Monotone measures with bad tangential behavior in the plane

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Abstract. We show that for every $\varepsilon > 0$, there is a set $A \subset \mathbb{R}^2$ such that $\mathcal{H}^1 \llcorner A$ is a monotone measure, the corresponding tangent measures at the origin are not unique and $\mathcal{H}^1 \llcorner A$ has the 1-dimensional density between 1 and $3+\varepsilon$ everywhere on the support.

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1. Introduction

In this paper, we study the existence of monotone measures with bad tangential behavior satisfying some additional assumptions natural for minimal surfaces. The question about their existence is motivated by open problems on existence and regularity of minimal surfaces, see [6].

Definition 1.1. Let μ be a Radon measure on $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that μ is *k-monotone* if the function $r \mapsto \frac{\mu B(z,r)}{r^k}$ is nondecreasing on $(0,\infty)$ for every $z \in \mathbb{R}^n$.

Definition 1.2. Let μ be a Radon measure on $\mathbb{R}^n$, $z \in \operatorname{spt}\mu$ and $k \le n$. We say that ν is a *k-tangent measure of μ at z* (we write $\nu \in \operatorname{Tan}_z^k \mu$), if ν is a non-zero Radon measure on $\mathbb{R}^n$ and if there is a sequence $\{r_j\}_{j=1}^{\infty}$, $r_j > 0$, $r_j \to 0$ as $j \to \infty$ such that

$$\frac{1}{r_j^k} T_{z,r_j}(\mu) \to \nu \quad \text{vaguely as } j \to \infty, \quad \text{where } T_{z,r}(x) = \frac{x-z}{r}\,,$$

i.e. if every continuous function φ on $\mathbb{R}^n$ with a compact support satisfies

$$\lim_{j\to\infty} \frac{1}{r_j^k} \int_{\mathbb{R}^n} \varphi\Big(\frac{x-z}{r_j}\Big)\, d\mu(x) = \int_{\mathbb{R}^n} \varphi\, d\nu.$$

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Instead of 1-monotone and 1-tangent we simply write *monotone* and *tangent*.

The tangent measures were introduced by Preiss in [5]. If μ is a k-monotone measure, then μ has a finite k-dimensional density $\theta^k_z\mu = \lim_{r\to 0_+} \frac{\mu B(z,r)}{\omega_k r^k}$, where ω_k is the volume of the unit ball in $\mathbb{R}^k$. If the density satisfies $\theta^k_z\mu \in (0,\infty)$, then our definition of tangent measures coincides up to a multiplicative constant with the one in [4, 14.1].

For better understanding of the problems concerning the minimal surfaces, it is important to study monotone measures with non-unique tangent measures at a point of the support.

The first such a measure was given by Kolář in [3]. However, this measure does not satisfy the density assumption natural for minimal surfaces. Therefore there were further attempts to construct other k-monotone measures with bad tangential behavior, i.e. find for fixed $\varepsilon > 0$ a Radon measure μ on $\mathbb{R}^n$ satisfying the following additional properties (we suppose that the origin $0 \in \operatorname{spt}\mu$ is the point with non-unique k-tangent measures to μ):

(1)
$$\theta^k_z\mu \geq 1 \quad \text{for every} \quad z \in \operatorname{spt}\mu \qquad \text{(then } \mu \text{ is called a concentrated measure)},$$

(2)
$$\theta^k_z\mu = 1 \quad \text{for every} \quad z \in \operatorname{spt}\mu \setminus \{0\}$$

and

(3)
$$\theta^k_0\mu \leq 1 + \varepsilon.$$

A k-monotone measure with non-unique tangential behavior satisfying all assumptions (1), (2) and (3) has not been constructed yet. However it is believed that such a measure exists. Let us also note that its existence would disprove the conjecture that the monotonicity is a sufficient assumption for the Allard regularity theorem, see [6].

Let us recall one of the partial results concerning the above problem. A monotone measure with non-unique tangential behavior satisfying (1) and weakened versions of (2) and (3) was constructed by Kirchheim using the method from [3]. This result was not published. Let us give the main ideas of the construction.

Fix $a > 0$ and define a symmetrical pair of logarithmic spirals by

$$\Gamma^+_a(t) = (\exp(at)\cos t, \exp(at)\sin t), \quad t \in \mathbb{R}$$

and

$$\Gamma^-_a(t) = (-\exp(at)\cos t, -\exp(at)\sin t), \quad t \in \mathbb{R}.$$

Next, we define the measures

$$\mu^+_a = \mathcal{H}^1 \llcorner [\Gamma^+_a], \qquad \mu^-_a = \mathcal{H}^1 \llcorner [\Gamma^-_a] \qquad \text{and} \qquad \mu_a = \mu^+_a + \mu^-_a,$$

where we use the notation $[\Gamma^+_a] = \{\Gamma^+_a(t) : t \in \mathbb{R}\}$, etc.

One can easily see that μ_a has non-unique tangential behavior at the origin (see the third section for a detailed proof), the density assumptions are satisfied (for sufficiently large a), but unfortunately μ_a is not monotone (see the last section for the proof). However, using a careful Taylor expansion with a computer algebra package, Kirchheim proved the "local monotonicity" of μ_a, it is the existence of $\delta = \delta(a) > 0$ such that $t \mapsto \frac{\mu_a B(z,r)}{r}$ is nondecreasing as long as $r < \delta|z|$. Then he used the compensation method from [3] (one adds a suitable "very" monotone measure, see for example Lemma 2.3) showing that there is a finite number of lines passing through the origin such that $\mathcal{H}^1$ restricted to the union of these lines, $[\Gamma_a^+]$ and $[\Gamma_a^-]$, is monotone. It is, the final measure is monotone, it has non-unique tangential behavior, condition (1) is satisfied, condition (2) is satisfied up to the points of intersection of the spirals and the lines and we have a version of (3) with the upper bound slightly larger than one plus the number of lines.

The goal of this paper is to give the following three improvements concerning Kirchheim's result. First, we give a short proof of the "local monotonicity" of μ_a (see Proposition (5.1)). Second, we obtain an estimate concerning above mentioned $\delta(a)$ (not only the existence). Let us note that our estimate enables us to show that it is enough to use two lines only as a compensation for the monotonicity (see Theorem 1.3) which is in fact the smallest possible number of lines (see the last section). Third, using the Definition 1.1 for large radii, we conclude that our final measure is monotone.

Now, let us state our main result. Set

$$L_1 = \{(t\cos(\tfrac{\pi}{3}), t\sin(\tfrac{\pi}{3})) : t \in \mathbb{R}\} \quad \text{and} \quad L_2 = \{(t\cos(\tfrac{2\pi}{3}), t\sin(\tfrac{2\pi}{3})) : t \in \mathbb{R}\}.$$

Theorem 1.3. *Let $\varepsilon > 0$. Then there is $K = K(\varepsilon) > 0$ such that for every $a > K$, the measure μ_a satisfies*

$$\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2) \ \text{ is monotone,}$$

$$\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2) \ \text{ does not have a unique tangent measure at the origin,}$$

$$\theta^1_z(\mu_a) = 1 \qquad \text{for all } \ z \in \operatorname{spt} \mu_a \setminus \{(0,0)\},$$

$$\theta^1_z(\mathcal{H}^1 \llcorner (L_1 \cup L_2)) = 1 \qquad \text{for all } \ z \in L_1 \cup L_2 \setminus \{(0,0)\},$$

$$\theta^1_{(0,0)}(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2)) \le 3 + \varepsilon.$$

A similar problem is studied in [2], where a version of logarithmic spirals in $\mathbb{R}^3$ is given.

We refer to [4], [5] and [6] for other information concerning the geometry of measures and the Monotonicity Formula.

