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# Further remarks on KC and related spaces

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*Abstract.* A topological space is KC when every compact set is closed and SC when every convergent sequence together with its limit is closed. We present a complete description of KC-closed, SC-closed and SC minimal spaces. We also discuss the behaviour of the finite derived set property in these classes.

*Keywords:* compact space, KC space, SC space, minimal KC space, minimal SC space, KC-closed space, SC-closed space, sequentially compact space, finite derived set property, wD property

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## 0. Introduction

KC spaces are those in which compact sets are closed. SC spaces are those in which convergent sequences together with their limits are closed

$$T_2 \to KC \to SC \to T_1.$$

In the last years, SC and much more KC spaces have been investigated in quite a few papers (see for instance [1], [3], [4]). The class of KC spaces is in a sense a nice enlargement of the class of $T_2$ spaces and in some cases it allows us to obtain better results. For instance, in [9] it is shown that the smallest cardinality of a countably compact KC space which is not sequentially compact is exactly $\mathfrak{h}$, while the same question for $T_2$ spaces is still rather undetermined. If we let $\mu$ to be the smallest cardinality of a countably compact $T_2$ space which is not sequentially compact, then we only know that $\mathfrak{s} \leq \mu \leq \mathfrak{c}$.

Various problems posed in [4] have recently been solved in [7]. Among them, there is the construction of a compact KC space in which each non-empty open set is dense and the construction of a $T_2$ space which cannot be embedded in any compact KC space. Another non-trivial contribution is the final solution of an old problem attributed to Larson on the nature of a minimal KC space [8]. The result in [8] asserts that a space is minimal KC if and only if it is compact KC. Parallel to this, in [7] it was also established that a minimal SC space is sequentially compact, but this is not a characterization. On the other hand, nothing is known in the literature about KC- or SC-closed spaces.

The main goal of this short paper is just to fill this gap, by giving precise characterizations of KC-closed, SC-closed and minimal SC spaces. We complete the paper with some remarks about the finite derived set property in the class of KC and SC spaces.

## 1. KC-closed, SC-closed and minimal SC

A space is KC-closed (resp. SC-closed) if it is closed in every KC space (resp. SC space) in which it is embedded.

A space is minimal SC if it does not have any proper coarser SC topology.

**Theorem 1.1.** *If $X$ is a non-compact KC space, then there exists a simple KC estension $Y = X \cup \{p\}$.*

Proof: We distinguish between two cases.

*Case* 1. $X$ is locally compact. Let $Y = X \cup \{p\}$ be the one-point compactification of $X$, i.e. $X$ is an open subspace of $Y$ and a local base at $p$ in $Y$ consists of the sets $\{Y \setminus K : K \text{ is a compact subset of } X\}$. It is evident that $X$ is dense in $Y$. It remains to check that $Y$ is a KC space. To this end, let $C$ be a compact subset of $Y$. If $p \notin C$, then $C$ is a compact subset of $X$ and so, according to the hypothesis, it is closed in $X$. Therefore, every point $x \in X \setminus C$ has a neighbourhood missing $C$ and the same holds for $p$ by construction. So, $C$ is closed in $Y$. Now, assume $p \in C$ and fix $x \notin C$. Let $U$ be a compact neighbourhood of $x$ in $X$. $U$ is closed in $X$ and $p$ is not in the closure of $U$. Hence, $U$ is closed in $Y$ and consequently $U \cap C$ is a compact subset of $X$. As $U \cap C$ is actually closed in $X$, we see that $U \setminus C$ turns out to be a neighbourhood of $x$ missing $C$. Thus, $C$ is closed in $Y$.

