Study of a viscoelastic frictional contact problem with adhesion

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Abstract. We consider a quasistatic frictional contact problem between a viscoelastic body with long memory and a deformable foundation. The contact is modelled with normal compliance in such a way that the penetration is limited and restricted to unilateral constraint. The adhesion between contact surfaces is taken into account and the evolution of the bonding field is described by a first order differential equation. We derive a variational formulation and prove the existence and uniqueness result of the weak solution under a certain condition on the coefficient of friction. The proof is based on time-dependent variational inequalities, differential equations and Banach fixed point theorem.

Keywords: viscoelastic, normal compliance, adhesion, frictional, variational inequality, weak solution

Classification: 47J20, 49J40, 74M10, 74M15

1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws lead to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [11]. The mathematical, mechanical and numerical state of the art can be found in [29]. In this reference we find a detailed analysis and numerical studies of the adhesive contact problems. Recently a new book ([31, Chapter 7–11, pp. 127–209]) introduces the reader into the theory of variational inequalities with emphasis on the study of contact mechanics and more specifically, on antiplane frictional contact problems. Also, recently existence results were established in [1], [9], [12] in the study of unilateral and frictional contact problems for linear elastic materials. In [23], [24] quasistatic frictional contact problems with adhesion for linear elastic materials were studied and existence results were given under a smallness assumption on the coefficient of friction. Here as in [19], where a similar problem was resolved, we study a mathematical model which describes a frictional and adhesive contact problem between a viscoelastic body with long memory and a deformable foundation. The contact is modelled with normal compliance in such a way that the penetration is limited and restricted to unilateral constraints. The main novelty of the model considered is the coupling of memory effects with friction and adhesion effects. We recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [2], [3], [4], [5], [7], [8], [13], [19], [20], [22], [25], [26], [27], [28], [29], [30], [32], [33]. Following [14], [15] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable satisfies the restrictions $0 \leq \beta \leq 1$. At a point on the boundary contact surface, when $\beta = 1$ the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [6], [14], [15], [16], [23], [25], [28], [29]. According to [18], the method presented here considers a compliance model in which the compliance term does not represent necessarily a compact perturbation of the original problem without contact. This leads us to study such models, where a strictly limited penetration is permitted with the limit procedure to the Signorini contact problem. In [32], [33] frictionless unilateral contact problems with adhesion for elastic materials were studied. Also recently in [10] a dynamic contact problem with nonlocal friction and adhesion between two viscoelastic bodies of Kelvin-Voigt type was resolved. An existence result was proved without any assumption on the smallness of the coefficient of friction and the variational formulation was approximated. Moreover some numerical results were presented. In this work as in [32], [33] we derive a variational formulation of the mechanical problem written as the coupling between a variational inequality and a differential equation. We prove the existence of a unique weak solution if the coefficient of friction is sufficiently small, and obtain a partial regularity result for the solution. However, comparing this result to that obtained in [10] and keeping in mind the existence results found in [23], [24], we observe that in quasistatic frictional contact problems information about the solution (second derivative of u, initial velocity) are removed and this is paid by more restrictive assumptions on other data, particulary on the coefficient of friction. On the other hand when the latter is great, it has been proved for example in the study of some frictional static contact problems (see [17]) that we have nonuniqueness of the solution.

The paper is structured as follows. In Section 2 we present some notation and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 3.1.

2. Problem statement and variational formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a domain initially occupied by a viscoelastic body with long memory. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. We assume that Γ is composed of three sets $\bar{\Gamma}_1$, $\bar{\Gamma}_2$, and $\bar{\Gamma}_3$, with the mutually disjoint relatively open sets Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in adhesive frictional contact with a deformable foundation.

Thus, the classical formulation of the mechanical problem is written as follows.