The paper is organized as follows. In the third section we study the tangential behavior. The next two sections are devoted to the proof of the monotonicity which is the most difficult part of the proof of Theorem 1.3. We prove the monotonicity showing that the lower derivative of $r \mapsto \frac{(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2))B(z,r)}{r}$ is non-negative for every pair (z,r), $z \in \mathbb{R}^2$, $r > 0$. When checking this pointwise property, we distinguish several cases. In the fourth section we consider the cases concerning z and r such that the proof of the non-negativity of the lower derivative is just a straightforward computation. The fifth section is devoted to very small radii (this is the difficult case that Kirchheim's result concerns) where we apply a technique from [1]. In the last section we show that the measure μ_a is not monotone.

2. Preliminaries

Notation. The *scalar product* of $x, y \in \mathbb{R}^2$ is denoted by $x \cdot y$, the Euclidean *norm* of x is $|x|$. Further, x_1 and x_2 are the first and the second coordinates of x (this notation is used in the main part of the paper, while in the last section the meaning of the lower index is different as specified below). Set

$$B(z,r) = \{x \in \mathbb{R}^2 : |x-z| \le r\}, \qquad S(z,r) = \{x \in \mathbb{R}^2 : |x-z| = r\}.$$

When $z = (0,0)$, we simply write $B(r)$ and $S(r)$.

The *1-dimensional Hausdorff measure* is denoted by $\mathcal{H}^1$. If A is a Borel set and μ is a Radon measure, then $\mu \llcorner A$ is the restriction of μ to A, i.e. $(\mu \llcorner A)(M) = \mu(M \cap A)$. If I is an interval and $\Gamma : I \mapsto \mathbb{R}^n$ is a continuous curve, then $[\Gamma] = \{\Gamma(t) : t \in I\}$.

Next, for given z and r we are interested in the points of intersection of $S(z,r)$ and $[\Gamma_a^+]$ (or $[\Gamma_a^-]$) with the maximal or minimal distance from the center z. The following three points are important for us.
If $S(z,r) \cap [\Gamma_a^+] \neq \emptyset$, then let us denote

$$(4)\quad \xi = \Gamma_a^+(\tau) \in S(z,r) \cap [\Gamma_a^+] \quad \text{such that} \quad |\xi| \ge |\theta| \quad \text{for all} \quad \theta \in S(z,r) \cap [\Gamma_a^+],$$

and

$$(5)\quad \tilde{\xi} = \Gamma_a^+(\tilde{\tau}) \in S(z,r) \cap [\Gamma_a^+] \quad \text{such that} \quad |\tilde{\xi}| \le |\theta| \quad \text{for all} \quad \theta \in S(z,r) \cap [\Gamma_a^+].$$

If $S(z,r) \cap [\Gamma_a^-] \neq \emptyset$, then we pick

$$(6)\quad \eta = \Gamma_a^-(\sigma) \in S(z,r) \cap [\Gamma_a^-] \quad \text{such that} \quad |\eta| \ge |\theta| \quad \text{for all} \quad \theta \in S(z,r) \cap [\Gamma_a^-].$$

As $z \in S(1)$, there is $\vartheta \in [0, 2\pi)$ such that $z = (\cos\vartheta, \sin\vartheta)$. Let us further set $\varphi = \tau - \vartheta$, $\psi = \sigma - \vartheta$ where τ and σ are given above.

In the last section, we work with a sequence of radii $\{r_j\}$. In this case ξ_j, τ_j, φ_j, etc. correspond to the radius r_j (it is, ξ_j, etc. no longer denotes the j-th coordinate of a point but the j-th member of a sequence).

Some notes on the logarithmic spirals. As

$$\frac{|\dot{\Gamma}_a^+(t)|}{\frac{\partial|\Gamma_a^+(t)|}{\partial t}} = \frac{\sqrt{(ae^{at}\cos t - e^{at}\sin t)^2 + (ae^{at}\sin t + e^{at}\cos t)^2}}{ae^{at}} = \sqrt{1 + \frac{1}{a^2}}\,,$$

and similarly for Γ_a^-, we obtain for $0 \leq c_1 \leq c_2$

(7)

$$\mu_a^+(\{x \in \mathbb{R}^2 : c_1 \leq |x| \leq c_2\}) = \mu_a^-(\{x \in \mathbb{R}^2 : c_1 \leq |x| \leq c_2\}) = \sqrt{1+\frac{1}{a^2}}(c_2 - c_1).$$

Hence, for any $r > 0$, we have

$$\frac{\mu_a B(r)}{2r} = \sqrt{1+\frac{1}{a^2}}\,. \tag{8}$$

The logarithmic spirals are self-similar, in the sense that multiplication of the coordinates by the same positive number corresponds to some rotation. More precisely, if we define

$$\Gamma^+_{a,t_0}(t) = (\exp(a(t-t_0))\cos t, \exp(a(t-t_0))\sin t), \quad t \in \mathbb{R},$$
$$\Gamma^-_{a,t_0}(t) = (-\exp(a(t-t_0))\cos t, -\exp(a(t-t_0))\sin t), \quad t \in \mathbb{R}$$

and

$$\mu_{a,t_0} = \mathcal{H}^1 \llcorner ([\Gamma^+_{a,t_0}] \cup [\Gamma^-_{a,t_0}]),$$

then for every $\varrho > 0$ we have

$$\frac{1}{\varrho} T_{(0,0),\varrho}(\mu_a) = \mu_{a,t_0} \qquad \text{with} \qquad t_0 = \frac{\ln \varrho}{a}\,. \tag{9}$$

Some notes on monotonicity. Let us recall some well known facts concerning the monotonicity of Radon measures. Let $\Gamma : [a,b] \mapsto \mathbb{R}^n$ be a regular C^1-curve and let $\nu = \mathcal{H}^1 \llcorner [\Gamma]$. If we want to prove that $r \mapsto \frac{\nu B(z,r)}{r}$ is nondecreasing on $(0,\infty)$ for some $z \in \mathbb{R}^n$, then it is enough to show that

$$\underline{\mathrm{D}}_{\mathrm{r}} \frac{\nu B(z,r)}{r} = \frac{1}{r^2}\big(r\, \underline{\mathrm{D}}_{\mathrm{r}}\, \nu B(z,r) - \nu B(z,r)\big) \tag{10}$$

is nonnegative on $(0,\infty)$. Here we use the notation $\underline{\mathrm{D}}_{\mathrm{r}}\, f(r) = \liminf_{\delta \to 0} \frac{f(r+\delta)-f(r)}{\delta}$.

Notice that the condition $\underline{\mathrm{D}}_{\mathrm{r}} \frac{\nu B(z,r)}{r} \geq 0$ is satisfied when $\nu B(z,r) \leq 2r$ and $\Gamma(a), \Gamma(b) \notin B(z,r)$ (if $\nu B(z,r) = 0$ then the proof is trivial and if $0 < \nu B(z,r) \leq 2$, then there are at least two points of intersection $S(z,r) \cap \Gamma((a,b))$ and the contribution of each of them to $\underline{\mathrm{D}}_{\mathrm{r}}\, \nu B(z,r)$ is at least 1). We use this criterion very often.

We say that a measure ν is *monotone at* (z,r) if $\underline{\mathrm{D}}_{\mathrm{r}} \frac{\nu B(z,r)}{r} \geq 0$. The superadditivity of the lower derivative $\underline{\mathrm{D}}_{\mathrm{r}}$ implies that a sum of monotone measures at (z,r) is again monotone at (z,r).

We also need the following result inspired by the proof of [1, Proposition 2.2] telling us when we have $\underline{\mathrm{D}}_{\mathrm{r}} \frac{\nu B(z,r)}{r} > 0$ for ν being $\mathcal{H}^1$ restricted to the graph of a function.