*Case* 2. There is a point $q \in X$ without compact neighbourhoods. Fix a point $p \notin X$ and define a topology on $Y = X \cup \{p\}$ by declaring $X$ an open subspace of $Y$ and by giving to $p$, as a local base in $Y$, the family $\{\{p\} \cup U \setminus K : U \text{ is a neighbourhood of } q \text{ and } K \text{ is a compact subset of } X\}$. It is evident that $X$ is dense in $Y$. To check that $Y$ is KC, fix a compact set $C \subseteq Y$. If $p \notin C$, then $C$ is actually a compact subset of $X$ and hence a closed subset of $X$. As, by construction the set $\{p\} \cup X \setminus C$ is a neighbourhood of $p$ missing $C$, we see that $C$ is actually closed in $Y$. Now, we assume $p \in C$. We are going to show that the set $C \setminus \{p\}$ is compact in $X$. Then, by the forgoing case, $C \setminus \{p\}$ will be closed in $X$ and $C$ in turn will be closed in $Y$. Let $\mathscr{V}$ be an open cover of $C \setminus \{p\}$ in $X$ and let $\mathscr{V}'$ be the subset of $\mathscr{V}$ consisting of those elements containing $q$. As the family $\{\mathscr{V} \setminus \mathscr{V}' \cup \{\{p\} \cup V : V \in \mathscr{V}'\}\}$ is an open cover of $C$ in $Y$, it contains a finite subcover $\mathscr{W}$. It is clear that the family $\mathscr{W} \setminus \mathscr{V}' \cup \{V : \{p\} \cup V \in \mathscr{W}\}$ is a finite subcollection of $\mathscr{V}$ which covers $C \setminus \{p\}$. This verifies the compactness of $C \setminus \{p\}$ and we are done. □

**Corollary 1.2.** *A space is KC-closed if and only if it is compact KC.*

Proof: It is clear that a compact space embeds as a closed subspace in a KC space. The converse follows from Theorem 1.1. □

**Theorem 1.3.** *A space is SC-closed if and only if it is the union of finitely many convergent sequences together with their limit points.*

Proof: Sufficiency follows directly from the definition of SC space. To prove necessity, let $X$ be an SC-closed space. For every $a \in X$, we will denote by $\tau(a)$ the collection of all open neighbourhoods of $a$.

*Step* 1. $X$ is sequentially compact. By contradiction, assume that there is a countable infinite set $A \subseteq X$ with no non-trivial convergent subsequences and let $\mathscr{U}$ be a free ultrafilter on $A$. Take a point $p \notin X$ and topologize $Y = X \cup \{p\}$ in such a way that $X$ is an open subspace of $Y$ and a local base at $p$ in $Y$ consists of the sets $\{\{p\} \cup V : V$ open in $X$ and $V \cap A \in \mathscr{U}\}$.

We claim that $Y$ is SC. To check this, it is enough to verify that if $S \subseteq X$ is a sequence converging to $x$ in $Y$ then $S \cup \{x\}$ is closed in $Y$. If $x \neq p$, then $S \cup \{x\}$ is closed in $X$. Moreover, as $A$ does not contain convergent subsequences, the set $S \cap A$ must be finite and so $A \setminus S \in \mathscr{U}$. This implies that $\{p\} \cup X \setminus (S \cup \{x\})$ is a neighbourhood of $p$ missing $S \cup \{x\}$ and thus $S \cup \{x\}$ is closed in $Y$ as well. The proof of the claim will be completed by showing that $x = p$ cannot occur. This is clear if some infinite $S' \subseteq S$ converges to some $a \in A$, as in this case $\{p\} \cup X \setminus (S' \cup \{a\})$ will be a neighbourhood of $p$ missing infinitely many points of $S$. To deal with the remaining case, put $A = \{a_n : n < \omega\}$ and let $V_0 \in \tau(a_0)$ be such that the set $S_0 = S \setminus V_0$ is infinite. Continuing by induction, let $V_{n+1} \in \tau(a_{n+1})$ be such that the set $S_{n+1} = S_n \setminus V_{n+1}$ is infinite. Next, let $S' \subseteq S$ be an infinite set satisfying $S' \subseteq^* S_n$ for each $n$. No matter if $S \cap A$ is finite or not, as $\mathscr{U}$ is an ultrafilter on $A$, there exists some $U \in \mathscr{U}$ such that $S'' = S' \setminus U$ is infinite. By construction, $V_n \cap S'$ is finite for each $n$ and therefore the set $\{p\} \cup \bigcup\{V_n \setminus S'' : a_n \in U\}$ is a neighbourhood of $p$ missing the infinite set $S''$. The claim is now verified. As $X$ is dense in $Y$, the SC-closedness of $X$ gives Step 1.

