alpha)$ is an open neighbourhood of the identity in H and $p^{-1}(W_\alpha)=p^{-1}\pi_\alpha^{-1}(V_\alpha)=p_\alpha^{-1}(V_\alpha)\subseteq U_\alpha$ for each $\alpha<\tau$. Hence the set $W=\bigcap_{\alpha<\tau}W_\alpha$ contains the identity of H and satisfies $p^{-1}(W)\subseteq O$. In particular, $\ker p\subseteq O$. Since $|H|\le 2^\lambda$, we can find a subset A of G such that $p(A)=H$ and $|A|\le 2^\lambda$. Then

$$G=A\cdot\ker p\subseteq A\cdot O\subseteq G,$$

that is, $A\cdot O=G$. This proves the inequality $\mathrm{in}((G)_\tau)\le 2^\lambda$.

By [7, Theorem 4.29], every topological group K satisfies $c(K)\le 2^{\mathrm{in}(K)}$. We apply this inequality with $(G)_\tau$ in place of K to conclude that $c((G)_\tau)\le 2^{2^\lambda}$. $\square$

Corollary 3.2. *Every topological group G satisfies $c((G)_\tau)\le 2^{2^{\tau\cdot c(G)}}$. In particular, $c((G)_\omega)\le 2^{2^{c(G)}}$.*

Proof: Since $\mathrm{in}(G) \le c(G)$ by [7, Proposition 3.3(b)], the conclusion follows from Theorem 3.1. $\square$

Let us show that the upper bounds for the cellularity given in Theorem 3.1 and Corollary 3.2 are exact. First, we need a lemma.

Lemma 3.3. *The free Abelian group $A_\mathfrak{c}$ with $\mathfrak{c}$ generators admits a second countable Hausdorff precompact group topology, where $\mathfrak{c} = 2^\omega$.*

Proof: Denote by $\mathcal{T}$ the maximal precompact group topology on $A_\mathfrak{c}$ (i.e., the Bohr topology of $A_\mathfrak{c}$, see [1, Section 9.9]). Since $|A_\mathfrak{c}| = \mathfrak{c}$, $\mathcal{T}$ contains a weaker metrizable group topology $\mathcal{T}_\omega$ by [1, Proposition 9.9.37]. Since every precompact group has countable cellularity, we conclude that the group $K = (A_\mathfrak{c}, \mathcal{T}_\omega)$ is Hausdorff, second countable, and precompact. $\square$

Example 3.4. *There exists a precompact Abelian topological group H such that $c((H)_\omega) = 2^\mathfrak{c}$.*

Proof: We apply Uspenskij's result in [8]: For every infinite cardinal τ, there exists a subgroup G_τ of $(A_{\tau,d})^{2^\tau}$ such that $c(G_\tau) = 2^\tau$, where $A_{\tau,d}$ is the free Abelian group A_τ with τ generators endowed with the discrete topology (the construction in [9] makes the use of the free group F_τ with τ generators instead of A_τ, but a similar argument works as well for A_τ, see [1, Example 5.4.13]).

By Lemma 3.3, the free Abelian group $A = A_\mathfrak{c}$ admits a second countable, Hausdorff, precompact group topology $\mathcal{T}_\omega$. Put $K = (A, \mathcal{T}_\omega)$ and $\lambda = 2^\mathfrak{c}$. It is clear that $(K)_\omega$ coincides with the discrete group A, say, A_d. Consider the identity isomorphism $\varphi\colon K^\lambda \to A_d^\lambda$ and let $H = \varphi^{-1}(G)$, where G is a subgroup of A_d^λ satisfying $c(G) = \lambda$. Then H is precompact being a subgroup of the precompact group K^λ and, therefore, $c(H) \le \omega$. In addition, $\varphi\colon (K^\lambda)_\omega \to (A_d^\lambda)_\omega$ is a topological isomorphism and the topology of $(A_d^\lambda)_\omega$ is finer than that of A_d^λ. Therefore, the restriction of $\varphi\colon (K^\lambda)_\omega \to A_d^\lambda$ to the subgroup $(H)_\omega$ of $(K^\lambda)_\omega$ is a continuous isomorphism of $(H)_\omega$ onto G and, hence, $2^\mathfrak{c} = c(G) \le c((H)_\omega)$. On the other hand, $c((H)_\omega) \le 2^\mathfrak{c}$ by Theorem 3.1, so $c((H)_\omega) = 2^\mathfrak{c}$. $\square$

4. Complexity of continuous real-valued functions on ω-narrow groups

Since $\mathbb{R}$-factorizable groups form a proper subclass of ω-narrow groups, it is natural to consider the following problem (see also [6, Problem 3.3]):

Problem 4.1. *Let G be an ω-narrow topological group and f be a continuous real-valued function on G. Does there exist a continuous homomorphism $\pi\colon G \to K$ onto a topological group K with $w(K) \le 2^\omega$ such that $\pi \prec f$?*

It turns out that the complexity of continuous real-valued functions on ω-narrow topological groups does not exceed $2^\mathfrak{c}$, where $\mathfrak{c} = 2^\omega$. We do not know, however, if this bound is exact.