Lemma 2.1. *Let $\delta_1, \delta_2 > 0$ and $f \in C^1([-\delta_1, \delta_2], \mathbb{R})$. Set $\mu_f = \mathcal{H}^1 \llcorner \{(x, f(x)) : x \in [-\delta_1, \delta_2]\}$. Fix $z = (0, h)$, with $h \in \mathbb{R}$, and fix $r > 0$ small enough so that $(-\delta_1, f(-\delta_1)), (\delta_2, f(\delta_2)) \notin B(z, r)$. Suppose that the following is satisfied.*

(i) *For all $x \in (-\delta_1, \delta_2)$ we have the inequality*

$$\frac{2|x|\sqrt{1+f'^2(x)}}{1+\sqrt{1+f'^2(x)}} - \mu_f(\{(t,s) : t \in I(0,x), s \in \mathbb{R}\}) > 0, \tag{11}$$

where $I(0, x)$ denotes the closed interval with the endpoints 0 and x.

(ii) *If $\mu_f B(z, r) > 0$, let $x_1, x_2 \in (-\delta_1, \delta_2)$ be such that $(x_1, f(x_1)), (x_2, f(x_2)) \in S(z, r)$ and $x_1 \leq x \leq x_2$ for every $x \in (-\delta_1, \delta_2)$ such that $(x, f(x)) \in S(z, r)$ and assume that x_1, x_2 have the following property: for both $i = 1, 2$ the angle between the tangent to the graph of f at $(x_i, f(x_i))$ and the line joining z and $(x_i, f(x_i))$ is less than $\frac{\pi}{2}$.*

Then μ_f is monotone at (z, r).

Proof: Since a sum of monotone measures at (z, r) is a monotone measure at (z, r), it is enough to consider even functions, $\delta_1 = \delta_2$ and $x > 0$. Suppose $h \in \mathbb{R}$, $r > 0$ are fixed and $\mu_f B(z, r) > 0$ (otherwise the proof is trivial by (10)).

We denote $x = \max\{t \in \mathbb{R} : (t, f(t)) \in B(z, r)\}$. Then obviously $x \in (0, \delta_2)$. Set $\eta(x) = \arctan f'(x)$, $\varphi_h(x) = \arctan \frac{f(x)-h}{x}$. Therefore $\cos(\eta(x) - \varphi_h(x)) > 0$ (see assumption (ii)) and we have

$$\frac{\partial \mu_f B(z,r)}{\partial r} \geq 2 \frac{1}{\cos(\eta(x) - \varphi_h(x))}.$$

As $r = \frac{x}{\cos(\varphi_h(x))}$, we obtain

$$\begin{aligned} \frac{\partial \mu_f B(z,r)}{\partial r} r &\geq \frac{2}{\cos(\eta(x) - \varphi_h(x))} \frac{x}{\cos(\varphi_h(x))} \\ &= \frac{4x}{\cos(\eta(x) - 2\varphi_h(x)) + \cos(\eta(x))} \\ &\geq \frac{4x}{1 + \cos(\eta(x))} = \frac{4x}{1 + \frac{1}{\sqrt{1+f'^2(x)}}} = \frac{4x\sqrt{1+f'^2(x)}}{1+\sqrt{1+f'^2(x)}} \end{aligned} \tag{12}$$

and the proof follows from (10), (12) and the assumptions of the lemma. □

Remark 2.2. If f satisfies $|f'| \leq \frac{1}{4}$ on $(-\delta_1, \delta_2)$, then Lemma 2.1 holds without assumption (ii).

Proof: Since a sum of monotone measures at (z, r) is a monotone measure at (z, r), it is enough to consider an even function and $\delta_1 = \delta_2$. Fix $r > 0$, $z = (0, h)$, with $h \in \mathbb{R}$, such that $(\delta_2, f(\delta_2)) \notin B(z, r)$. If $\mu_f B(z, r) = 0$, then μ_f is monotone at (z, r). Otherwise there is $x_0 \in (0, \delta_2)$ such that $(x_0, f(x_0)) \in S(z, r)$ and $(x, f(x)) \notin S(z, r)$ whenever $|x| \in (x_0, \delta_2)$.

Now, we distinguish two cases. First, if $|h-f(0)|<\frac{3}{4}x_0$ then condition (ii) is satisfied (since $|f'|<1$, the angle between $(1,0)$ and the tangent to the graph at $(x_0,f(x_0))$ is plainly less than $\frac{\pi}{4}$; since $|f'|\le\frac{1}{4}$ we have $|f(x_0)-f(0)|\le\frac{1}{4}x_0$, hence $|h-f(x_0)|\le|h-f(0)|+|f(0)-f(x_0)|<x_0$, thus the angle between $(1,0)$ and the vector $(x_0,f(x_0))-(0,h)$ is less than $\frac{\pi}{4}$).

Second, let $|h-f(0)|\ge\frac{3}{4}x_0$. Assumption $|f'|\le\frac{1}{4}$ implies $|f(x_0)-f(0)|\le\frac{1}{4}x_0$, hence

$$r=\sqrt{x_0^2+(h-f(x_0))^2}\ge\sqrt{x_0^2+\left(\frac{1}{2}x_0\right)^2}\ge\frac{11}{10}x_0. \tag{13}$$

Since $\mu_f B(z,r)>0$, there are at least two points in $S(z,r)\cap\operatorname{spt}\mu_f$. Hence $\underline{\mathrm{D}}_{\mathrm{r}}\,\mu_f B(z,r)\ge 2$ and thus from (13) and $|f'|\le\frac{1}{4}$ we obtain

$$\begin{aligned}\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{\mu_f B(z,r)}{r}&=\frac{1}{r^2}\big(r\,\underline{\mathrm{D}}_{\mathrm{r}}\,\mu_f B(z,r)-\mu_f B(z,r)\big)\\&\ge\frac{1}{r^2}\Big(2\frac{11}{10}x_0-2\sqrt{1+\left(\frac{1}{4}\right)^2}\,x_0\Big)\ge 0.\end{aligned}$$

Therefore μ_f is monotone at (z,r). □

Our last auxiliary result concerns the monotonicity of $\mathcal{H}^1\llcorner(L_1\cup L_2)$.

Lemma 2.3. *The measure $\mathcal{H}^1\llcorner(L_1\cup L_2)$ is monotone. Moreover, if $z\in S(1)$ and $r\ge\frac{9}{10}$, then*

$$\frac{\partial}{\partial r}\frac{(\mathcal{H}^1\llcorner(L_1\cup L_2))B(z,r)}{r}\ge\frac{1}{200r^3}\,.$$

Proof: For a line L and a center z, we denote $d=\operatorname{dist}(z,L)$. If $r>d$, then

$$\frac{\partial}{\partial r}\frac{(\mathcal{H}^1\llcorner L)\,B(z,r)}{r}=\frac{\partial}{\partial r}\frac{2\sqrt{r^2-d^2}}{r}=2\frac{d^2}{r^2\sqrt{r^2-d^2}}\ge 2\frac{d^2}{r^3}\,. \tag{14}$$

Since, in addition, $(\mathcal{H}^1\llcorner L)B(z,r)=0$ for $0<r\le d$, we see that $\mathcal{H}^1\llcorner L$ is monotone and the first assertion of the lemma follows.