*Step* 2. $X$ is the union of finitely many convergent sequences. Assume the contrary and topologize the set $Y = X \cup \{p\}$ in such a way that $X$ is an open subspace of $Y$ and a local base at $p$ consists of the sets $\{p\} \cup X \setminus \bigcup \mathscr{F}$, where $\mathscr{F}$ is a finite set of convergent sequences in $X$ together with their limit points. We claim that $Y$ is an SC space. Let $S \subseteq X$ be a sequence converging to $x$ in $Y$. If $x \neq p$, then $S \cup \{x\}$ is closed in $X$ and the set $Y \setminus (S \cup \{x\})$ is a neighbourhood of $p$ missing $S \cup \{x\}$. So $S \cup \{x\}$ is closed in $Y$. The proof of the claim will be completed by showing that $x = p$ cannot occur. Since by Step 1, $X$ is sequentially compact, there exists a sequence $S' \subseteq S$ converging to some $x \in X$. As $Y \setminus (S' \cup \{x\})$ is a neighbourhood of $p$ missing infinitely many points of $S$, we are done.

Again the SC-closedness of $X$ gives Step 2 and the proof of the theorem is complete. □

**Theorem 1.4.** *A space is minimal SC if and only if it is an SC space in which each proper closed set is the union of finitely many convergent sequences together with their limit points.*

Proof: If $X$ is minimal SC then, by [7, Theorem 3.8], it is sequentially compact. On the other hand, [1, Theorem 2.2] asserts that a sequentially compact SC space remains SC by throwing away all proper closed sets which are not a finite union of convergent sequences. Thus, a combination of the previous two facts gives the first part of the theorem.

For the converse, let $(X,\tau)$ be an SC space in which each proper closed set is the union of finitely many convergent sequences together with their limit points. If $\sigma$ is an SC topology weaker than $\tau$, then any convergent sequence in $\tau$ is also convergent in $\sigma$ and therefore such a sequence together with the limit point is closed in $\sigma$ (take into account that $(X,\sigma)$ is SC). This means that each closed set in $\tau$ is closed as well in $\sigma$ and we are done. □

**Lemma 1.5.** *A space in which each proper closed set is the union of finitely many convergent sequences together with their limit points is compact and sequentially compact.*

Proof: Since every non-empty open set has compact complement, the compactness of the space is clear. To check the sequential compactness, let $S$ be an infinite sequence and fix some point $x$ in the space. If $S$ converges to $x$ we stop, otherwise there is a open neighbourhood of $x$ missing an infinite set $S' \subseteq S$. As $S'$ is the union of convergent sequences, the requirement for the sequential compactness is fully satisfied. □

Adding the previous results to what we already know, we get the following complete and (perhaps surprising) antisymmetric picture:

(1) Compact $T_2 \to$ minimal $T_2 \to T_2$-closed;
(2) compact KC $=$ minimal KC $=$ KC-closed;
(3) compact SC $\leftarrow$ minimal SC $\leftarrow$ SC-closed.

None of the previous arrows is reversible.

## 2. The FDS property in the class of SC spaces

The small uncountable cardinals $\mathfrak{h}, \mathfrak{s}$, and $\mathfrak{t}$ play a major role below. The cardinal $\mathfrak{t}$ is the least cardinality of a complete tower on $\omega$. By a *complete tower* we mean a collection of sets well-ordered with respect to reverse almost containment ($A \leq B$ iff $B \setminus A$ is finite, written $A \subseteq^* B$) such that no infinite set is almost contained in every member of the collection. The cardinals $\mathfrak{s}$ and $\mathfrak{h}$ are defined with the help of the following concepts. A set $S$ is said to **split** a set $A$ if both $A \cap S$ and $A \setminus S$ are infinite. A *splitting family on* $\omega$ is a family of subsets of $\omega$ such that every infinite subset of $\omega$ is split by some member of the family. We will call a splitting family *a splitting tree* if any two members are either almost disjoint, or one is almost contained in the other; thus it is a tree by reverse almost inclusion.

