A note on coclones of topological spaces

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Abstract. The clone of a topological space is known to have a strictly more expressive first-order language than that of the monoid of continuous self-maps. The current paper studies coclones of topological spaces (i.e. clones in the category dual to that of topological spaces and continuous maps) and proves that, in contrast to clones, the first-order properties of coclones cannot express anything more than those of the monoid, except for the case of discrete and indiscrete spaces.

Keywords: clone, coclone, monoid of continuous self-maps, clone theory, monoid theory

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1. Introduction

The monoid $\mathrm{Mon}(X)$ of a topological space X is the set of all continuous maps from X to X, together with the composition operator. This algebraic structure has been extensively studied and is known to reflect many of the topological properties of the underlying space. For some quite large classes of spaces — e.g. the class of completely regular T_1 spaces which contain an arc, this monoid completely describes the topology of the space; see [9].

In his monograph [11], W. Taylor introduced a multi-sorted extension of the monoid of continuous self-maps, namely, the clone of a topological space. More specifically, Taylor studied those properties of spaces which can be described by a formula in the first order language of the theory of clones. Since then, a number of publications have been devoted to this topic; see [1] for a survey of these.

Let ω denote the set of all finite ordinals.

Generally, in a category $\mathfrak{K}$ with finite products, the *clone* of an object $X \in \operatorname{obj} \mathfrak{K}$ is the ω-sorted algebra (in the sense of [2])

$$\mathrm{Clo}(X) = \langle C_n; S^n_m; \pi^n_i \rangle_{m,n \in \omega, i<n}$$

with

$$\begin{aligned} &C_n = \mathfrak{K}(X^n, X) && n \in \omega, \\ &S^n_m : C_n \times (C_m)^n \to C_m && m, n \in \omega, \\ &\pi^n_i \in C_n && i < n \in \omega, \end{aligned}$$

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where π_i^n is the i-th projection of X^n onto X and S_m^n composes a morphism $f \in \mathfrak{K}(X^n, X)$ with the diagonal product of an n-tuple $g_0, \dots, g_{n-1} \in \mathfrak{K}(X^m, X)$ to obtain

$$S_m^n(f; g_0, \dots, g_{n-1}) = f \circ (g_0 \bigtriangleup \cdots \bigtriangleup g_{n-1}) \in \mathfrak{K}(X^m, X).$$

The k-segment $\mathrm{Clo}_k(X)$ is the reduction of $\mathrm{Clo}(X)$ to the first k sorts:

$$\mathrm{Clo}_k(X) = \langle C_n; S_m^n; \pi_i^n \rangle_{m,n<k,i<n}.$$

The term clone was apparently coined by P. Hall in [4]. The above definition of a clone of an object is tightly related to F.W. Lawvere's algebraic theories (see [7], [8]).

By the clone of a topological space X we mean its clone in the category of all topological spaces and continuous maps. Clearly, the clone of a topological space extends the monoid of continuous maps in the sense that its 1-st sort $\langle C_1; S_1^1; \pi_0^1 \rangle$ is exactly $\mathrm{Mon}(X)$. Thus, every topological property of the space which can be described by an algebraic or first-order property of the monoid of continuous maps can also be described by a corresponding property of the clone. The opposite is not true: there are topological properties which can be described by a first order property of the clone and yet which cannot be described by properties of the monoid. The following, much stronger result was proved by Sichler and Trnková in [10]:

Theorem. *For any triple $2 \le n_1 \le n_2 \le n_3$ of finite ordinals there exist metrizable topological spaces X and Y on the same carrier set such that*

- $\mathrm{Clo}_n(X)$ *and* $\mathrm{Clo}_n(Y)$ *coincide if and only if* $n \le n_1$*;*
- $\mathrm{Clo}_n(X)$ *is isomorphic to* $\mathrm{Clo}_n(Y)$ *if and only if* $n \le n_2$*;*
- $\mathrm{Clo}_n(X)$ *is elementarily equivalent to* $\mathrm{Clo}_n(Y)$ *if and only if* $n \le n_3$*.*

It should be noted, however, that for a lot of spaces the monoid of continuous self-maps possesses strong enough properties to determine properties of the clone of continuous maps. An obvious example of this was already presented above: for completely regular T_1 spaces containing an arc the monoid of continuous maps determines the topology of the space. Hence, in the class of completely regular T_1 spaces which contain an arc, isomorphism of monoids implies isomorphism of clones.

Another example can be found in [5], [6]. Consider a rigid space, i.e. one with only identical and constant continuous self-maps (for example, a Cook continuum [3]). Under the assumption that the space is Hausdorff, Herrlich proved that the only continuous operations on such a space are projections and constants. In [11], Taylor noted that Herrlich's proof is valid in any concrete category with constants as long as the object has at least 3 elements. Thus, the clone of a rigid space with at least 3 elements consists solely of projections and constants. On the other hand, being rigid is a first order property of the monoid of continuous self-maps. The monoid of a rigid space hence completely describes its clone. In contrast to the case of completely regular T_1 spaces containing an arc, the monoid of a rigid space is not strong enough to describe the topology of the space, yet it is strong enough to fully determine the space's clone.