Let us prove estimate (14). Recall $z=(\cos\vartheta,\sin\vartheta)$. In case $\vartheta\in[0,\frac{\pi}{3}-\frac{\pi}{60}]$, we have

$$\operatorname{dist}(z,L_1)=\sin(\tfrac{\pi}{3}-\vartheta)\in[\sin(\tfrac{\pi}{60}),\sin(\tfrac{\pi}{3})]\subset[\tfrac{1}{20},\tfrac{9}{10}]$$

and thus for $r\ge\frac{9}{10}$ we obtain from (14)

$$\frac{\partial}{\partial r}\frac{(\mathcal{H}^1\llcorner(L_1\cup L_2))B(z,r)}{r}\ge\frac{\partial}{\partial r}\frac{(\mathcal{H}^1\llcorner L_1)B(z,r)}{r}\ge 2\frac{\sin^2(\frac{\pi}{60})}{r^3}\ge\frac{1}{200r^3}\,.$$

If $\vartheta\in[\frac{\pi}{3}-\frac{\pi}{60},\frac{\pi}{2}]$, then

$$\operatorname{dist}(z,L_2)=\sin(\tfrac{2\pi}{3}-\vartheta)\in[\sin(\tfrac{\pi}{6}),\sin(\tfrac{\pi}{3}+\tfrac{\pi}{60})]\subset[\tfrac{1}{2},\tfrac{9}{10}],$$

hence for $r \geq \frac{9}{10}$ we have by (14)

$$\frac{\partial}{\partial r}\frac{(\mathcal{H}^1 \llcorner (L_1 \cup L_2))B(z,r)}{r} \geq \frac{\partial}{\partial r}\frac{(\mathcal{H}^1 \llcorner L_2)B(z,r)}{r} \geq 2\frac{\sin^2\left(\frac{\pi}{6}\right)}{r^3} \geq \frac{1}{2r^3}\,.$$

Thus, we are done in the first quadrant. In any other quadrant the proof is similar (see the definition of L_1 and L_2). □

3. Tangential behavior

Proposition 3.1.

$$\mathrm{Tan}^1_{(0,0)}(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2)) = \{\mu_{a,t_0} + \mathcal{H}^1 \llcorner (L_1 \cup L_2) : 0 \leq t_0 < \pi\}.$$

Proof: Using (9) one can easily prove that

$$\mathrm{Tan}^1_{(0,0)}(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2)) \supset \{\mu_{a,t_0} + \mathcal{H}^1 \llcorner (L_1 \cup L_2) : 0 \leq t_0 < \pi\}.$$

Indeed, since we plainly have for any $t_0 \in [0,\pi)$ the identity

$$\mu_{a,t_0} = \mu_{a,t_0+k\pi} \qquad \text{for all} \quad k \in \mathbb{Z}, \tag{15}$$

it is enough to take the sequence of blow-ups corresponding to $\varrho_j = \exp(a(t_0 - j\pi))$, $j \in \mathbb{N}$.

The opposite inclusion is obtained by a suitable choice of a test function. Assume $\varrho_j > 0$ for $j \in \mathbb{N}$, $\varrho_j \to 0$ and $\frac{1}{\varrho_j}T_{(0,0),\varrho_j}(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2))$ vaguely converges. Set $t_j = \frac{\ln \varrho_j}{a}$. Hence from (9) and an obvious identity

$$\frac{1}{\varrho_j}T_{(0,0),\varrho_j}(\mathcal{H}^1 \llcorner (L_1 \cup L_2)) = \mathcal{H}^1 \llcorner (L_1 \cup L_2)$$

we see that μ_{a,t_j} vaguely converges. Let $\psi : [0,\infty) \mapsto \mathbb{R}$ be a continuous function with a compact support satisfying $\psi \geq 0$, $\psi(0) = 0$ and $\int_0^\infty \psi(t)\,dt = 1$. We define on $\mathbb{R}^2$ a continuous test function with a compact support by $\varphi_1(0,0) = 0$ and

$$\varphi_1(x) = \psi(|x|)\left|\frac{x_1}{|x|}\cos\left(\frac{\ln|x|}{a}\right) + \frac{x_2}{|x|}\sin\left(\frac{\ln|x|}{a}\right)\right| \qquad \text{for} \quad |x| > 0.$$

If $t \in \mathbb{R}$ and $x = \Gamma^+_{a,t_j}(t)$ or $x = \Gamma^-_{a,t_j}(t)$ we have

$$\varphi_1(x) = \psi(|x|)|\cos t\cos(t-t_j) + \sin t\sin(t-t_j)| = \psi(|x|)|\cos t_j|.$$

Hence, we obtain from (7)

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi_1\,d\mu_{a,t_j} &= |\cos t_j|\int_{\mathbb{R}^2}\psi(|x|)\,d\mu_{a,t_j} = |\cos t_j|\int_0^\infty 2\sqrt{1+\frac{1}{a^2}}\psi(t)\,dt\\ &= 2\sqrt{1+\frac{1}{a^2}}|\cos t_j|.\end{aligned}$$

Therefore $|\cos t_j|$ converges. If $|\cos t_j| \to 1$, then from (15) we see that $\mu_{a,t_j} \to \mu_{a,0} = \mu_a$ vaguely. Similarly, if $|\cos t_j| \to 0$, then $\mu_{a,t_j} \to \mu_{a,\frac{\pi}{2}}$ vaguely.

Finally, if $|\cos t_j| \to c \in (0,1)$, then there is $t_0 \in (0, \frac{\pi}{2})$ such that $|\cos t_j| \to |\cos t_0| = |\cos(\pi - t_0)|$. Let us set $\varphi_2(0,0) = 0$ and

$$\varphi_2(x) = \psi(|x|)\Big|\frac{x_1}{|x|}\cos\Big(\frac{\ln|x|}{a} + t_0\Big) + \frac{x_2}{|x|}\sin\Big(\frac{\ln|x|}{a} + t_0\Big)\Big| \qquad \text{for} \quad |x| > 0,$$

where the function ψ is the same as above. This time we obtain for $x \in \operatorname{spt}\mu_{a,t_j}$

$$\varphi_2(x) = \psi(|x|)|\cos t\cos(t - t_j + t_0) + \sin t\sin(t - t_j + t_0)| = \psi(|x|)|\cos(t_j - t_0)|.$$

The vague convergence implies the same way as above that $|\cos(t_j - t_0)|$ converges. If $|\cos(t_j - t_0)| \to 1$, then (15) implies $\mu_{a,t_j} \to \mu_{a,t_0}$ vaguely. Otherwise, since $|\cos(t_j - t_0)| \to d \neq 1$ and $|\cos t_j| \to |\cos(\pi - t_0)|$, we have $\cos t_j \to \cos(\pi - t_0)$. Thus using (15) we obtain $\mu_{a,t_j} \to \mu_{a,\pi - t_0}$ vaguely. Hence we have the remaining inclusion

$$\operatorname{Tan}^1_{(0,0)}(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2)) \subset \{\mu_{a,t_0} + \mathcal{H}^1 \llcorner (L_1 \cup L_2) : 0 \leq t_0 < \pi\}.$$

$\square$

4. Large radii: monotonicity by compensation

Because of the self-similarity of the logarithmic spirals it is enough to prove monotonicity at (z,r) only for $z \in S(1) \cup \{(0,0)\}$ and $r > 0$. In case of large radii, we carefully estimate each term on the right hand side of (10) for $\nu = \mu_a$.

Throughout the rest of the paper we will often use the notation defined in Preliminaries, in particular the one used in (4)–(6) without further notice.

Proposition 4.1. *There is $K_1 > 0$ such that if $a > K_1$, $z \in S(1)$ and $r \geq \frac{9}{10}$, then $\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2)$ is monotone at (z,r).*

Proof of Proposition 4.1: case $r \in [\frac{9}{10}, 8]$. If $r \in [\frac{9}{10}, 8]$ and $z \in S(1)$, then the proof of the monotonicity at (z,r) is obtained directly from formula (10). The main ingredient of the proof is the estimate concerning $\mu_a B(z,r)$ given in Lemmata 4.2, 4.3 and 4.4, respectively.