The least cardinality of a splitting family is denoted $\mathfrak{s}$, while least height of a splitting tree is denoted $\mathfrak{h}$. It is easy to show that $\omega_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s} \leq \mathfrak{c}$ $(= 2^\omega)$. For more about the relationships of these cardinals see [13] and (except for $\mathfrak{h}$) [10].

The seminal paper on $\mathfrak{h}$ is [5], where it is also shown that $\mathfrak{h}$ is the smallest cardinal $\kappa$ such that there exists a splitting family $\mathscr{S} = \bigcup\{\mathscr{M}_\alpha : \alpha < \kappa\}$, where each $\mathscr{M}_\alpha$ is a MAD family.

The Novak (or Baire) number $\mathfrak{n}$ of $\omega^*(= \beta\omega \setminus \omega)$ is the smallest cardinality of a cover of $\omega^*$ by nowhere dense sets. A good reference for this cardinal is again [5]. Recall that $\max\{\mathfrak{t}^+, \mathfrak{h}\} \leq \mathfrak{n} \leq 2^{\mathfrak{c}}$ and the equality $\mathfrak{h} = \mathfrak{n}$ holds if and only if there is a splitting tree of height $\mathfrak{h}$ without long chains. A chain in a tree is long if its cardinality equals the height of the tree.

A space has the **Finite Derived Set** (FDS) property if every infinite set has an infinite subset with at most finitely many accumulation points.

After the introduction of this notion in [12], 2004, some work has been done to find conditions for its validity. It turned out that the FDS property is influenced by the sort of separation axioms we are assuming. Some non-trivial results are:

**Theorem A** ([6]). *The smallest cardinality of a Urysohn space without the FDS property is* $\mathfrak{c}$.

**Theorem A'** ([6]). *The smallest weight of an SC space without the FDS property is* $\mathfrak{s}$.

**Theorem B** ([6]). *A Hausdorff space of cardinality less than* $\mathfrak{s}$ *has the FDS property.*

**Theorem C** ([4]). *A KC space* $X$ *satisfying* $hL(X) < \mathfrak{t}$ *has the FDS property.*

**Theorem D** ([2]). *A compact KC space of cardinality less than* $2^{\mathfrak{t}}$ *has the FDS property.*

**Theorem E** ([6]). *A Lindelöf SC space of cardinality not exceeding* $\mathfrak{t}$ *has the FDS property.*

A very easy, but useful, fact is in the following:

**Proposition 2.1.** *A sequentially compact SC space has the FDS property.*

Theorems A and A' are obviously definitive, but we cannot say the same about Theorem B. Indeed, if we denote by $\mu$ the smallest cardinality of a Hausdorff space without the FDS property, then we may only assert that $\mathfrak{s} \leq \mu \leq \mathfrak{c}$ and that it is consistent to have $\mu < \mathfrak{c}$.

**Problem 2.1.** *Is* $\mu = \mathfrak{s}$ *true in ZFC?*

Because of Proposition 1, it turns out that any condition which forces a countably compact space to be sequentially compact can be in general adapted to have a theorem ensuring the validity of the FDS property. For instance, using some results in [9], we may formulate a definitive conclusion for KC or SC space analogous to Theorem A.