As usual in set theory, we will assume any natural number n is the set of smaller numbers $\{0, 1, \dots, n-1\}$. For a topological space X, denote by $n \times X = nX$ the direct sum of n copies of X. The i-th copy of X in the sum nX is $\{i\} \times X$ for $i = 0, 1, \dots, n-1$.

The *coclone* of a topological space X is the clone of X in the category dual to that of topological spaces and continuous maps. Thus, the coclone $\mathrm{Coclo}(X)$ of the space X is the ω-sorted algebra

$$\langle C_n; S^n_m; \iota^n_i \rangle_{m,n\in\omega, i<n},$$

where

- C_n is the set of all continuous maps from X to nX;
- ι^n_i is the identical injection

$$\iota^n_i : X \hookrightarrow \{i\} \times X \subseteq nX$$

  for $i < n \in \omega$; and
- $S^n_m : nX \times (mX)^n \to mX$ is the composition operation which maps any $(n+1)$-tuple $(f; g_0, \dots, g_{n-1})$ to the continuous function $F : X \to mX$ defined as follows:

$$F(x) = g_i(f(x)) \quad \text{if} \quad f(x) \in \{i\} \times X.$$

The first sort $\langle C_1; S^1_1; \iota^1_0 \rangle$ of $\mathrm{Coclo}(X)$ is again a monoid — one which is dual to $\mathrm{Mon}(X)$. Let us denote this monoid by $\mathrm{Mon}^d(X)$.

In this paper, we show that results similar to those obtained by Sichler and Trnková cannot be proved for coclones of topological spaces. In Section 3, we prove that for non-indiscrete spaces, isomorphism of monoids implies isomorphism of coclones. Similarly, in Section 4 we prove that elementary equivalence of monoids of non-indiscrete spaces implies elementary equivalence of their coclones.

Note that a map $F : \mathrm{Mon}(X) \to \mathrm{Mon}(Y)$ is an isomorphism of $\mathrm{Mon}(X)$ and $\mathrm{Mon}(Y)$ if and only if it is an isomorphism of $\mathrm{Mon}^d(X)$ and $\mathrm{Mon}^d(Y)$. Also, the bijection $\varphi \mapsto \varphi^d$ of the set of all monoid-theoretic first-order formulas onto itself which turns the order of multiplication around has the property that $\mathrm{Mon}(X) \models \varphi$ if and only if $\mathrm{Mon}^d(X) \models \varphi^d$. In light of these facts, in the later sections we will not distinguish between $\mathrm{Mon}(X)$ and $\mathrm{Mon}^d(X)$ and will thus identify the first sort of $\mathrm{Coclo}(X)$ with $\mathrm{Mon}(X)$.

2. Preliminaries

In Section 4, we will construct transformations of first order formulas of the theory of (co)clones. For this reason, we need to exactly specify the first order language of this theory that we will be using.

As we are talking about an ω-sorted theory, we need infinitely many variables of each sort $n \in \omega$. We will take symbols f_i^n to be the n-th sort variables of the theory of coclones, $i \in \omega$. In each sort n we have n constants: $\iota_0^n, \dots, \iota_{n-1}^n$. For any pair $m, n \in \omega$ of finite ordinals, S_m^n is a heterogeneous operation symbol of type $n \times m^n \to m$. And, of course, in addition to these the alphabet of the theory of coclones contains parentheses, comma, equation sign $=$, logical operations $\&$ and $\neg$ and the universal quantifier $\forall$. All other logical operations, as well as the existence quantifier, are defined in terms of $\&$, $\neg$ and $\forall$ in the traditional way.

Terms in the first order theory of coclones are constructed by the following scheme:

- each variable f_i^n and each constant ι_i^n is an n-th sort term;
- if t is an n-th sort term and $t_0, \dots, t_{n-1}$ are m-th sort terms, then the sequence $S_m^n(t, t_0, \dots, t_{n-1})$ is an m-th sort term.

Note that we usually write $S_m^n(t; t_0, \dots, t_{n-1})$ instead of $S_m^n(t, t_0, \dots, t_{n-1})$. The semicolon has neither syntactic nor semantical meaning here — we use it merely for better visual separation of the two parts of the composition.

Formulas of the first order theory of coclones are again constructed according to the usual scheme:

- if t and t' are terms of the same sort, $t = t'$ is a formula (an *elementary formula*);
- if φ and ψ are formulas, so are $(\varphi \ \& \ \psi)$ and $(\neg\varphi)$;
- if φ is a formula, so is $(\forall f_i^n)\varphi$.

And again as usual, we will omit parentheses in formulas if doing so does not introduce ambiguity in the meaning of the formula, and will add unnecessary parentheses if they improve readability.

The 1-st sort of the clone is simply a monoid. Similarly, the reduction of the above specified language to the 1-st sort is the language of the theory of monoids. Hence we will consider monoid-theoretical formulas to also be clone-theoretical formulas.

However, we will allow a richer set of variables for the first order language of the theory of monoids and will usually denote monoid-theoretic variables by small letters of the Latin alphabet. Also, we will write $g \circ f$ instead of $S_1^1(f, g)$.