Problem P_1. Find a displacement $u : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to [0,1]$ such that, for all $t \in [0,T]$,

$$\sigma(t) = F\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\,ds \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div}\sigma(t) + \varphi_1(t) = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$u(t) = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma(t)\nu = \varphi_2(t) \quad \text{on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + p\left(u_\nu(t)\right) - c_\nu\beta^2(t)R_\nu\left(u_\nu\left(t\right)\right) \le 0 \\ \left(\sigma_\nu(t) + p\left(u_\nu(t)\right) - c_\nu\beta^2\left(t\right)R_\nu\left(u_\nu(t)\right)\right)\left(u_\nu\left(t\right) - g\right) = 0 \end{array}\right\} \text{ on } \Gamma_3, \tag{2.5}$$

$$\left.\begin{array}{l} \left|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau\left(u_\tau\left(t\right)\right)\right| \le \mu p\left(u_\nu\left(t\right)\right) \\ \left|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau\left(u_\tau\left(t\right)\right)\right| < \mu p\left(u_\nu(t)\right) \Rightarrow u_\tau = 0 \\ \left|\sigma_\tau\left(t\right) + c_\tau\beta^2(t)R_\tau\left(u_\tau\left(t\right)\right)\right| = \mu p\left(u_\nu\left(t\right)\right) \Rightarrow \\ \exists\lambda \ge 0 \text{ s.t. } u_\tau = -\lambda\left(\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau\left(u_\tau(t)\right)\right) \end{array}\right\} \text{ on } \Gamma_3, \tag{2.6}$$

$$\dot\beta(t) = -\left[\beta(t)\left(c_\nu\left(R_\nu\left(u_\nu\left(t\right)\right)\right)^2 + c_\tau\left|R_\tau\left(u_\tau(t)\right)\right|^2\right) - \varepsilon_a\right]_+ \quad \text{on } \Gamma_3, \tag{2.7}$$

$$\beta(0) = \beta_0 \ \text{ on } \Gamma_3. \tag{2.8}$$

Equation (2.1) represents the viscoelastic constitutive law with long memory of the material; F is the elasticity operator and $\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\,ds$ is the memory term in which $\mathcal{F}$ denotes the tensor of relaxation; the stress $\sigma(t)$ at current instant t depends on the whole history of strains up to this moment of time. Equation (2.2) represents the equilibrium equation while (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. The conditions (2.5) represent the unilateral contact with adhesion in which c_ν is a given adhesion coefficient and R_ν, R_τ are truncation operators defined in (2.5) and (2.6), respectively, by

$$R_\nu(s)=\begin{cases}L & \text{if } s<-L\\ -s & \text{if } -L\le s\le 0\\ 0 & \text{if } s>0\end{cases},\quad R_\tau(v)=\begin{cases}v & \text{if } |v|\le L,\\ L\frac{v}{|v|} & \text{if } |v|>L.\end{cases}$$

Here $L>0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [23], [29]) and p is a normal compliance function which satisfies the assumption (2.16); g denotes the maximum value of the penetration which satisfies $g\ge 0$. When $u_\nu<0$ i.e. when there is separation between the body and the foundation then the condition (2.5) combined with hypothesis (2.16) and definition of R_ν shows that $\sigma_\nu=c_\nu\beta^2R_\nu(u_\nu)$ and does not exceed the value $L\|c_\nu\|_{L^\infty(\Gamma_3)}$. When $g>0$, the body may interpenetrate into the foundation, but the penetration is limited, that is $u_\nu\le g$. In this case of penetration (i.e. $u_\nu\ge 0$), when $0\le u_\nu<g$ then $-\sigma_\nu=p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu\le 0$. Since p is an increasing function, the reaction is increasing with the penetration. If $u_\nu=g$ then $-\sigma_\nu\ge p(g)$ and σ_ν is not uniquely determined. If $g>0$ and $p=0$, conditions (2.5) become the Signorini's contact conditions with a gap and adhesion

$$u_\nu\le g,\ \sigma_\nu-c_\nu\beta^2R_\nu\left(u_\nu\right)\le 0,\ \left(\sigma_\nu-c_\nu\beta^2R_\nu\left(u_\nu\right)\right)\left(u_\nu-g\right)=0.$$

If $g=0$, the conditions (2.5) combined with hypothesis (2.16) lead to the Signorini contact conditions with adhesion, with zero gap, given by

$$u_\nu\le 0,\ \sigma_\nu-c_\nu\beta^2R_\nu\left(u_\nu\right)\le 0,\ \left(\sigma_\nu-c_\nu\beta^2R_\nu\left(u_\nu\right)\right)u_\nu=0.$$