**Theorem 2.2.** *A SC space* $X$ *satisfying* $|X| < \mathfrak{h}$ *has the FDS property.*

Proof: Fix an infinite set $A \subseteq X$ and let $S = \{x_n : n < \omega\} \subseteq A$. Since $X$ is an SC space, if $S$ has a convergent subsequence then we are done. So, we assume that $S$ has no convergent subsequence. For any $x \in X$ let $\mathscr{A}_x$ be the collection of all $A \in [\omega]^\omega$ such that there exists an open neighbourhood $U$ of $x$ satisfying $x_n \notin U$ for each $n \in A$. Fix a maximal almost disjoint subcollection $\mathscr{B}_x \subset \mathscr{A}_x$. As we are assuming that $S$ does not have any subsequence converging to $x$, it follows that $\mathscr{B}_x$ is actually a MAD family on $\omega$. Since $|X| < \mathfrak{h}$, the collection $\{\mathscr{B}_x : x \in X\}$ is not splitting and so there exists a set $C \in [\omega]^\omega$ such that $C$ is almost contained in same member of $\mathscr{B}_x$ for each $x \in X$. According to the way we have chosen $\mathscr{B}_x$, this means that $x$ is not an accumulation point of the set $\{x_n : n \in C\}$. Thus, the set $\{x_n : n \in C\} \subseteq A$ does not have accumulation points and we are done. $\square$

The next example is a really minor modification of a construction presented in [9]. We repeat it in detail for the reader's convenience.

**Example 2.3.** *A KC space of cardinality $\mathfrak{h}$ which does not have the FDS property.*

Proof: Let $\mathbb{N}$ be the set of positive integers, defined in such a way as to be disjoint from the class of ordinals. Let $X$ have $\mathbb{N} \cup \mathfrak{h}$ as an underlying set. We will define the topology on $X$ with the help of a splitting tree $\mathscr{T} = \bigcup\{\mathscr{M}_\alpha : \alpha < \mathfrak{h}\}$, where each $\mathscr{M}_\alpha$ is an infinite MAD family on $\mathbb{N}$ and $\mathscr{M}_\alpha$ refines $\mathscr{M}_\mathbf{t}$ whenever $\mathbf{t} < \alpha$. Points of $\mathbb{N}$ are isolated. If $\alpha, \mathbf{t} \in \mathfrak{h} \cup \{-1\}$ let $(\mathbf{t}, \alpha] = \{\xi : \mathbf{t} < \xi \leq \alpha\}$. Let a base for the neighborhoods of $\alpha$ be all sets of the form

$$N(\alpha, \mathbf{t}, \mathscr{F}, F) = (\mathbf{t}, \alpha] \cup \mathbb{N} \setminus \left(\bigcup \mathscr{F} \cup F\right)$$

such that $\mathbf{t} < \alpha$, and $\mathscr{F}$ is a finite subcollection of $M_\alpha$ and $F$ is a finite subset of $\mathbb{N}$.

**Claim 1.** *This defines a topology.*

**Claim 2.** *$X$ is a KC space.*

Proof: We show that every compact subset of $X$ meets $\mathbb{N}$ in a finite set. Since the relative topology on $\mathfrak{h}$ is the usual (Hausdorff) order topology and $\mathfrak{h}$ is closed, Claim 2 will then follow.

Let $K$ be a compact subset of $X$. Then $K \cap \mathfrak{h}$ is compact in $\mathfrak{h}$, hence has a greatest element $\alpha$. Suppose $K \cap \mathbb{N}$ is infinite. Let $M \in \mathscr{M}_\alpha$ hit $K$; then $\{N(\alpha, -1, \{M\}, \emptyset)\} \cup \{\{n\} : n \in \mathbb{N}\}$ is an open cover of $K$ without a finite subcover, contradicting the compactness of $K$. $\square$

We conclude the proof of the example by showing that every infinite subset of $\mathbb{N}$ has infinitely many accumulation points in $X$.

Let $A$ be an infinite subset of $\mathbb{N}$. Since $\mathscr{T}$ is splitting, there is $\beta_0 < \mathfrak{h}$ such that at least two elements of $\mathscr{M}_{\beta_0}$, say $M$ and $M'$, hit $A$. Next, we may find $\beta_1 > \beta_0$ such that $\mathscr{M}_{\beta_1}$ splits both $M \cap A$ and $M' \cap A$. Continuing in this way, at stage $n$ we find $\beta_n < \mathfrak{h}$ such that $2^{n+1}$ elements of $M_{\beta_n}$ hit $A$. Now, letting