Let us consider the following example of a monoid-theoretic formula:

$$\forall f (x \circ f = x). \tag{1}$$

Obviously, (1) claims that x is a left zero. Now, if we are considering a monoid of continuous self-maps of some topological space, left zeroes coincide with constant maps. Taking this into account, we denote formula (1) by $\mathrm{Const}(x)$. We will abbreviate the formula

$$\forall x \forall y \dots \forall z (\mathrm{Const}(x) \ \& \ \mathrm{Const}(y) \ \& \ \cdots \ \& \ \mathrm{Const}(z) \to \varphi)$$

as

$$(\forall\, x, y, \ldots, z \in \mathrm{Const})\ \varphi.$$

In the same way, we will abbreviate

$$\exists x \exists y \ldots \exists z (\mathrm{Const}(x)\ \ \&\ \ \mathrm{Const}(y)\ \ \&\ \ \cdots\ \ \&\ \ \mathrm{Const}(z)\ \ \&\ \ \varphi)$$

as

$$(\exists x, y, \ldots, z \in \mathrm{Const})\ \varphi.$$

We will also sometimes write $f(x)$ instead of $f \circ x$, in cases when x is assumed to be constant.

Lastly, to avoid symbol overloading, we denote formula equality by $\equiv$.

Let us consider another example of a monoid-theoretical formula:

$$\begin{aligned} \star\,\mathrm{Discr} \equiv \forall\, x, y \in \mathrm{Const}\ \exists f (f(x) = y\ \ \& \\ \forall\, z \in \mathrm{Const}\ (z \neq x \to f(z) = x)). \end{aligned} \tag{2}$$

Note that this formula holds for both discrete and indiscrete spaces.

Conversely, let X be a non-indiscrete space for which $\mathrm{Mon}(X) \models \star\,\mathrm{Discr}$. As X is non-indiscrete, it contains a non-empty proper open subset $U \subsetneq X$. Pick any $x \notin U$ and any $y \in U$; then there is a continuous map $f : X \to X$ which maps x into U and the complement outside of U. Thus, the singleton $\{x\}$ is open as the preimage of the open set U under the continuous map f. Applying the same reasoning to $\{x\}$ instead of U, we get that every other singleton in X is also open. Thus, X is discrete.

To summarize, the monoid-theoretical formula (2) characterizes discrete and indiscrete spaces in the class of all topological spaces.

Note that discrete and indiscrete topologies on a set with at least two elements have the same monoid of continuous self-maps and are thus indistinguishable by their monoids. Coclones of these spaces, on the other hand, are different. The coclone of the indiscrete topological space satisfies the following formula:

$$\mathrm{Conn} \equiv \forall\, f_0^2 \exists f_0^1 \{ f_0^2 = S_2^1(f_0^1; \iota_0^2) \vee f_0^2 = S_2^1(f_0^1; \iota_1^2) \}. \tag{3}$$

The coclone of a discrete non-trivial space clearly does not satisfy this formula. Thus, the formula $\star\,\mathrm{Discr}\ \ \&\ \ \mathrm{Conn}$ identifies indiscrete spaces, and $\star\,\mathrm{Discr}\ \ \&\ \ \neg\,\mathrm{Conn}$ identifies discrete non-trivial spaces in the first-order language of the theory of coclones. The monoid of continuous self-maps is only able to identify spaces with discrete and indiscrete topologies but is not able to distinguish between those.

A subset of a topological space which is both open and closed is called *clopen*. A partition of a topological space into n clopen subsets is called a *clopen n-partition*. A central aspect of our paper is the representation of clopen partitions of a space X in $\mathrm{Mon}(X)$. As the only objects we possess in $\mathrm{Mon}(X)$ are maps, we have to represent clopen partitions as collections of continuous self-maps.

Let C be a non-empty clopen subset of a topological space X. Choose an arbitrary point $c \in C$. The map $p_{C,c}$ defined by the following formula is obviously continuous:

$$p_{C,c}(x) = \begin{cases} x; & \text{if } x \in C; \\ c; & \text{if } x \notin C. \end{cases} \tag{4}$$

It is also a projection:

$$p_{C,c} \circ p_{C,c} = p_{C,c}.$$

The set C can be determined as the set of fixed points of $p_{C,c}$.

On the other hand, every continuous map p identifies a subset

$$C_p = \{x \in X;\ p(x) = x\}.$$

If p is a projection, then $C_p = \Im(p)$ is non-empty.

Continuous projections are objects which can be described in the first-order theory of monoids (and clones). Taking this into account, we will represent non-empty clopen sets by the projections as defined by (4). Empty sets are, curiously, more tricky to deal with. In the case of disconnected spaces, we will represent them by continuous maps with no fixed points. Note, however, that the sets of fixed points in general do not need to be open or closed.