These contact conditions were used in [30], [32]. It follows from (2.5) that there is no penetration between the body and the foundation, since $u_\nu\le 0$ during the process. Also, note that when the bonding field vanishes, then the contact conditions (2.5) become the classical Signorini contact conditions with zero gap, that is,

$$u_\nu\le 0,\ \sigma_\nu\le 0,\ \sigma_\nu u_\nu=0.$$

Conditions (2.6) represent Coulomb's law of dry friction with adhesion where μ denotes the coefficient of friction and c_τ is a given adhesion coefficient. Equation (2.7) represents the ordinary differential equation which describes the evolution of the bonding field, in which $r_+=\max\{r,0\}$, and it was already used in [7]. Since $\dot\beta\le 0$ on $\Gamma_3\times(0,T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [21] it must be pointed out clearly that condition (2.7) does not allow for complete debonding in finite time. Finally, (2.8) is the initial condition, in which β_0 denotes the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We denote by S_d the space of second order symmetric tensors on

$\mathbb{R}^d$ $(d=2,3)$ while $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u,v\in\mathbb{R}^d$, $u\cdot v=u_iv_i$, $|v|=(v\cdot v)^{\frac{1}{2}}$, and for every $\sigma,\tau\in S_d$, $\sigma\cdot\tau=\sigma_{ij}\tau_{ij}$, $|\tau|=(\tau\cdot\tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H=\left(L^2(\Omega)\right)^d,\ H_1=\left(H^1(\Omega)\right)^d,\ Q=\left\{\tau=(\tau_{ij}):\tau_{ij}=\tau_{ji}\in L^2(\Omega)\right\},$$
$$Q_1=\{\sigma\in Q:\operatorname{div}\sigma\in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u,v)_H=\int_\Omega u_iv_i\,dx,\quad \langle\sigma,\tau\rangle_Q=\int_\Omega \sigma_{ij}\tau_{ij}\,dx.$$

The strain tensor is

$$\varepsilon(u)=(\varepsilon_{ij}(u))=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right);$$

$\operatorname{div}\sigma=(\sigma_{ij,j})$ is the divergence of σ. For every element $v\in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu=v\cdot\nu,\quad v_\tau=v-v_\nu\nu.$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma\in Q_1$, and when σ is a regular function then

$$\sigma_\nu=(\sigma\nu)\cdot\nu,\quad \sigma_\tau=\sigma\nu-\sigma_\nu\nu,$$

and the following Green's formula holds:

$$\langle\sigma,\varepsilon(v)\rangle_Q+(\operatorname{div}\sigma,v)_H=\int_\Gamma \sigma\nu\cdot v\,da\quad\forall\, v\in H_1,$$

where da is the surface measure element. Now, let V be the closed subspace of H_1 defined by

$$V=\{v\in H_1:v=0\text{ on }\Gamma_1\},$$

and denote the convex subset of admissible displacements given by

$$K=\{v\in V:v_\nu\le g\ \ a.e.\text{ on }\Gamma_3\}.$$

Since $\operatorname{meas}(\Gamma_1)>0$, the following Korn's inequality holds [11]:

$$\|\varepsilon(v)\|_Q\ge c_\Omega\|v\|_{H_1}\quad\forall\, v\in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u,v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Thus, $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall\, v \in V. \tag{2.10}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0,T;V)$. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in C([0,T];H), \quad \varphi_2 \in C\left([0,T]; \left(L^2\left(\Gamma_2\right)\right)^d\right). \tag{2.11}$$

We define the function $f : [0,T] \to V$ by

$$(f(t), v)_V = \int_\Omega \varphi_1(t) \cdot v\, dx + \int_{\Gamma_2} \varphi_2(t) \cdot v\, da \quad \forall\, v \in V,\ t \in [0,T], \tag{2.12}$$

and we note that (2.11) and (2.12) imply

$$f \in C\left([0,T];V\right).$$

In the study of the mechanical problem P_1 we assume that the elasticity operator $F : \Omega \times S_d \to S_d$, satisfies

$$\left\{\begin{array}{l}
\text{(a) there exists } M > 0 \text{ such that} \\
\quad |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \leq M\,|\varepsilon_1 - \varepsilon_2| \quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d, \\
\quad a.e.\ x \in \Omega; \\
\text{(b) there exists } m > 0 \text{ such that} \\
\quad (F(x,\varepsilon_1) - F(x,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m\,|\varepsilon_1 - \varepsilon_2|^2, \\
\quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d,\ a.e.\ x \in \Omega; \\
\text{(c) the mapping } x \to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\
\quad \text{for any } \varepsilon \text{ in } S_d; \\
\text{(d) } x \to F(x,0) \in Q.
\end{array}\right. \tag{2.13}$$