Lemma 4.2. *Assume $a \geq 9$, $z \in S(1)$ and $r \in [1 + \frac{1}{a}, 8]$. Then*

$$\mu_a B(z,r) \leq 2\Big(1 + 4\frac{\ln^2 a}{a^2}\Big)r.$$

Proof: As $r \in [1 + \frac{1}{a}, 8]$ and $|z| = 1$, we have $S(z,r) \cap [\Gamma_a^+] \neq \emptyset \neq S(z,r) \cap [\Gamma_a^-]$ and $|\xi|, |\eta| \in [\frac{1}{a}, 9]$. Hence $\tau, \sigma \in [-\frac{\ln a}{a}, \frac{\ln 9}{a}]$, and thus $a \geq 9$ implies $|\frac{\sin\tau}{\cos\tau}| \leq 2\frac{\ln a}{a}$, $|\frac{\sin\sigma}{\cos\sigma}| \leq 2\frac{\ln a}{a}$, $\xi_1 > 0$ and $\eta_1 < 0$. Therefore

$$|\xi_1| + |\eta_1| = \xi_1 - \eta_1 = |\xi_1 - \eta_1| \leq |\xi_1 - z_1| + |z_1 - \eta_1| \leq 2r$$

and thus (7) implies

$$\begin{aligned}\mu_a B(z,r) &= \mu_a^+ B(z,r) + \mu_a^- B(z,r)\\ &\le \sqrt{1+\frac{1}{a^2}}(|\xi|+|\eta|) = \sqrt{1+\frac{1}{a^2}}\Big(\sqrt{1+\Big(\frac{\xi_2}{\xi_1}\Big)^2}|\xi_1| + \sqrt{1+\Big(\frac{\eta_2}{\eta_1}\Big)^2}|\eta_1|\Big)\\ &= \sqrt{1+\frac{1}{a^2}}\Big(\sqrt{1+\Big(\frac{\sin\tau}{\cos\tau}\Big)^2}|\xi_1| + \sqrt{1+\Big(\frac{\sin\sigma}{\cos\sigma}\Big)^2}|\eta_1|\Big)\\ &\le 2\sqrt{1+\frac{1}{a^2}}\sqrt{1+4\frac{\ln^2 a}{a^2}}\,r < 2\Big(1+4\frac{\ln^2 a}{a^2}\Big)r.\end{aligned}$$

□

Lemma 4.3. *Assume $a \ge 9$, $z \in S(1)$ and $r \in [1-\frac{3}{a}, 1+\frac{1}{a}]$. Then*

$$\mu_a B(z,r) \le 2\Big(1+4\frac{\ln^2 a}{a^2}\Big)\Big(1+\frac{1}{a}\Big) \le 2r\Big(1+4\frac{\ln^2 a}{a^2}\Big)\Big(1+\frac{6}{a}\Big).$$

Proof: Since $B(z,r) \subset B(z, 1+\frac{1}{a})$, the first inequality follows from Lemma 4.2. The second estimate follows from the assumptions concerning r and a. □

Lemma 4.4. *Assume $a \ge 9$, $z \in S(1)$ and $r \in [\frac{9}{10}, 1-\frac{3}{a}]$. Then*

$$\mu_a B(z,r) \le 2\sqrt{1+\frac{1}{a^2}}\,r + \frac{1}{a}\sqrt{1+\frac{1}{a^2}}\,.$$

Proof: By the symmetry between Γ_a^+ and Γ_a^- we can suppose $z_1 \ge 0$. Since every $x \in B(z,r)$ satisfies

$$1-r = |z|-r \le |x| \le |z|+r = 1+r,$$

we obtain from (7)

$$\mu_a^+ B(z,r) \le \sqrt{1+\frac{1}{a^2}}\Big((1+r)-(1-r)\Big) = 2\sqrt{1+\frac{1}{a^2}}\,r. \tag{16}$$

Next, let us estimate $\mu_a^- B(z,r)$. Plainly $B(z,r) \subset B(2)$. Further, if $t \in [\frac{\ln\frac{1}{a}}{a}, \frac{\ln 2}{a}]$ and $\theta = \Gamma_a^-(t)$, then we have $|\theta| \in [\frac{1}{a}, 2]$, $\theta_1 < 0$, $|\theta_2| \le |e^{at}t| \le \frac{2}{a}$ (because the function $g(s) = e^{as}s$ satisfies $g'(s) = e^{as}(1+as)$, $g(\frac{\ln\frac{1}{a}}{a}) = -\frac{\ln a}{a^2} > -\frac{2}{a}$, $g(-\frac{1}{a}) = -\frac{1}{ea}$ and $g(\frac{\ln 2}{a}) = \frac{2\ln 2}{a}$, which implies $|g(s)| \le \frac{2}{a}$ on $[\frac{\ln\frac{1}{a}}{a}, \frac{\ln 2}{a}]$). Therefore

$$|z-\theta|^2 = (z_1-\theta_1)^2 + (z_2-\theta_2)^2 \ge z_1^2 + z_2^2 - 2|z_2||\theta_2| \ge 1 - \frac{4}{a} > \Big(1-\frac{3}{a}\Big)^2 \ge r^2.$$

It follows that $\theta \notin B(z,r)$. Hence we have $B(z,r) \cap [\Gamma_a^-] \subset B(\frac{1}{a})$ and thus from (7) we obtain

$$\mu_a^- B(z,r) \le \mu_a^- B(\tfrac{1}{a}) = \frac{1}{a}\sqrt{1+\frac{1}{a^2}}. \tag{17}$$

As $\mu_a = \mu_a^+ + \mu_a^-$, the proof follows from (16) and (17). □

Proof of Proposition 4.1: case $\frac{9}{10} \le r \le 8$: Let us suppose that $\mu_a B(z,r) > 0$, otherwise the proof is trivial. Since there are at least two points in the intersection $S(z,r) \cap \operatorname{spt} \mu_a$, we have

$$\underline{\mathrm{D}}_{\mathrm{r}}\, \mu_a B(z,r) \ge 2. \tag{18}$$

If $r \in [\frac{9}{10}, 1-\frac{3}{a}]$, using (10), (18), Lemma 4.4 and Lemma 2.3 we obtain

$$\underline{\mathrm{D}}_{\mathrm{r}}\, \frac{(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2))B(z,r)}{r} \ge \frac{2}{r} - \frac{2r+\frac{1}{a}}{r^2}\sqrt{1+\frac{1}{a^2}} + \frac{1}{200r^3}\,.$$

If $r \in [1-\frac{3}{a}, 1+\frac{1}{a}]$, then (10), (18), Lemma 4.3 and Lemma 2.3 give

$$\underline{\mathrm{D}}_{\mathrm{r}}\, \frac{(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2))B(z,r)}{r} \ge \frac{2}{r} - \frac{2}{r}\Big(1+4\frac{\ln^2 a}{a^2}\Big)\Big(1+\frac{6}{a}\Big) + \frac{1}{200r^3}\,.$$

Finally, if $r \in [1+\frac{1}{a}, 8]$, then (10), (18), Lemma 4.2 and Lemma 2.3 imply

$$\underline{\mathrm{D}}_{\mathrm{r}}\, \frac{(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2))B(z,r)}{r} \ge \frac{2}{r} - \frac{2}{r}\Big(1+4\frac{\ln^2 a}{a^2}\Big) + \frac{1}{200r^3}\,.$$

Now, if a is sufficiently large, then the right hand side is positive in all three cases. □

Proof of Proposition 4.1: case $r \ge 8$. For large radii, our estimates have to be much more careful then in the previous case. Let us briefly outline our strategy. Since there are always at least two points of the intersection $B(z,r) \cap \operatorname{spt} \mu_a$ (recall $r \ge 8$), from (7) and (10) we obtain

$$\underline{\mathrm{D}}_{\mathrm{r}}\, \frac{\mu_a B(z,r)}{r} \ge \frac{\sqrt{1+\frac{1}{a^2}}}{r^2}\Big(\Big(\frac{\partial|\xi|}{\partial r} + \frac{\partial|\eta|}{\partial r}\Big)r - |\xi| - |\eta|\Big).$$

Next, we estimate all the terms on the right hand side using the identities from Lemma 4.6. Notice, that when estimating $\frac{\partial|\xi|}{\partial r}$ (and similarly $\frac{\partial|\eta|}{\partial r}$), we do not use the explicit formula (21) (which is not convenient to work with), but we proceed in the following way. First, we obtain a rough estimate (see Lemma 4.7). Then we use formula (20) together with this rough estimate on the right hand side (where $\frac{\partial|\xi|}{\partial r}$ is multiplied by $\frac{1}{a}$, which can be made very small).