Theorem 4.2. *Let f be a continuous real-valued function on an ω-narrow topological group G. Then there exists a continuous homomorphism $\pi\colon G \to H$ onto a topological group H satisfying $w(H) \leq 2^{\mathfrak{c}}$ such that $\pi \prec f$.*

Proof: By [7, Theorem 4.29], the cellularity of G is not greater than $\mathfrak{c}$. Hence, according to [1, Theorem 8.1.18], one can find a continuous homomorphism $\varphi\colon G \to K$ onto a topological group K with $\psi(K) \leq \mathfrak{c}$ such that $\varphi \prec f$. Take a continuous real-valued function g on K satisfying $f = g \circ \varphi$. Clearly, the group K is ω-narrow as a continuous homomorphic image of the ω-narrow group G. We can now apply [7, Theorem 4.6] according to which $|K| \leq 2^{\mathrm{in}(K)\cdot\psi(K)} \leq 2^{\mathfrak{c}}$. In particular, $nw(K) \leq |K| \leq 2^{\mathfrak{c}}$. Now we use the following weak form of Shakhmatov's theorem in [5] (with $\tau = 2^{\mathfrak{c}}$): If K is a topological group with $nw(K) \leq \tau$ and $g\colon K \to \mathbb{R}$ is a continuous function, then there exist a continuous isomorphism $i\colon K \to H$ onto a topological group H with $w(H) \leq \tau$ and a continuous function $h\colon H \to \mathbb{R}$ such that $g = h \circ i$.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, "\pi"] & \mathbb{R} \\ K \arrow[r, "i"] & H \arrow[u, "h"'] \end{tikzcd}$$

Then the continuous homomorphism $\pi = i \circ \varphi$ of G onto H and the function h satisfy the equality $f = h \circ \pi$, i.e., $\pi \prec f$. Since $w(H) \leq 2^{\mathfrak{c}}$, this finishes the proof. □

The following result provides a partial solution to Problem 4.1 in the special case when H is weakly Lindelöf. As usual we denote by $\mathfrak{c}$ the power of the continuum.

Proposition 4.3. *Let $f\colon G \to X$ be a continuous mapping, where G is a weakly Lindelöf topological group and X is a Tychonoff space with $w(X) \leq \mathfrak{c}$. Then there exists a continuous homomorphism $\pi\colon G \to L$ onto a topological group L with $w(L) \leq \mathfrak{c}$ such that $\pi \prec f$.*

Proof: Clearly X is homeomorphic to a subspace of $\mathbb{R}^{\mathfrak{c}}$. Taking compositions of f with projections of $\mathbb{R}^{\mathfrak{c}}$ to the factors, we can assume that $X = \mathbb{R}$. Then by [1, Theorem 8.1.18], one can find a continuous homomorphism $\varphi\colon G \to K$ onto a topological group K of countable pseudocharacter and a continuous real-valued function $g\colon K \to \mathbb{R}$ such that $f = g \circ \varphi$. The group G is ω-narrow since it is weakly Lindelöf [7, Proposition 4.4], so K is also ω-narrow as a continuous homomorphic image of G. Therefore, $|K| \leq 2^{\mathrm{in}(K)\cdot\psi(K)} = \mathfrak{c}$ by [7, Theorem 4.6]. In particular, $nw(K) \leq \mathfrak{c}$. By a theorem in [5], there exist a continuous isomorphism $i\colon K \to L$ onto a topological group L with $w(L) \leq \mathfrak{c}$ and a continuous function $h\colon L \to \mathbb{R}$ such that $g = h \circ i$. Hence the homomorphism $\pi = i \circ \varphi\colon G \to L$ is as required. □

We are now in the position to present another subclass of ω-narrow groups where Problem 4.1 is solved in the affirmative.

Proposition 4.4. *Let G be an ω-narrow topological group. If G is $\mathfrak{c}$-steady, then for every continuous real-valued function f on G there exists a continuous homomorphism $\pi\colon G \to H$ onto a topological group H with $w(H) \le \mathfrak{c}$ such that $\pi \prec f$.*

Proof: Given a continuous real-valued function f on G, we can find, as in the proof of Theorem 4.2, a continuous homomorphism $\varphi\colon G \to K$ onto a topological group K with $\psi(K) \le \mathfrak{c}$ and a continuous real-valued function g on K such that $f = g \circ \varphi$. Since G is $\mathfrak{c}$-steady, the group K satisfies $nw(K) \le \mathfrak{c}$. Applying Shakhmatov's theorem in [5] once again, we find a continuous isomorphism $i\colon K \to H$ of K onto a topological group H with $w(H) \le \mathfrak{c}$ and a continuous real-valued function h on H such that $g = h \circ i$. Therefore, the continuous homomorphism $\pi = i \circ \varphi$ of G onto H satisfies $\pi \prec f$. □

5. Open problems

There exist ω-narrow groups H satisfying $c(H) = \mathfrak{c}$ [9]. In fact, there are even Lindelöf groups with the same property [2, Example 8]. We do not know, however, whether large pairwise disjoint families of open sets in ω-narrow groups can be discrete:

Problem 5.1. *Does there exist an ω-narrow topological group which contains a discrete family γ of open sets with $|\gamma| = \mathfrak{c}$?*

Another related problem concerns regular closed subsets of Lindelöf groups:

Problem 5.2. *Is every regular closed subset of a Lindelöf topological group the intersection of at most 2^ω open sets?*

Example 3.4 leaves the following open problem.

Problem 5.3. *Let γ be a family of G_δ-sets in a precompact topological group K. Does there exist a subfamily γ_0 of γ such that $|\gamma_0| \le \mathfrak{c}$ and $\bigcup \gamma_0$ is dense in $\bigcup \gamma$? What if the group K is ω-narrow?*

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(*Received* November 3, 2010)