Let X be a non-indiscrete and non-trivial space; then there exist points $a, b \in X$ such that a has a neighborhood which does not contain b. If $A \subseteq X$ is a clopen set, the function which maps A to a and the complement to b is continuous. So is the function mapping A to b and the complement to a. Conversely, if there are continuous functions mapping A and its complement to a and b and vice-versa, then A is necessarily a clopen set. This condition can be expressed in the first order theory of monoids. The following formula is equivalent to the set of fixed points of p being a clopen set:

$$\begin{aligned} \text{Clopen}(p) \equiv & \forall\, x, y \in \text{Const}\ \ \exists f\ \ \forall\, z \in \text{Const} \\ & \qquad \{(p(z) = z \to f(z) = x)\ \ \&\ \ (p(z) \neq z \to f(z) = y)\}. \end{aligned} \tag{5}$$

It is easily seen that the requirement of the space X being non-trivial can be relaxed. Really, the only self-map of a trivial space satisfies (5), and its set of fixed points is clopen. Thus, the formula Clopen describes maps which represent clopen sets for any non-indiscrete space.

Let p and q be two continuous maps. The sets they represent are complementary if and only if they satisfy the following conditions:

- p and q do not have common fixed points;
- each point is a fixed point for either p or q.

These conditions are described by the following monoid-theoretic formula:

$$\text{Compl}(p, q) \equiv \forall\, x \in \text{Const}\ (p(x) = x \leftrightarrow q(x) \neq x).$$

This formula can be easily generalized to describe an arbitrary partition of the set into finitely many sets:

$$\begin{aligned}(6)\qquad &\mathrm{Compl}(p_0,p_1,\dots,p_{n-1}) \equiv \\ &\qquad \forall x\in \mathrm{Const}\ \{(p_0(x)=x \vee \dots \vee p_{n-1}(x)=x)\ \ \& \\ &\qquad\qquad \mathop{\&}_{i\neq j}(p_i(x)=x \to p_j(x)\neq x)\}.\end{aligned}$$

Combining Compl and Clopen, we get a formula which describes maps representing a clopen n-partition for non-indiscrete spaces:

$$\begin{aligned}(7)\qquad \mathrm{Clopen\,Part}(p_0,p_1,\dots,p_{n-1}) &\equiv \mathrm{Compl}(p_0,\dots,p_{n-1})\ \ \& \\ &\qquad \mathrm{Clopen}(p_0)\ \ \&\ \ \cdots\ \ \&\ \ \mathrm{Clopen}(p_{n-1}).\end{aligned}$$

3. The isomorphism case

Theorem 1. *Suppose X and Y are non-indiscrete spaces. Further, suppose that* $\mathrm{Mon}(X)$ *is isomorphic to* $\mathrm{Mon}(Y)$. *Then* $\mathrm{Coclo}(X)$ *is isomorphic to* $\mathrm{Coclo}(Y)$.

Proof: Let $F:\mathrm{Mon}(X)\to\mathrm{Mon}(Y)$ be a monoid isomorphism. We will extend F to an isomorphism of coclones of X and Y.

Suppose $f : X \to nX = \{0,\dots,n-1\}\times X$ is a continuous function. As a function into the product of two sets, the function f is uniquely determined by its continuous components $f_n : X\to n$ and $f_X : X\to X$. The first of these is in its turn uniquely determined by the clopen n-partition $\{X_i;\ i=0,\dots,n-1\}$, where

$$X_i=\{x\in X;\ f_n(x)=i\}=\{x\in X;\ f(x)\in\{i\}\times X\}.$$

Let I be the set of those indices i for which X_i is not empty. For any collection of symbols $\sigma_0,\dots,\sigma_{n-1}$, let $\overrightarrow{\sigma_I}$ denote the sequence

$$\overrightarrow{\sigma_I}=(\sigma_i)_{i\in I}.$$

For $i\notin I$ we define $Y_i=\emptyset$. For $i\in I$, we define the set $Y_i\subseteq Y$ below.

As in the previous section, we represent X_i $(i\in I)$ by a continuous projection p_i. We know that the collection $\overrightarrow{p_I}$ satisfies the formula $\mathrm{Clopen\,Part}(\overrightarrow{p_I})$. As F is a monoid isomorphism, the images $\overrightarrow{F(p_I)}$ are also projections and also satisfy $\mathrm{Clopen\,Part}(\overrightarrow{F(p_I)})$. Thus, these projections represent sets $Y_i\subseteq Y$ for $i\in I$ which, together with the empty sets Y_i for $i\notin I$, form a clopen n-partition $Y_0,\dots,Y_{n-1}$ of Y.

We now denote $g_Y=F(f_X)$ and construct the continuous map $\widetilde{F}(f)=g$ as follows:

$$g(y)=(i,g_Y(y))\quad \text{for}\quad y\in Y_i.$$

We will show that $\widetilde{F}:\mathrm{Coclo}(X)\to\mathrm{Coclo}(Y)$ is a clone isomorphism.

First, note that $\widetilde{F}(f)$ does not depend on the choice of projections p_i. Indeed, F is a bijection of the set of left zeroes of $\mathrm{Mon}(X)$ onto that of $\mathrm{Mon}(Y)$, i.e. F

maps constants bijectively onto constants. Thus, is we identify constant maps and their values, $y \in Y_i$ if and only if $F(p_i) \circ y = y$, which in turn is equivalent to $p_i \circ F^{-1}(y) = F^{-1}(y)$, and the latter is the same as $F^{-1}(y) \in X_i$. So the sets Y_i do not depend on the choice of p_i and thus neither does $\widetilde{F}(f)$.