Lemma 4.5. *Assume $\tau,\sigma\in\mathbb{R}$, $z\in S(1)$ and $r\geq 8$. Then*

$$\sqrt{\cos^2\varphi+r^2-1}\geq\Big(1-\frac{1}{100}\Big)r,\qquad \sqrt{\cos^2\psi+r^2-1}\geq\Big(1-\frac{1}{100}\Big)r.$$

Proof: Since $r\geq 8$, we have $\sqrt{r^2-1}\geq\frac{99}{100}r$. Now, both estimates follow easily. □

Lemma 4.6. *Assume $a\geq 30$ and $z\in S(1)$. The function $r\mapsto|\xi|$ is continuously differentiable on $(8,\infty)$ and satisfies*

$$|\xi|=\cos\varphi+\sqrt{\cos^2\varphi+r^2-1}, \tag{19}$$

$$\frac{\partial|\xi|}{\partial r}=\frac{-\frac{1}{a}\sin\varphi\frac{\partial|\xi|}{\partial r}+r}{\sqrt{\cos^2\varphi+r^2-1}} \tag{20}$$

and

$$\frac{\partial|\xi|}{\partial r}=\frac{r}{\sqrt{\cos^2\varphi+r^2-1}+\frac{\sin\varphi}{a}}\,. \tag{21}$$

The function $r\mapsto|\eta|$ is continuously differentiable on $(8,\infty)$ and satisfies

$$|\eta|=-\cos\psi+\sqrt{\cos^2\psi+r^2-1}, \tag{22}$$

$$\frac{\partial|\eta|}{\partial r}=\frac{\frac{1}{a}\sin\psi\frac{\partial|\eta|}{\partial r}+r}{\sqrt{\cos^2\psi+r^2-1}} \tag{23}$$

and

$$\frac{\partial|\eta|}{\partial r}=\frac{r}{\sqrt{\cos^2\psi+r^2-1}-\frac{\sin\psi}{a}}\,. \tag{24}$$

Proof: Using $\xi=\Gamma_a^+(\tau)=(|\xi|\cos\tau,|\xi|\sin\tau)$ we set

$$\begin{aligned}F(r,\tau)=|\xi-z|^2-r^2&=(|\xi|\cos\tau-z_1)^2+(|\xi|\sin\tau-z_2)^2-r^2\\&=|\xi|^2-2|\xi|(z_1\cos\tau+z_2\sin\tau)+1-r^2\\&=|\xi|^2-2|\xi|\cos\varphi+1-r^2.\end{aligned}$$

Solving the equation $F(r,\tau)=0$ with respect to nonnegative $|\xi|$ we obtain (19).

The smoothness, (20) and (21) follow from the Implicit Function Theorem. Indeed, $\frac{\partial\varphi}{\partial\tau}=\frac{\partial(\tau-\vartheta)}{\partial\tau}=1$ and $\frac{\partial|\xi|}{\partial\tau}=\frac{\partial\exp(a\tau)}{\partial\tau}=a\exp(a\tau)=a|\xi|$ imply

$$\frac{\partial F}{\partial\tau}=2a|\xi|^2-2a|\xi|\cos\varphi+2|\xi|\sin\varphi=2a|\xi|\Big(|\xi|-\cos\varphi+\frac{\sin\varphi}{a}\Big).$$

Hence applying (19) and Lemma 4.5 we obtain

$$\frac{\partial F}{\partial \tau} = 2a|\xi|\Big(\sqrt{\cos^2\varphi + r^2 - 1} + \frac{\sin\varphi}{a}\Big) > 0.$$

Further $\frac{\partial F}{\partial r} = -2r$, $\frac{\partial|\xi|}{\partial \tau} = a|\xi|$, above formula for $\frac{\partial F}{\partial \tau}$ imply

$$\frac{\partial|\xi|}{\partial r} = \frac{\partial|\xi|}{\partial \tau}\frac{\partial \tau}{\partial r} = \frac{\partial|\xi|}{\partial \tau}\cdot(-1)\frac{\frac{\partial F}{\partial r}}{\frac{\partial F}{\partial \tau}} = a|\xi|\frac{2r}{2a|\xi|(\sqrt{\cos^2\varphi + r^2 - 1} + \frac{\sin\varphi}{a})}.$$

This is (21). As $\alpha = \frac{\beta}{\gamma+\delta}$ is equivalent to $\alpha = \frac{\beta-\delta\alpha}{\gamma}$ provided $\gamma + \delta \neq 0 \neq \gamma$, (20) follows from (21). For the point of intersection η, the proof is similar. $\square$

Lemma 4.7. *Assume $a \geq 30$, $z \in S(1)$ and $r \geq 8$. Then*

$$\frac{\partial|\xi|}{\partial r} \leq 2 \qquad \text{and} \qquad \frac{\partial|\eta|}{\partial r} \leq 2.$$

Proof: The estimate concerning $\frac{\partial|\xi|}{\partial r}$ follows from (21) and Lemma 4.5. For $\frac{\partial|\eta|}{\partial r}$ we use (24) and Lemma 4.5. $\square$

Lemma 4.8. *Assume $a \geq 30$, $z \in S(1)$ and $r \geq 8$. Then*

$$\Big(\frac{\partial|\xi|}{\partial r} + \frac{\partial|\eta|}{\partial r}\Big)r - |\xi| - |\eta| \geq -\frac{3}{a(r-1)}.$$

Proof: Since $r > 1$ and $z \in S(1)$, we have $|\xi|, |\eta| \in [r-1, r+1]$. Set $\delta = |\tau - \sigma|$. We observe

$$0 \leq \delta = \Big|\frac{1}{a}\ln|\xi| - \frac{1}{a}\ln|\eta|\Big| \leq \frac{1}{a}\ln\Big(\frac{r+1}{r-1}\Big) = \frac{1}{a}\ln\Big(1 + \frac{2}{r-1}\Big) \leq \frac{2}{a(r-1)}. \tag{25}$$

Using (19), (20), (22) and (23) we obtain

$$\begin{aligned}
&\Big(\frac{\partial|\xi|}{\partial r} + \frac{\partial|\eta|}{\partial r}\Big)r - |\xi| - |\eta| \\
&= \Big(\frac{-\frac{1}{a}\sin\varphi\frac{\partial|\xi|}{\partial r} + r}{\sqrt{\cos^2\varphi + r^2 - 1}} + \frac{\frac{1}{a}\sin\psi\frac{\partial|\eta|}{\partial r} + r}{\sqrt{\cos^2\psi + r^2 - 1}}\Big)r - \cos\varphi - \sqrt{\cos^2\varphi + r^2 - 1} \\
&\qquad + \cos\psi - \sqrt{\cos^2\psi + r^2 - 1}.
\end{aligned} \tag{26}$$

Further, we have

$$|\cos\varphi - \cos\psi| \leq |\varphi - \psi| = |\tau - \sigma| = \delta, \tag{27}$$

$$|\sin\varphi - \sin\psi| \leq |\varphi - \psi| = |\tau - \sigma| = \delta, \tag{28}$$

$$
\begin{aligned}
&\frac{r^2}{\sqrt{\cos^2\varphi + r^2 - 1}} + \frac{r^2}{\sqrt{\cos^2\psi + r^2 - 1}} \\
&\quad - \sqrt{\cos^2\varphi + r^2 - 1} - \sqrt{\cos^2\psi + r^2 - 1} \\
&= \frac{1-\cos^2\varphi}{\sqrt{\cos^2\varphi + r^2 - 1}} + \frac{1-\cos^2\psi}{\sqrt{\cos^2\psi + r^2 - 1}} \geq 0
\end{aligned}
\tag{29}
$$

and by Lemma 4.5 and (27)