Also we need to introduce the space of the tensors of fourth order defined by

$$Q_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) : \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega)\},$$

which is a real Banach space with the norm

$$\|\mathcal{E}\|_{Q_\infty} = \max_{0\le i,j,k,l\le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

We assume that the tensor of relaxation $\mathcal{F}$ satisfies

$$\mathcal{F} \in C([0,T]; Q_\infty). \tag{2.14}$$

The adhesion coefficients satisfy

$$c_\nu, c_\tau, \varepsilon_a \in L^\infty(\Gamma_3) \text{ and } c_\nu, c_\tau, \varepsilon_a \ge 0 \ a.e. \text{ on } \Gamma_3, \tag{2.15}$$

and we assume that the initial bonding field satisfies

$$\beta_0 \in L^2(\Gamma_3);\ 0 \le \beta_0 \le 1 \ a.e. \text{ on } \Gamma_3. \tag{2.16}$$

Next, we define respectively the functionals

$$j_c : V \times V \to \mathbb{R}, \quad j_\tau : V \times V \to \mathbb{R}$$

by

$$j_c(u,v) = \int_{\Gamma_3} p(u_\nu) v_\nu \, da, \quad j_\tau(u,v) = \int_{\Gamma_3} \mu p(u_\nu)|v_\tau| \, da,$$

and let

$$j = j_c + j_\tau.$$

We also define the functional

$$r : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$$

by

$$r(\beta,u,v) = \int_{\Gamma_3} \left(-c_\nu \beta^2 R_\nu(u_\nu)\, v_\nu + c_\tau \beta^2 R_\tau(u_\tau)\cdot v_\tau\right) da$$
$$\forall (\beta,u,v) \in L^2(\Gamma_3) \times V \times V.$$

As in [18] we assume that the normal compliance function p satisfies

$$\begin{cases} \text{(a) } p :]-\infty, g] \to \mathbb{R}; \\ \text{(b) there exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2|, \text{ for all } r_1, r_2 \le g; \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0, \text{ for all } r_1, r_2 \le g; \\ \text{(d) } p(r) = 0 \text{ for all } r < 0. \end{cases} \tag{2.17}$$

We assume that the coefficient of friction μ satisfies

$$\mu \in L^{\infty}(\Gamma_3) \text{ and } \mu \geq 0 \text{ } a.e. \text{ on } \Gamma_3. \tag{2.18}$$

Finally we need to introduce the following set of the bonding field,

$$B = \left\{\theta : [0,T] \to L^2(\Gamma_3) : 0 \leq \theta(t) \leq 1, \ \forall\, t \in [0,T], \ a.e. \text{ on } \Gamma_3\right\}.$$

Below, c is a generic positive constant which does not depend on $t \in [0,T]$, whose value may change from place to place.

Now using Green's formula, we obtain that the problem P_1 has the following variational formulation.

Problem P_2. Find a displacement field $u \in C([0,T];K)$ and a bonding field $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B$ such that

$$\begin{aligned} &\langle F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t))\rangle_Q + \left\langle \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\,ds, \varepsilon(v) - \varepsilon(u(t))\right\rangle_Q \\ &\quad + r(\beta(t), u(t), v - u(t)) + j(\beta(t), u(t), v) - j(\beta(t), u(t), u(t)) \\ &\quad \geq (f(t), v - u(t))_V \quad \forall\, v \in K, \ t \in [0,T], \end{aligned} \tag{2.19}$$

(2.20)

$$\dot{\beta}(t) = -\left[\beta(t)\left(c_\nu\left(R_\nu(u_\nu(t))\right)^2 + c_\tau\left|R_\tau(u_\tau(t))\right|^2\right) - \varepsilon_a\right]_+ \ a.e. \ t \in (0,T),$$

$$\beta(0) = \beta_0. \tag{2.21}$$

3. Existence and uniqueness of the solution

Our main result in this section is the following theorem.

Theorem 3.1. *Let* (2.11), (2.13), (2.14), (2.15), (2.16), (2.17) *and* (2.18) *hold. Then, there exists a constant $\mu_0 > 0$ such that Problem P_2 has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

The proof of Theorem 3.1 is carried out in several steps. In the first step, let $k > 0$ and consider the space X defined as

$$X = \left\{\beta \in C\left([0,T]; L^2(\Gamma_3)\right) : \sup_{t\in[0,T]} \left[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}\right] < +\infty\right\}.$$

It is well known that X is a Banach space with the norm

$$\|\beta\|_X = \sup_{t\in[0,T]} \left[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}\right].$$

Next for a given $\beta \in X$, we consider the following variational problem.