The map $\widetilde{F}$ is inverse to $\widetilde{F^{-1}}$. Indeed, $\widetilde{F}(f) = g$ is uniquely determined by the n-partition $Y_0, \ldots, Y_{n-1}$ and the map $g_Y : Y \to Y$. Each non-empty Y_i can be represented by the projection $F(p_i)$ which F^{-1} maps to p_i. The latter represents the set X_i. The function $\widetilde{F^{-1}}(g)$ is then given by the partition $X_0, \ldots, X_{n-1}$ and the function $F^{-1}(g_Y) = f_X$ and hence coincides with f. Thus the bijection of $\widetilde{F}$ is established.

Now, let $f : X \to nX$ and $g^0, \ldots, g^{n-1} : X \to mX$ be continuous maps. Denote $h = S^n_m(f; g^0, \ldots, g^{n-1})$. As above, let f_X be the second component of f and let $X_0, \ldots, X_{n-1}$ be the clopen partition associated with its first component. Similarly, for each g^i let g^i_X denote its second component and let $X^i_0, \ldots, X^i_{m-1}$ denote the partition corresponding to its first component. Denote

$$X^i_{i,j} = X_i \cap f_X^{-1}(X^i_j)$$

for $j = 0, 1, \ldots, m-1$. Evidently $X^i_{i,0}, \ldots, X^i_{i,m-1}$ is a clopen partition of X_i.

Note that the first component of h is given by the clopen partition

$$\bigcup_{i=0}^{n-1} X^i_{i,0}, \ldots, \bigcup_{i=0}^{n-1} X^i_{i,m-1}.$$

Denote $X^j = \bigcup_{i=0}^{n-1} X^i_{i,j}$. On each component X^j, the second component of h equals

$$h_X \mid X^j = \bigcup_{i=0}^{n-1} g^i_X \circ f_X | X^i_{i,j}. \tag{8}$$

Suppose each non-empty X_i is represented by the projection p_i, each non-empty $X^i_{i,j}$ is represented by the projection $p^i_{i,j}$, and each non-empty X^j is represented by p^j. Evidently $p^j \circ p^i_{i,j} = p^i_{i,j}$, thus also $F(p^j) \circ F(p^i_{i,j}) = F(p^i_{i,j})$. Hence, if we denote by $Y^i_{i,j}$ and Y^j the clopen sets represented by $F(p^i_{i,j})$ and $F(p^j)$, respectively, then $Y^i_{i,j} \subseteq Y^j$. Similarly, if Y_i is the set represented by the projection $F(p_i)$, we get $Y^i_{i,j} \subseteq Y_i$. Now, as

$$\{X^i_{i,j}; i = 0, \ldots, n-1,\ j = 0, \ldots, m-1\}$$

is a clopen partition of X, the same is true of

$$\{Y^i_{i,j}; i = 0, \ldots, n-1,\ j = 0, \ldots, m-1\}$$

and Y (here we take $Y^i_{i,j} = \emptyset$ if $X^i_{i,j} = \emptyset$).

It is now sufficient to prove that $\widetilde{F}(h)$ and $S^n_m(\widetilde{F}(f);\widetilde{F}(g^0),\dots,\widetilde{F}(g^{n-1}))$ coincide on each nonempty set $Y^i_{i,j}$. According to the definition,

$$\widetilde{F}(h)(y)=(j,F(h_X)(y))\quad \text{for}\ \ y\in Y^i_{i,j}.$$

Taking into account the equation (8) and the fact that $y\in Y^i_{i,j}$ is equivalent to $F(p^i_{i,j})(y)=y$, for $y\in Y^i_{i,j}$ we get

$$\begin{aligned}\widetilde{F}(h)(y)&=(j,F(h_X)(F(p^i_{i,j})(y)))\\&=(j,F(h_X\circ p^i_{i,j})(y))\\&=(j,F(g^i_X\circ f_X\circ p^i_{i,j})(y))\\&=(j,F(g^i_X\circ f_X)(y)).\end{aligned}\tag{9}$$

On the other hand, as $Y^i_{i,j}\subseteq Y_i$, we have $\widetilde{F}(f)(y)=(i,F(f_X)(y))$ for $y\in Y^i_{i,j}$. Let p^i_j be a projection representing X^i_j and let Y^i_j be the set represented by $F(p^i_j)$. As $f_X(X^i_{i,j})\subseteq X^i_j$, we get $F(f_X)(Y^i_{i,j})\subseteq Y^i_j$ and again, according to the definition of $\widetilde{F}$,

$$\widetilde{F}(g^i)(F(f_X)(y))=(j,F(g^i_X)(F(f_X)(y))).\tag{10}$$

Comparing (9) and (10) completes the proof. □

Remark. Suppose the isomorphism F in Theorem 1 is identical, i.e. X and Y are defined on the same set and their monoids of continuous self-maps coincide. Then the clone isomorphism $\widetilde{F}$ constructed in the above proof is again identical. In other words, for non-indiscrete spaces monoid coincidence implies coclone coincidence.

4. The elementary equivalence case

We have chosen to represent clopen sets as sets of fixed points of continuous maps. Thus, we cannot represent empty sets in spaces with the fixed point property. Such spaces are necessarily connected; hence, in this section, we will consider the cases of connected and disconnected spaces separately.