$\alpha = \sup\{\beta_n : n < \omega\}$, we see that infinitely many members of $\mathscr{M}_\alpha$ hit $A$. If $\alpha < \xi < \mathfrak{h}$, then infinitely many members of $\xi$ hit $A$, thus the closure of $A$ includes a terminal segment of $\mathfrak{h}$. □

**Theorem 2.4.** *The smallest cardinality of a KC (or SC) space without the FDS property is* $\mathfrak{h}$.

PROOF: The result follows from Theorem 2.2 and the previous example. □

Mimicking another result in [9], we can strengthen Theorem C as follows:

**Theorem 2.5.** *Let $X$ be an SC space. If every splitting tree has a chain of cardinality $hL(X)^+$, then $X$ has the FDS property.*

PROOF: In what follows, $S^\bullet$ denotes the set of all accumulation points of $S$. Note that $S^\bullet$ is closed in $X$ and that $S^\bullet \subseteq R^\bullet$ whenever $S$ is almost contained in $R$.

Let us assume by contradiction that there exists a set $A \in [X]^\omega$ such that each infinite subset of it has infinitely many accumulation points. As we are assuming that $X$ is an SC space, the set $A$ has no non-trivial convergent subsequence. This means that for any $B \in [A]^\omega$ and any $x \in \overline{B}$ there exists some open set $U_x$ such that $x \in U_x$ and $C = B \setminus U_x$ is infinite. Clearly, $\overline{C}$ is a proper subset of $\overline{B}$, and $C^\bullet$ is a proper subset of $B^\bullet$.

For $\alpha < \mathfrak{h}$ let us suppose to have already defined a collection $\{\mathscr{A}_\gamma : \gamma < \alpha\}$ of MAD families contained in $[A]^\omega$ satisfying:

> if $\beta < \gamma < \alpha$ then $\mathscr{A}_\gamma$ "strongly refines" $\mathscr{A}_\beta$, *i.e.*, each member $C \in \mathscr{A}_\gamma$ is almost contained in some $B \in \mathscr{A}_\beta$ and $C^\bullet$ is a proper subset of $B^\bullet$.

If $\alpha = t + 1$ and $\mathscr{A}_t$ has been defined, then for each $B \in \mathscr{A}_t$ we let $\mathscr{E}(B) \subset [B]^\omega$ be an almost disjoint family maximal with respect to the property that $C^\bullet$ is a proper subset of $B^\bullet$ for any $C \in \mathscr{E}(B)$. Put $\mathscr{A}_\alpha = \bigcup\{\mathscr{E}(B) : B \in \mathscr{A}_t$. Taking into account the properties of $A$, it is easy to check that $\mathscr{A}_\alpha$ is a MAD family on $A$.

If $\alpha$ is a limit ordinal then, in order to define $\mathscr{A}_\alpha$, observe first that, as $|\alpha| < \mathfrak{h}$, there exists an infinite subset $S$ of $A$ which is almost contained in some (unique) member of $\mathscr{A}_\gamma$ for each $\gamma < \alpha$. Let $\mathscr{S}$ be the collection of all such $S$ and let $\mathscr{A}_\alpha$ be a maximal almost disjoint family of members of $\mathscr{S}$. By the induction hypothesis, $\mathscr{A}_\alpha$ strongly refines all $\mathscr{A}_t$, $t < \alpha$. It is also a MAD family on $A$: for any $B \in [A]^\omega$ the trace of the tree $\{\mathscr{A}_\gamma : \gamma < \alpha\}$ on $B$ cannot be splitting and so there exist infinite subsets of $B$ in $\mathscr{S}$.

The tree $\bigcup\{\mathscr{A}_\alpha : \alpha < \mathfrak{h}\}$ has a chain $\mathscr{C}$ of cardinality $hL(X)^+$. As $hL(X)^+ \leq \mathfrak{h}$, this is obvious if the tree is not splitting and follows from our hypothesis in the other case. Then the family $\{C^\bullet : C \in \mathscr{C}\}$ is a strictly decreasing collection of closed sets, in contrast with the definition of $hL(X)$. □

Since a splitting tree has always a chain of length $\mathfrak{t}$, Theorem C is clearly a corollary of Theorem 2.5.