(30)

$$
\begin{aligned}
&\Big|\frac{1}{\sqrt{\cos^2\varphi + r^2 - 1}} - \frac{1}{\sqrt{\cos^2\psi + r^2 - 1}}\Big| \\
&= \frac{|(\cos^2\psi + r^2 - 1) - (\cos^2\varphi + r^2 - 1)|}{\sqrt{\cos^2\varphi + r^2 - 1}\sqrt{\cos^2\psi + r^2 - 1}(\sqrt{\cos^2\varphi + r^2 - 1} + \sqrt{\cos^2\psi + r^2 - 1})} \\
&\leq \frac{|\cos^2\psi - \cos^2\varphi|}{2(\frac{99}{100}r)^3} \leq \frac{2|\cos\psi - \cos\varphi|}{r^3} \leq \frac{2\delta}{r^3}\,.
\end{aligned}
$$

Lemma 4.5, Lemma 4.7, (20), (23), (25), (28) and (30) imply

$$
\begin{aligned}
\Big|\frac{\partial|\xi|}{\partial r} - \frac{\partial|\eta|}{\partial r}\Big| &= \Big|\Big(\frac{r}{\sqrt{\cos^2\varphi + r^2 - 1}} - \frac{r}{\sqrt{\cos^2\psi + r^2 - 1}}\Big) \\
&\qquad + \frac{1}{a}\Big(\Big(\frac{-\sin\varphi\frac{\partial|\xi|}{\partial r}}{\sqrt{\cos^2\varphi + r^2 - 1}} + \frac{\sin\varphi\frac{\partial|\xi|}{\partial r}}{\sqrt{\cos^2\psi + r^2 - 1}}\Big) \\
&\qquad + \frac{(-\sin\varphi + \sin\psi)\frac{\partial|\xi|}{\partial r}}{\sqrt{\cos^2\psi + r^2 - 1}} + \frac{\sin\psi(-\frac{\partial|\xi|}{\partial r} - \frac{\partial|\eta|}{\partial r})}{\sqrt{\cos^2\psi + r^2 - 1}}\Big)\Big| \\
&\leq \frac{2\delta}{r^2} + \frac{1}{a}\Big(2\frac{2\delta}{r^3} + \frac{2\delta}{\frac{99}{100}r} + \frac{4}{\frac{99}{100}r}\Big) \leq \frac{5}{ar}.
\end{aligned}
\tag{31}
$$

Finally, from Lemma 4.5, Lemma 4.7, (25), (28), (30) and (31) we obtain

(32)

$$
\begin{aligned}
&\Big|\Big(\frac{-\frac{1}{a}\sin\varphi\frac{\partial|\xi|}{\partial r}}{\sqrt{\cos^2\varphi + r^2 - 1}} + \frac{\frac{1}{a}\sin\psi\frac{\partial|\eta|}{\partial r}}{\sqrt{\cos^2\psi + r^2 - 1}}\Big)r\Big| \\
&\qquad = \frac{r}{a}\Big|\frac{-\sin\varphi\frac{\partial|\xi|}{\partial r}}{\sqrt{\cos^2\varphi + r^2 - 1}} + \frac{\sin\varphi\frac{\partial|\xi|}{\partial r}}{\sqrt{\cos^2\psi + r^2 - 1}} \\
&\qquad\qquad - \frac{(\sin\varphi - \sin\psi)\frac{\partial|\xi|}{\partial r}}{\sqrt{\cos^2\psi + r^2 - 1}} - \frac{\sin\psi\Big(\frac{\partial|\xi|}{\partial r} - \frac{\partial|\eta|}{\partial r}\Big)}{\sqrt{\cos^2\psi + r^2 - 1}}\Big| \\
&\qquad \leq \frac{r}{a}\Big(2\frac{2\delta}{r^3} + \frac{2\delta}{\frac{99}{100}r} + \frac{1}{\frac{99}{100}r}\frac{5}{ar}\Big) \leq \frac{1}{ar}\,.
\end{aligned}
$$

Now, the proof follows from (26) combined with estimates (27) (see also (25)), (29) and (32). ☐

Proof of Proposition 4.1: case $r \geq 8$. Let us suppose that $\mu_a B(z,r) > 0$, otherwise the proof is trivial. Hence there are at least two points in the intersection $S(z,r) \cap \operatorname{spt} \mu_a$. Using in addition (7), (10), Lemma 2.3 and Lemma 4.8 we obtain

$$\begin{aligned}
&\underline{\mathrm{D}}_{\mathrm{r}} \frac{(\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2))B(z,r)}{r} \\
&\qquad \geq \frac{\sqrt{1+\frac{1}{a^2}}}{r^2}\Big(\Big(\frac{\partial|\xi|}{\partial r} + \frac{\partial|\eta|}{\partial r}\Big) r - |\xi| - |\eta|\Big) + \underline{\mathrm{D}}_{\mathrm{r}} \frac{(\mathcal{H}^1 \llcorner (L_1 \cup L_2))B(z,r)}{r} \\
&\qquad \geq -\frac{\sqrt{1+\frac{1}{a^2}}}{r^2} \frac{3}{a(r-1)} + \frac{1}{200 r^3}.
\end{aligned}$$

If a is sufficiently large, then the right hand side is positive and we are done. $\square$

5. Small radii

For very small radii we cannot rely on any compensation, because some balls centered on $S(1)$ with small radii do not intersect $L_1 \cup L_2$.

Proposition 5.1. *There is $K_2 > 0$ such that if $a > K_2$, $0 < r \leq \frac{9}{10}$ and $z \in S(1)$, then μ_a is monotone at (z,r).*

For the proof of this proposition we need some auxiliary lemmata.

If the center $z \in S(1)$ is relatively far from $[\Gamma_a^+]$ or $[\Gamma_a^-]$, then the proof is easy.

Lemma 5.2. *There is $K_3 > 0$ such that if $a > K_3$, $0 < r \leq \frac{9}{10}$, $z \in S(1)$ and $|z-(1,0)| \geq \frac{1}{20}$, then μ_a^+ is monotone at (z,r).*

Proof: Let us use the logarithmic parameterization

$$\tilde{\Gamma}_a^+(t) = \Big(t\cos\Big(\frac{\ln t}{a}\Big), t\sin\Big(\frac{\ln t}{a}\Big)\Big), \qquad t \in (0,\infty).$$

We observe that there is $K_3 > 0$ such that for every $a > K_3$ and $t \in [\frac{1}{20}, 2]$ we have

$$(\tilde{\Gamma}_a^+)_1(t) > 0, \qquad |(\tilde{\Gamma}_a^+)_2(t)| \leq \frac{1}{80} \qquad \text{and} \qquad \sqrt{1+\frac{1}{a^2}} \leq \sqrt{1+\frac{1}{80^2}}. \tag{33}$$

Now, we distinguish two cases. First, let $z_1 \leq 0$. For $t \in (0, \frac{1}{20}]$ we have

$$|z - \tilde{\Gamma}_a^+(t)| \geq |z| - |\tilde{\Gamma}_a^+(t)| \geq 1 - \frac{1}{20} > \frac{9}{10} \geq r.$$

For $t \in [\frac{1}{20}, 2]$, we see that (33) and $z_1 \leq 0 \leq (\tilde{\Gamma}_a^+)_1(t)$ imply

$$
\begin{aligned}
|z-\tilde{\Gamma}_a^+(t)| &= \sqrt{(z_1-(\tilde{\Gamma}_a^+)_1(t))^2+(z_2-(\tilde{\Gamma}_a^+)_2(t))^2} \geq \sqrt{z_1^2+z_2^2-2|z_2||(\tilde{\Gamma}_a^+)_2(t)|} \\
&\geq \sqrt{1-2\frac{1}{80}} > \frac{9}{10} \geq r.
\end{aligned}
$$

Finally, for $t>2$ we have

$$
|z-\tilde{\Gamma}_a^+(t)| \geq |\tilde{\Gamma}_a^+(t)|-|z| \geq 2-1 > \frac{9}{10} \geq r.
$$

Hence if $z_1 \leq 0$, we always have $\mu_a^+ B(z,r)=0$ and thus μ_a^+ is monotone at (z,r).