Problem $P_{1\beta}$**.** Find $u_\beta \in C([0,T];K)$ such that

$$\begin{aligned}(3.1)\qquad &\langle F\varepsilon\left(u_\beta(t)\right), \varepsilon(v) - \varepsilon\left(u_\beta(t)\right)\rangle_Q \\ &\quad + \left\langle \int_0^t \mathcal{F}(t-s)\varepsilon\left(u_\beta(s)\right)\, ds, \varepsilon(v) - \varepsilon\left(u_\beta(t)\right)\right\rangle_Q \\ &\quad + r\left(\beta\left(t\right), u_\beta\left(t\right), v - u_\beta(t)\right) + j\left(u_\beta(t), v\right) - j\left(u_\beta\left(t\right), u_\beta\left(t\right)\right) \\ &\geq \left(f(t), v - u_\beta(t)\right)_V \quad \forall\, v \in K,\ t \in [0,T].\end{aligned}$$

We have the following result.

Proposition 3.2. *There exists a constant* $\mu_1 > 0$ *such that Problem* $P_{1\beta}$ *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1.$$

For the proof of this proposition we consider the following problem.

Problem $P_{1\beta\eta}$**.** For $\eta \in C([0,T];Q)$, find $u_{\beta\eta} \in C([0,T];K)$ such that

$$\begin{aligned}(3.2)\qquad &\langle F\varepsilon\left(u_{\beta\eta}(t)\right), \varepsilon\left(v - u_{\beta\eta}(t)\right)\rangle_Q + \langle \eta(t), \varepsilon\left(v - u_{\beta\eta}(t)\right)\rangle_Q \\ &+ r\left(\beta(t), u_{\beta\eta}\left(t\right), v - u_{\beta\eta}(t)\right) + j\left(u_{\beta\eta}(t), v\right) - j\left(u_{\beta\eta}\left(t\right), u_{\beta\eta}(t)\right) \\ &\geq \left(f\left(t\right), v - u_{\beta\eta}\left(t\right)\right)_V \ \forall\, v \in K,\ t \in [0,T].\end{aligned}$$

Riesz's representation theorem leads to the existence of an element $f_\eta \in C([0,T];V)$ such that

$$\left(f_\eta\left(t\right), v\right)_V = (f(t), v)_V - \langle \eta(t), \varepsilon(v)\rangle_Q \quad \forall\, v \in V.$$

Then it is clear that Problem $P_{1\beta\eta}$ is equivalent to the following problem.

Problem $P_{2\beta\eta}$**.** For $\eta \in C([0,T];Q)$, find $u_{\beta\eta} \in C([0,T];K)$ such that

$$\begin{aligned}(3.3)\qquad &\langle F\varepsilon\left(u_{\beta\eta}(t)\right), \varepsilon\left(v - u_{\beta\eta}(t)\right)\rangle_Q + r\left(\beta\left(t\right), u_{\beta\eta}(t), v - u_{\beta\eta}(t)\right) \\ &+ j\left(u_{\beta\eta}\left(t\right), v\right) - j\left(u_{\beta\eta}(t), u_{\beta\eta}\left(t\right)\right) \geq \left(f_\eta(t), v - u_{\beta\eta}(t)\right)_V \\ &\forall\, v \in K,\ t \in [0,T].\end{aligned}$$

We have the following result.

Lemma 3.3. *There exists a constant* $\mu_1 > 0$ *such that Problem* $P_{2\beta\eta}$ *has a unique solution if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$.

Proof: Let $t \in [0,T]$ and let $A_t : V \to V$ be the operator defined by

$$(A_t u, v)_V = \langle F\varepsilon\left(u\right), \varepsilon(v)\rangle_Q + r(\beta(t), u, v) + j_c(u, v) \quad \forall\, u, v \in V.$$