First, for a non-indiscrete space, its connectedness can be expressed by the following monoid-theoretical formula:

$$\text{Connected}\equiv\forall p\forall q(p\circ p=p\ \ \&\ \ q\circ q=q\to\neg\,\text{Clopen Part}(p,q)).\tag{11}$$

For connected spaces, the coclone differs very little from the monoid. Each continuous map $F:X\to nX$ is given by an index $i<n$ and a continuous map $f:X\to X$, for which $F(x)=(i,f(x))$ for every $x\in X$. If $F:X\to nX$ is given by the pair (i,f) and $G_0,\dots,G_{n-1}:X\to mX$ are given by the pairs $(j_0,g_0),\dots,(j_{n-1},g_{n-1})$, respectively, then the composition $S^n_m(F;G_0,\dots,G_{n-1})$ is given by the pair $(j_i,g_i\circ f)$.

The above gives an idea for translating first order properties of coclones of connected topological spaces into first order properties of their monoids. For any variable f_i^n in the language of clones, let us have n new "shadow" variables $f_{i,0}^n, \dots . f_{i,n-1}^n$. Let $\varphi(f_i^n)$ be an arbitrary first order clone-theoretic formula. The variable f_i^n may or may not actually occur in φ. By $\overline{\varphi}(f_i^n)$ we denote the formula

$$\overline{\varphi}(f_i^n) = \mathop{\&}_{j<n} \varphi(f_{i,j}^n). \tag{12}$$

For every clone-theoretic formula φ in the original language we define another clone-theoretic formula φ' in the language enriched with shadow variables. The formula φ' is constructed by recursion on the complexity of the formula φ:

- if φ is an elementary formula, then $\varphi' \equiv \varphi$;
- if $\varphi \equiv \psi \ \& \ \theta$, then $\varphi' \equiv \psi' \ \& \ \theta'$;
- if $\varphi \equiv \neg\psi$, then $\varphi' \equiv \neg\psi'$;
- if $\varphi \equiv \forall f_i^n(\psi(f_i^n))$, then $\varphi' \equiv \forall f_i^n(\overline{\psi}(f_i^n))$.

Note that if φ is a closed formula, then φ' only contains non-shadow variables as parts of quantifiers (e.g. $\forall f_i^n$). In other words, each variable occurring in a term is annotated with an index. We can now translate clone-theoretic formulas having the latter property into the first order language of monoids. To do that, we first define a monoid-theoretic term and an integer index for every clone-theoretic term having no non-shadow variables. For any variable f_i^n, pick a monoid-theoretic variable $f_{n,i}$. Clone-theoretic terms are translated according to the following rules:

- if $t \equiv \iota_j^n$, take $t^{Mc} \equiv 1$ and $i(t) = j$;
- if $t \equiv f_{i,j}^n$, take $t^{Mc} \equiv f_{n,i}$ and $i(t) = j$;
- if $t \equiv S_m^n(\tau; t_0, \dots, t_{n-1})$, take $t^{Mc} \equiv t_{i(\tau)}^{Mc} \circ \tau^{Mc}$ and $i(t) = i(t_{i(\tau)})$.

The translation of formulas φ with no non-shadow variables in terms is constructed as follows:

- if φ is the elementary formula $t = \tau$ with $i(t) = i(\tau)$, we take $\varphi^{Mc} \equiv t^{Mc} = \tau^{Mc}$;
- if φ is the elementary formula $t = \tau$ with $i(t) \neq i(\tau)$, we take $\varphi^{Mc} \equiv 1 \neq 1$;
- if $\varphi \equiv \psi \ \& \ \theta$, we take $\varphi^{Mc} \equiv \psi^{Mc} \ \& \ \theta^{Mc}$;
- if $\varphi \equiv \neg\psi$, we take $\varphi^{Mc} \equiv \neg\psi^{Mc}$;
- if $\varphi \equiv \forall f_i^n(\psi)$, take $\varphi^{Mc} \equiv \forall f_{n,i}(\psi^{Mc})$.

The composition $\varphi \mapsto \varphi' \mapsto (\varphi')^{Mc}$ achieves the sought translation of first order properties of coclones to those of monoids for connected spaces. We will henceforth write φ^{Mc} instead of $(\varphi')^{Mc}$.

Theorem 2. *Let X be a connected non-indiscrete space and let φ be an arbitrary closed clone-theoretic formula. Then* $\mathrm{Coclo}(X) \models \varphi$ *if and only if* $\mathrm{Mon}(X) \models \varphi^{Mc}$.

Proof: The proof closely follows the construction of the mapping $\varphi \mapsto \varphi^{Mc}$.