**Corollary 2.6** ($\mathfrak{h} < \mathfrak{n}$).

$$\mathfrak{h} = \min\{hL(X)\colon X \textit{ is an SC (or KC) space without the FDS property}\}.$$

Proof: The previous example provides a KC space $X$ without the FDS property satisfying $hL(X) = |X| = \mathfrak{h}$. On the other hand, the assumption $[\mathfrak{h} < \mathfrak{n}]$ is equivalent to say that every splitting tree has a chain of length $\mathfrak{h}$ and therefore we may apply Theorem 2.5 to conclude that every SC space $X$ satisfying $hL(X) < \mathfrak{h}$ has the FDS property. □

**Problem 2.2.** *Does an SC (or KC) space $X$ satisfying $hL(X) < \mathfrak{h}$ have the FDS property?*

In contrast with Theorem C, it seems difficult to weaken KC to SC in Theorem D. However, another result in [9] enables us to say something non-trivial for compact SC space. In fact, a simple application of Proposition 1 to Theorem 5 in [9] leads to:

**Theorem 2.7.** *A compact SC space $X$ satisfying $|X| < \mathfrak{n}$ has the FDS property.*

As there are models of ZFC where $2^{\mathfrak{t}} \leq \mathfrak{n}$, the above theorem provides a consistent positive answer to the following:

**Problem 2.3.** *Does a compact SC space $X$ such that $|X| < 2^{\mathfrak{t}}$ have the FDS property?*

By remaining in the class of KC spaces, Theorem D may be strengthened in another direction.

Recall that a space $X$ has property wD if for every infinite closed discrete set $A \subseteq X$ there exists an infinite discrete family $\mathscr{W}$ of open sets such that $|W \cap A| = 1$ for each $W \in \mathscr{W}$. A countably compact space has trivially property wD. Consequently, compact implies Lindelöf wD.

**Theorem 2.8.** *A KC Lindelöf wD space of cardinality less than $2^{\mathfrak{t}}$ has the FDS property.*

Proof: Let $X$ be a Lindelöf KC and wD space and assume that $X$ does not have the FDS property. So, we may fix a countable infinite set $A \subseteq X$ such that every infinite subset of $A$ has infinitely many accumulation points. For any $\alpha \in \mathfrak{t}$ and any $f \in {}^{\alpha}2$ we define an infinite set $A_f \subseteq A$ in such a way that:

(1) if $\beta < \alpha$ and $f \in {}^{\alpha}2$ then $A_f \subseteq^* A_{f\restriction\beta}$;

(2) if $f, g \in {}^{\alpha}2$ and $f \neq g$ then $A_f^{\bullet} \cap A_g^{\bullet} = \emptyset$.

Put $A_{\emptyset} = A$ and assume to have defined everything for each $\beta < \alpha$. If $\alpha$ is a limit ordinal and $f \in {}^{\alpha}2$, then take as $A_f$ any infinite pseudointersection of the family $\{A_{f\restriction\beta} : \beta < \alpha\}$. If $\alpha = \gamma + 1$, fix some $g \in {}^{\gamma}2$ and choose an accumulation point $x$ of the set $A_g$. As $A_g$ cannot be a sequence converging to $x$, there exists an open neighbourhood $U$ of $x$ such that $A_g \setminus U$ is infinite. We claim that the set $C = \overline{A_g \setminus U}$ is countably compact. Assume the contrary and let $D$ be an infinite closed discrete subset of $C$. The set $D$ is also closed in $X$ and therefore property wD ensures the existence of an infinite discrete family $\mathscr{W}$ of open sets satisfying $|W \cap D| = 1$ for each $W \in \mathscr{W}$. Now, the set $\bigcup \mathscr{W} \cap A_g$ is an infinite set with no accumulation point. This contradicts the choice of $A$ and the claim is proved. As