As in [29], using (2.13)(a), (2.15), (2.17)(b) and the properties of R_ν and R_τ, we see that the operator A_t is Lipschitz continuous. Also using (2.13)(b), (2.15), (2.17)(c) and the properties of R_ν and R_τ, we have

$$(A_t u - A_t v, u - v)_V \geq m\|u - v\|_V^2 \quad \forall\, u, v \in V.$$

Then the operator A_t is strongly monotone. Next, we can easily check that, for a given $u \in K$, the functional $j_\tau(u, \cdot) : K \to \mathbb{R}$ is convex and lower semicontinuous. Let $\mu_1 = m/L_p d_\Omega^2$, then for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, since K is a nonempty closed convex subset of V, using a standard existence and uniqueness result for elliptic variational inequalities (see [2]), it follows that there exists a unique element $u_{\beta\eta}(t) \in K$ which satisfies the inequality (3.3). Moreover according again to [29], using (3.3), we have

$$\begin{aligned} &\|u_{\beta\eta}(t_1) - u_{\beta\eta}(t_2)\|_V \\ &\leq c\left(\|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)} + \|f_\eta(t_1) - f_\eta(t_2)\|_V\right) \quad \forall\, t_1, t_2 \in [0, T]. \end{aligned}$$

Hence the regularity $f_\eta \in C([0,T];V)$ and $\beta \in C([0,T];L^2(\Gamma_3))$ imply that $u_{\beta\eta} \in C([0,T];K)$. □

Now to end the proof of Proposition 3.2, we introduce the operator

$$\Lambda_\beta : C([0,T];Q) \to C([0,T];Q)$$

defined by

$$\Lambda_\beta \eta(t) = \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta\eta}(s))\, ds \quad \forall\, \eta \in C([0,T];Q),\ t \in [0,T]. \tag{3.4}$$

Lemma 3.4. *The operator Λ_β has a unique fixed point η_β.*

Proof: Let $\eta_1, \eta_2 \in C([0,T];Q)$. Using (3.3), (3.4) and (2.14) we obtain

$$\|\Lambda_\beta \eta_1(t) - \Lambda_\beta \eta_2(t)\|_Q \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_Q \, ds \quad \forall\, t \in [0,T].$$

Reiterating this inequality n times, yields

$$\left\|\Lambda_\beta^n \eta_1 - \Lambda_\beta^n \eta_2\right\|_{C([0,T];Q)} \leq \frac{(cT)^n}{n!} \|\eta_1 - \eta_2\|_{C([0,T];Q)}.$$

As $\lim_{n\to+\infty} \frac{(cT)^n}{n!} = 0$, it follows that for a positive integer n sufficiently large, Λ_β^n is a contraction; then, by using the Banach fixed point theorem, it admits a unique fixed point η_β which is also a unique fixed point of Λ_β i.e.,

$$\Lambda_\beta \eta_\beta(t) = \eta_\beta(t) \quad \forall\, t \in [0,T]. \tag{3.5}$$

Then by (3.3) and (3.5) we conclude that $u_{\beta\eta_\beta}$ is the unique solution of (3.1) and Proposition 3.2 is proved. □

Next denote $u_\beta = u_{\beta\eta_\beta}$. In the step below we consider the following problem.

Problem $P_{2\beta}$**.** Find $\beta^* : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot{\beta}^*(t) = -\left[\beta^*(t)\left(c_\nu\left(R_\nu\left(u_{\beta^*\nu}(t)\right)\right)^2 + c_\tau\left|R_\tau\left(u_{\beta^*\tau}\right)\right|^2\right) - \varepsilon_a\right]_+ \quad a.e.\ t \in (0,T), \tag{3.6}$$

$$\beta^*(0) = \beta_0. \tag{3.7}$$

We obtain the following result.

Proposition 3.5. *Problem* $P_{2\beta}$ *has a unique solution* β^* *which satisfies*

$$\beta^* \in W^{1,\infty}\left(0,T; L^2\left(\Gamma_3\right)\right) \cap B.$$

Proof: Consider the mapping $\Lambda : X \to X$ given by

$$\Lambda\beta(t) = \beta_0 - \int_0^t \left[\beta(s)\left(c_\nu(R_\nu\left(u_{\beta\nu}(s)\right))^2 + c_\tau\left|R_\tau\left(u_{\beta\tau}\right)\right|^2\right) - \varepsilon_a\right]_+ ds,$$

where u_β is the solution of Problem $P_{1\beta}$. Then we have

$$\begin{aligned}
&\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \\
&\le c\int_0^t \left\|\beta_1(s)\left(R_\nu\left(u_{\beta_1\nu}(s)\right)\right)^2 - \beta_2(s)\left(R_\nu\left(u_{\beta_2\nu}(s)\right)\right)^2\right\|_{L^2(\Gamma_3)} ds \\
&\quad + c\int_0^t \left\|\beta_1(s)\left|R_\tau\left(u_{\beta_1\tau}(s)\right)\right|^2 - \beta_2(s)\left|R_\tau\left(u_{\beta_2\tau}(s)\right)\right|^2\right\|_{L^2(\Gamma_3)} ds.
\end{aligned}$$