Suppose X is a connected non-indiscrete space. Let

$$\varphi(f_{i_1}^{n_1}, \dots, f_{i_k}^{n_k})$$

be a clone-theoretic formula with free variables among $f_{i_1}^{n_1}, \dots, f_{i_k}^{n_k}$ and let $F_1 : X \to n_1 X$, …, $F_k : X \to n_k X$ be any continuous maps. Suppose F_s is given by the index j_s and the continuous map $f_s : X \to X$, i.e. $F_s(x) = (j_s, f_s(x))$ for $x \in X$, $s = 1, \dots, k$. The formula

$$\varphi_I \equiv \varphi(f_{i_1, j_1}^{n_1}, \dots, f_{i_k, j_1}^{n_k})$$

only contains bound non-shadow variables. This implies that φ_I' only contains indexed variables in terms and hence $(\varphi_I')^{Mc}$ can be constructed. Again, we will write φ_I^{Mc} instead of $(\varphi_I')^{Mc}$. We will prove that $\mathrm{Coclo}(X) \models \varphi(F_1, \dots, F_k)$ if and only if $\mathrm{Mon}(X) \models \varphi_I^{Mc}(f_1, \dots, f_k)$.

The proof of the latter claim is constructed by induction on the complexity of φ.

If φ is the elementary formula $t = \tau$, then $\mathrm{Coclo}(X) \models \varphi(F_1, \dots, F_k)$ if and only if $t(F_1, \dots, F_k)$ and $\tau(F_1, \dots, F_k)$ are represented by the same index and continuous map $X \to X$. Due to the way the translation of terms is done, this is true if and only if t_I^{Mc} and τ_I^{Mc} evaluate to the same index and map on the k-tuple $(f_1, \dots, f_k)$, that is $\mathrm{Mon}(X) \models \varphi_I^{Mc}(f_1, \dots, f_k)$.

If $\varphi \equiv \psi \;\;\&\;\; \theta$ or $\varphi \equiv \neg\psi$, and the claim is true for ψ and θ, then it is also clearly true for φ.

If $\varphi \equiv \forall\, f_i^n(\psi(f_i^n, f_{i_1}^{n_1}, \dots, f_{i_k}^{n_k}))$, then $\mathrm{Coclo}(X) \models \varphi(F_1, \dots, F_k)$ if and only if for every continuous $F : X \to nX$ $\mathrm{Coclo}(X) \models \psi(F, F_1, \dots, F_k)$. According to the induction hypothesis, the latter means that $\mathrm{Mon}(X) \models \psi_I^{Mc}(f, f_1, \dots, f_k)$ for each index j chosen for f_i^n and for every continuous $f : X \to X$, or, equivalently, $\mathrm{Mon}(X) \models (\overline{\psi}(f_i^n))_I^{Mc}(f, f_1, \dots, f_k)$. The latter is equivalent to $\mathrm{Mon}(X) \models \varphi_I^{Mc}(f_1, \dots, f_k)$. This completes the proof of the claim.

Now, if φ is a closed formula, then φ_I is identical with φ, hence φ_I^{Mc} is identical with φ^{Mc} and the above claim reduces to $\mathrm{Coclo}(X) \models \varphi$ if and only if $\mathrm{Mon}(X) \models \varphi^{Mc}$. □

We will now do a similar construction for disconnected spaces. Disconnected spaces necessarily have continuous self-maps with no fixed points, which will have the task of representing empty sets in clopen partitions.

In Section 3, for every continuous map $f : X \to nX$ we constructed a clopen partition X_0, …, X_{n-1} of the space X and a continuous self-map f_X. We have a monoid-theoretic formula (7) which describes clopen partitions. Hence, for disconnected spaces, we can represent a continuous map $f : X \to nX$ by a collection of n maps which satisfy Clopen Part, and another map which represents the second component of f.

Every clone-theoretic formula is equivalent to one with simple elementary formulas, where a simple elementary formula is one of the form

$$\tau_1 = \tau_2$$

or

$$t' = S^n_m(t; t_0, \ldots, t_{n-1})$$

with τ_1, τ_2 being variables or constants and $t', t, t_0, \ldots, t_{n-1}$ being variables. We will construct a translation of clone-theoretical formulas which only contain simple elementary subformulas to the language of the theory of monoids.

For each variable f^n_i take $n+1$ distinct monoid-theoretic variables $p_{n,i,0}, \ldots, p_{n,i,n-1}, f_{n,i}$.

First, we construct a monoid-theoretic formula φ^{Md} for every simple elementary formula φ.

Let φ denote the formula $\iota^n_i = \iota^n_j$. If $i = j$, put $\varphi^{Md} \equiv 1 = 1$; if $i \neq j$, put $\varphi^{Md} \equiv 1 \neq 1$. Let φ be the formula $\iota^n_i = f^n_j$ or $f^n_j = \iota^n_i$. In this case, put $\varphi^{Md} \equiv p_{n,j,i} = 1 \ \& \ f_{n,j} = 1$. For $\varphi \equiv f^n_i = f^n_j$, put

$$\varphi^{Md} \equiv \forall x \in \mathrm{Const} \left\{ \mathop{\&}_{l=0,\ldots,n-1} [p_{n,i,l}(x) = x \to p_{n,j,l}(x) = x] \right\} \ \& \ f_{n,i} = f_{n,j}. \tag{13}$$