$X$ is Lindelöf, the set $C \cup A$, which is actually the union of a compact set with a countable set, is Lindelöf. Since $C \cup A$ is not closed in $X$, it cannot be compact and, being Lindelöf, it is indeed not countably compact. Let $B$ be an infinite closed discrete subset of $C \cup A$ and observe that the countable compactness of $C$ implies $B \cap C$ finite and moreover all the accumulation points of $B$ in $X$ are outside $C \cup A$. Now, to complete the induction it suffices to define $A_{g^\frown 0} = A_g \setminus U$ and $A_{g^\frown 1} = B \setminus C$. By the Lindelöfness of $X$, for any $f \in {}^{\mathfrak{t}}2$ we may pick a point $x_f \in \bigcap\{A^\bullet_{f\restriction\alpha} : \alpha \in \mathfrak{t}\}$. As the mapping $f \mapsto x_f$ is injective, we see that $|X| \geq 2^{\mathfrak{t}}$. □

Gryzlov's theorem that every compact $T_1$ space of countable pseudocharacter is of cardinality $\leq \mathfrak{c}$ [11] leads to a rather interesting consequence.

**Corollary 2.9** ($\mathfrak{c} < 2^{\mathfrak{t}}$). *A Lindelöf KC and wD space of countable pseudocharacter has the FDS property.*

Proof: It suffices to observe that the set $\overline{A}$ in the proof of Theorem 2.8 has cardinality $\leq \mathfrak{c}$ and by construction each $x_f$ belongs to $\overline{A}$. □

The above corollary can be viewed as a partial answer to the following problem, already raised in [4].

**Problem 2.4.** *Does a Lindelöf KC space of countable pseudocharacter have the FDS property?*

## References


[1] Alas O.T., Wilson R.G., *Minimal properties between $T_1$ and $T_2$*, Houston J. Math. **32** (2006), no. 2, 493–504.

[2] Alas O.T., Wilson R.G., *When is a compact space sequentially compact?*, Topology Proc. **29** (2005), no. 2, 327–335.

[3] Alas O.T., Wilson R.G., *Spaces in which compact subsets are closed and the lattice of $T_1$ topologies on a set*, Comment. Math. Univ. Carolin. **43** (2002), no. 4, 641–652.

[4] Alas O.T., Tkachenko M.G., Tkachuk V., Wilson R.G., *The FDS-property and the spaces in which compact sets are closed*, Sci. Math. Jpn. **61** (2005), no. 3, 473–480.

[5] Balcar B., Pelant J., Simon P., *The space of ultrafilters on N covered by nowhere dense sets*, Fund. Math. **110** (1980), no. 1, 11–24.

[6] Bella A., *Remarks on the finite derived set property*, Appl. Gen. Topol. **6** (2005), no. 1, 101–106.

[7] Baldovino C., Costantini C., *On some questions about KC and related spaces*, Topology Appl. **156** (2009), no. 17, 2692–2703.

[8] Bella A., Costantini C., *Minimal KC spaces are compact*, Topology Appl. **155** (2008), no. 13, 1426–1429.

[9] Bella A., Nyikos P.J., *Sequential compactness vs. countable compactness*, Colloq. Math. **120** (2010), no. 2, 165–189.

[10] van Douwen E.K., *The integers and topology*, in Handbook of Set-Theoretic Topology, K. Kunen and J. Vaughan, editors, North-Holland, Amsterdam, 1984, Chapter 3, pp. 111–167.

[11] Gryzlov A.A., *Two theorems on the cardinality of topological spaces*, Dokl. Akad. Nauk SSSR **251** (1980), no. 4, 780–783; Soviet Math. Dokl. **21** (1980), no. 2, 506–509.

[12] Shakhmatov D., Tkachenko M.G., Wilson R.G., *Transversal and $T_1$-independent topologies*, Houston J. Math. **30** (2004), no. 2, 421–433.

[13] Vaughan J.E., *Small uncountable cardinals and topology*, in Open Problems in Topology, J. van Mill and G.M. Reed, editors, North-Holland, Amsterdam, 1990, pp. 195–218.

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