We use the definition of the truncation operators R_ν and R_τ and write

$$\beta_1 = \beta_1 - \beta_2 + \beta_2.$$

It follows after some algebra calculus that

$$\begin{aligned}
&\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \\
&\le c\textstyle\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds + c\int_0^t \|u_{\beta_1\nu}(s) - u_{\beta_2\nu}(s)\|_{L^2(\Gamma_3)}\, ds.
\end{aligned}$$

Moreover using (2.10), we get

$$\begin{aligned}
&\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \\
&\le c\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds + cd_\Omega\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V\, ds.
\end{aligned} \tag{3.8}$$

Now we need to show the following lemma.

Lemma 3.6. *There exists a constant* $\mu_0 \in]0,\mu_1[$ *such that for* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *we have*

$$\|u_{\beta_2}(t) - u_{\beta_1}(t)\|_V \le c\,\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \quad \forall\, t \in [0,T].$$

PROOF: Let $t \in [0,T]$. Take $u_{\beta_2}(t)$ in the inequality (3.1) satisfied by $u_{\beta_1}(t)$, then take $u_{\beta_1}(t)$ in the same inequality satisfied by $u_{\beta_2}(t)$. After adding the resulting inequalities we find that

$$\begin{aligned}
&\langle F\varepsilon(u_{\beta_1}(t)) - F\varepsilon(u_{\beta_2}(t)), \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\rangle_Q \\
&+\left\langle \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\,ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\right\rangle_Q \\
&+r(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + r(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\
&+j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) \\
&-j(u_{\beta_2}(t), u_{\beta_2}(t)) \geq 0.
\end{aligned}$$

Using the assumption (2.13)(b) on F we deduce from the previous inequality that

$$\begin{aligned}
&m\,\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \\
&\leq \left\langle \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\,ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\right\rangle_Q \\
&\quad +r(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + r(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\
&\quad +j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) \\
&\quad -j(u_{\beta_2}(t), u_{\beta_2}(t)).
\end{aligned} \tag{3.9}$$

Using the properties of R_ν and R_τ (see [29]), we find that

$$\begin{aligned}
&r(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + r(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\
&\leq \left(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)}\right) Ld_\Omega\,\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}\,\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V .
\end{aligned}$$

On the other hand as in [31] we have

$$\begin{aligned}
&\left\langle \int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\right) ds, \varepsilon(u_{\beta_2}(t) - u_{\beta_1}(t))\right\rangle_Q \\
&\leq \left(\int_0^t \|\mathcal{F}(t-s)\|_{Q_\infty}\|u_{\beta_2}(s) - u_{\beta_1}(s)\|_V\,ds\right)\|u_{\beta_2}(t) - u_{\beta_1}(t)\|_V \\
&\leq c\left(\int_0^t \|u_{\beta_2}(s) - u_{\beta_1}(s)\|_V\,ds\right)\|u_{\beta_2}(t) - u_{\beta_1}(t)\|_V .
\end{aligned}$$

Using the elementary inequality

$$cab \leq c^2\frac{a^2}{2m} + m\frac{b^2}{2},$$

where the constant $m > 0$ is introduced in (2.13)(b), we find that

$$\begin{aligned} &\left\langle \int_0^t \mathcal{F}(t-s)\left(\varepsilon\left(u_{\beta_1}(s)\right)-\varepsilon\left(u_{\beta_2}(s)\right)\right)\, ds, \varepsilon\left(u_{\beta_2}(t)-u_{\beta_1}(t)\right)\right\rangle_Q \\ &\leq \frac{c^2}{2m}\left(\int_0^t \left\|u_{\beta_2}(s)-u_{\beta_1}(s)\right\|_V\, ds\right)^2+\frac{m}{2}\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V^2 . \end{aligned} \tag{3.10}$$