Now, let $\varphi \equiv f^m_j = S^n_m(f^n_i; f^m_{j_0}, \ldots, f^m_{j_{n-1}})$. In this case, put

$$\varphi^{Md} \equiv \forall x \in \mathrm{Const} \left\{ \mathop{\&}_{\substack{k=0,\ldots,m-1\\ l=0,\ldots,n-1}} [p_{m,j,k}(x) = x \ \& \ p_{n,i,l}(x) = x \to p_{m,j_l,k}(f_{n,i}(x)) = f_{n,i}(x) \ \& \ f_{m,j_l}(f_{n,i}(x)) = f_{m,j}(x)] \right\}. \tag{14}$$

By recursion on the complexity of φ, we extend the above construction to arbitrary clone-theoretic formulas with only simple elementary subformulas. If $\varphi \equiv \psi \ \& \ \theta$, put $\varphi^{Md} \equiv \psi^{Md} \ \& \ \theta^{Md}$. If $\varphi \equiv \neg\psi$, put $\varphi^{Md} \equiv \neg\psi^{Md}$. If $\varphi \equiv \forall f^n_i(\psi)$, put

$$\varphi^{Md} \equiv \forall p_{n,i,0} \ldots \forall p_{n,i,n-1} \{\mathrm{Clopen\,Part}(p_{n,i,0}, \ldots, p_{n,i,n-1}) \to \forall f_{n,i}(\psi^{Md})\}. \tag{15}$$

The following theorem holds for the mapping $\varphi \mapsto \varphi^{Md}$:

Theorem 3. *Let X be a disconnected space and let*

$$\varphi(f^{n_1}_{i_1}, \ldots, f^{n_k}_{i_k})$$

be a clone-theoretic formula containing only simple elementary subformulas, with free variables among $f_{i_1}^{n_1},\dots,f_{i_k}^{n_k}$. Evidently, the free variables of φ^{Md} are then among $p_{n_1,i_1,i},\dots,p_{n_k,i_k,i}$, $f_{n_1,i_1},\dots,f_{n_k,i_k}$.

Suppose $F_1 : X \to n_1X$, …, $F_k : X \to n_kX$ are continuous maps with their first components given by projections $P_{n_1,i_1,i},\dots,P_{n_k,i_k,i}$ and second components $F_{n_1,i_1},\dots,F_{n_k,i_k}$. Then

$$\mathrm{Coclo}(X) \models \varphi(F_1,\dots,F_k)$$

if and only if $\mathrm{Mon}(X)$ *satisfies the formula φ^{Md}, with maps $P_{n_j,i_j,i}$ and F_{n_j,i_j} substituted for each of the variables $p_{n_j,i_j,i}$ and f_{n_j,i_j}, respectively.*

Proof: Again, we prove this theorem by induction on the complexity of the formula φ.

It is easy to see that the claim of the theorem holds for simple elementary formulas. The same is true for formulas of type $\psi\ \&\ \theta$ and $\neg\psi$ when the claim of the theorem is known to be true for ψ and θ.

Let now φ be the formula $\forall f_i^n(\psi)$ and suppose the claim of the theorem is true for $\psi(f_i^n, f_{i_1}^{n_1},\dots,f_{i_k}^{n_k})$. Thus, for every continuous function $F : X \to nX$ given by projections $P_{n,i,0},\dots,P_{n,i,n-1}$ and a map $F_{n,i}$,

$$\mathrm{Coclo}(X) \models \psi(F, F_1,\dots,F_k) \tag{16}$$

if and only if

$$\mathrm{Mon}(X) \models \psi^{Md}(P_{n,i,0},\dots,P_{n,i,n-1},F_{n,i},\dots). \tag{17}$$

Now, $\mathrm{Coclo}(X) \models \varphi(F_1,\dots,F_k)$ if and only if (16) holds for any continuous $F : X \to nX$, which is the case if and only if (17) holds for any continuous $P_{n,i,0},\dots,P_{n,i,n-1}$ and $F_{n,i}$ such that $\mathrm{Clopen\,Part}(P_{n,i,0},\dots,P_{n,i,n-1})$ is satisfied. The latter is equivalent to $\mathrm{Mon}(X) \models \varphi^{Md}(\dots)$. □

We can easily extend the mapping $\varphi \mapsto \varphi^{Md}$ to arbitrary formulas. For any clone-theoretic formula φ we can choose a clone-theoretically equivalent formula φ' with only simple equivalent subformulas and then take $\varphi^{Md} \equiv (\varphi')^{Md}$. We get an easy consequence of Theorem 3 for this extension:

Consequence. If X is a disconnected space and if φ is a closed formula, then $\mathrm{Coclo}(X) \models \varphi$ if and only if $\mathrm{Mon}(X) \models \varphi^{Md}$.

We can now prove the main theorem of this paper:

Theorem 4. *There is a mapping $\varphi \mapsto \varphi^M$ of first order formulas of clones to those of monoids which satisfies the following condition: for any non-indiscrete topological space X and any closed formula φ of the theory of clones,*

$$\mathrm{Coclo}(X) \models \varphi$$

if and only if

$$\mathrm{Mon}(X) \models \varphi^M.$$

Proof: Simply take

$$\varphi^M \equiv (\text{Connected} \to \varphi^{Mc}) \ \ \& \ \ (\neg\,\text{Connected} \to \varphi^{Md}).$$

Theorems 2 and 3 complete the proof. □

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