Also using the assumptions (2.17)(b) and (2.17)(c) on the function p yields

$$\begin{aligned} & j\left(u_{\beta_1}(t), u_{\beta_2}(t)\right)-j\left(u_{\beta_1}(t), u_{\beta_1}(t)\right) \\ &\quad +j\left(u_{\beta_2}(t), u_{\beta_1}(t)\right)-j\left(u_{\beta_2}(t), u_{\beta_2}(t)\right) \\ &\leq L_p d_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V^2 . \end{aligned} \tag{3.11}$$

Now, we combine inequalities (3.9), (3.10) and (3.11) to obtain

$$\begin{aligned} & m\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 \leq L_p d_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V^2 \\ &\quad +\frac{c^2}{2m}\left(\int_0^t\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V\, ds\right)^2+\frac{m}{2}\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V^2 \\ &\quad +\left(\left\|c_\nu\right\|_{L^\infty(\Gamma_3)}+\left\|c_\tau\right\|_{L^\infty(\Gamma_3)}\right) L d_\Omega\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)} \\ &\quad \times\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V . \end{aligned} \tag{3.12}$$

Using Young's inequality we get

$$\begin{aligned} & \left(\left\|c_\nu\right\|_{L^\infty(\Gamma_3)}+\left\|c_\tau\right\|_{L^\infty(\Gamma_3)}\right) L d_\Omega\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)} \\ &\quad \times\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V \\ &\leq c\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}^2+\frac{m}{4}\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V^2 . \end{aligned} \tag{3.13}$$

Then we deduce from (3.12) and (3.13) that

$$\begin{aligned} \frac{m}{4}\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V^2 &\leq L_p d_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V^2 \\ &+\frac{c^2}{2m}\left(\int_0^t\left\|u_{\beta_2}(s)-u_{\beta_1}(s)\right\|_V\, ds\right)^2+c\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}^2 . \end{aligned}$$

Let

$$\mu_0=\frac{m}{4 L_p d_\Omega^2}=\frac{\mu_1}{4} .$$

Then if

$$\|\mu\|_{L^\infty(\Gamma_3)}<\mu_0,$$

we deduce that

$$\|u_{\beta_2}(t) - u_{\beta_1}(t)\|_V^2 \leq c\left(\int_0^t \|u_{\beta_2}(s) - u_{\beta_1}(s)\|_V^2\, ds + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2\right).$$

Hence Gronwall's argument implies that

$$\|u_{\beta_2}(t) - u_{\beta_1}(t)\|_V \leq c\,\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}. \tag{3.14}$$ □

Now to end the proof of Proposition 3.5 we use (3.8) and (3.14) to get

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \leq c\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds.$$

On the other hand we have

$$\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds \leq \|\beta_1 - \beta_2\|_X \frac{\exp(kt)}{k}.$$

Therefore

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \leq c\,\|\beta_1 - \beta_2\|_X \frac{\exp(kt)}{k} \quad \forall\, t \in [0,T],$$

which yields

$$\exp(-kt)\,\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \leq \frac{c}{k}\,\|\beta_1 - \beta_2\|_X \quad \forall\, t \in [0,T].$$

Hence we obtain

$$\|\Lambda\beta_1 - \Lambda\beta_2\|_X \leq \frac{c}{k}\,\|\beta_1 - \beta_2\|_X. \tag{3.15}$$

The inequality (3.15) shows that for k sufficiently large Λ is a contraction. Then it has a unique fixed point β^* which satisfies (3.6) and (3.7). To prove that $\beta^* \in B$, we use (2.17) and we refer the reader to [30, Remark 3.1]. □

Lemma 3.7. *(u_{β^*}, β^*) is a unique solution of Problem P_2.*

Proof: *Existence.* Let $\beta = \beta^*$ and let u_{β^*} the solution of Problem $P_{1\beta}$. We conclude by (3.1), (3.6) and (3.7) that (u_{β^*}, β^*) is a solution to Problem P_2.

Uniqueness. Suppose that (u, β) is a solution of Problem P_2 which satisfies (2.19), (2.20) and (2.21). It follows from (2.19) that u is a solution to Problem $P_{1\beta}$, and from Proposition 3.2 that $u = u_\beta$. Take $u = u_\beta$ in (2.19) and use the initial condition (2.21), we deduce that β is a solution to Problem $P_{2\beta}$. Therefore, we obtain from Proposition 3.5 that $\beta = \beta^*$ and then we conclude that (u_{β^*}, β^*) is a unique solution to Problem P_2. □

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