In the second case we have $z_1>0$. We can further suppose that $\mu_a^+ B(z,r)>0$, otherwise the proof is trivial. In this case one can easily check that $|z_2|>\frac{1}{40}$, the points $\xi,\tilde{\xi}\in S(z,r)\cap[\Gamma_a^+]$ are well defined, $|\xi|,|\tilde{\xi}|\in[\frac{1}{20},2]$ and $\underline{\mathrm{D}}_{\mathrm{r}}\,\mu_a^+ B(z,r)\geq 2$. Hence using (33) we arrive to the estimate

$$
\begin{aligned}
r &= \sqrt{\Big(\frac{|\xi-\tilde{\xi}|}{2}\Big)^2+\Big|z-\frac{\xi+\tilde{\xi}}{2}\Big|^2} \geq \frac{|\xi-\tilde{\xi}|}{2}\sqrt{1+\frac{4|z_2-\frac{\xi_2+\tilde{\xi}_2}{2}|^2}{(|\xi|+|\tilde{\xi}|)^2}} \\
&\geq \frac{|\xi-\tilde{\xi}|}{2}\sqrt{1+\frac{4(\frac{1}{40}-\frac{1}{80})^2}{4}} \geq \frac{|\xi-\tilde{\xi}|}{2}\sqrt{1+\frac{1}{a^2}} \geq \frac{|\xi|-|\tilde{\xi}|}{2}\sqrt{1+\frac{1}{a^2}}\,.
\end{aligned}
$$

Therefore we obtain from (7) and (10)

$$
\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{\mu_a^+ B(z,r)}{r} = \frac{\underline{\mathrm{D}}_{\mathrm{r}}\,\mu_a^+ B(z,r)r-\mu_a^+ B(z,r)}{r^2} \geq \frac{1}{r^2}\Big(2r-\sqrt{1+\frac{1}{a^2}}(|\xi|-|\tilde{\xi}|)\Big) \geq 0.
$$

□

Our next goal is to obtain the local monotonicity for the measure μ_a^+ (see Lemma 5.4). We start with the following auxiliary result.

Lemma 5.3. *There is $K_4>0$ such that for $a>K_4$ the function*

$$
\Phi_a(t)=\exp(at)\Big((a^2-1)\cos t+2a\sin t-(1+a^2)\Big)+(1+a^2)(1+\cos t)-2a^2
$$

satisfies $\Phi_a(t)\geq 0$ on $[0,\frac{1}{a}]$ and $\Phi_a(t)\leq 0$ on $[-\frac{12}{5a},0]$.

Proof: We have

$$
\begin{aligned}
\Phi_a'(t) &= \exp(at)\Big(a(1+a^2)\cos t+(1+a^2)\sin t-a(1+a^2)\Big)-(1+a^2)\sin t, \\
\Phi_a''(t) &= \exp(at)\Big((1+a^2)^2\cos t-a^2(1+a^2)\Big)-(1+a^2)\cos t.
\end{aligned}
$$

As $\Phi_a(0)=\Phi'_a(0)=0$ it is enough to find $K_4>0$ such that the function

$$\Psi_a(t)=\exp(at)\Big((1+a^2)\cos t-a^2\Big)-\cos t=\frac{1}{1+a^2}\Phi''_a(t)$$

satisfies the following inequalities for $a>K_4$

$$\Psi_a(t)\le 0 \quad \text{on} \quad \Big[-\frac{12}{5a},0\Big] \qquad \text{and} \qquad \Psi_a(t)\ge 0 \quad \text{on} \quad \Big[0,\frac{1}{a}\Big]. \tag{34}$$

First, there is $M_1>0$ such that $\cos t\ge 0$ for $a>M_1$ and $t\in[-\frac{12}{5a},0]$, and thus

$$\Psi_a(t)\le \exp(at)\Big((1+a^2)\cos t-a^2-\cos t\Big)=a^2\exp(at)(\cos t-1)\le 0.$$

This is the first inequality in (34). Let us prove the second one. There is $M_2>M_1$ such that for $a>M_2$ and $t\in[0,\frac{1}{a}]$ we have

$$\exp(at)\ge 1+at+\frac{a^2t^2}{2}, \tag{35}$$

$$\cos t\ge 1-\frac{t^2}{2} \tag{36}$$

and

$$\begin{aligned}(1+a^2)\cos t-a^2 &\ge (1+a^2)\Big(1-\frac{t^2}{2}\Big)-a^2\ge(1+a^2)\Big(1-\frac{1}{2a^2}\Big)-a^2\\ &=\frac{1}{2}-\frac{1}{2a^2}\ge 0.\end{aligned} \tag{37}$$

Using (35) and (37) we obtain

$$\begin{aligned}\Psi_a(t)&\ge\Big((1+a^2)\cos t-a^2\Big)\Big(1+at+\frac{a^2t^2}{2}\Big)-\cos t\\ &=a^2(\cos t-1)+\Big((1+a^2)\cos t-a^2\Big)\Big(at+\frac{a^2t^2}{2}\Big).\end{aligned}$$

Hence estimate (36) implies

$$\Psi_a(t)\ge -\frac{a^2t^2}{2}+\Big(1-\frac{(1+a^2)t^2}{2}\Big)\Big(at+\frac{a^2t^2}{2}\Big)=at\Big(1-\frac{(1+a^2)t^2}{2}-\frac{a(1+a^2)t^3}{4}\Big).$$

Finally, there is $K_4>M_2$ such that for $a>K_4$ and $t\in[0,\frac{1}{a}]$ we have

$$1-\frac{(1+a^2)t^2}{2}-\frac{a(1+a^2)t^3}{4}\ge 1-\frac{(1+a^2)}{2a^2}-\frac{(1+a^2)}{4a^2}=\frac{1}{4}-\frac{3}{4a^2}\ge 0$$

and thus $\Psi_a(t)\ge 0$ on $[0,\frac{1}{a}]$ for $a>K_4$. We have (34) and we are done. $\square$

Lemma 5.4. *There is $K_5 > 0$ such that if $a > K_5$, $0 < r \le \frac{9}{10}$, $z \in S(1)$ and $|z - (1,0)| \le \frac{1}{20}$, then μ_a^+ is monotone at (z, r).*

Proof: Suppose $\mu_a^+ B(r,z) > 0$, otherwise the proof is trivial. We find $\theta \in [\Gamma_a^+]$ such that $|z - \theta| = \operatorname{dist}(z, [\Gamma_a^+])$. As $(1,0) \in [\Gamma_a^+]$ and $|z-(1,0)| \le \frac{1}{20}$, we obtain

$$||\theta| - |z|| \le |\theta - z| \le |(1,0) - z| \le \frac{1}{20},$$

hence $|\theta| \in [\frac{19}{20}, \frac{21}{20}]$. Assumption $r \le \frac{9}{10}$ implies $|\tilde{\xi}|, |\xi| \in [\frac{1}{10}, \frac{19}{10}] \subset [\frac{|\theta|}{11}, 2|\theta|]$. Now we would like to parameterize a suitable part of $[\Gamma_a^+]$ as a graph of a function so that we could use Lemma 2.1.

Because of the self-similarity of the logarithmic spirals, our case is equivalent to the case with the nearest point $\theta_0 = \Gamma_a^+(t_0)$, where $t_0 = -\arctan(\frac{1}{a})$, the points of intersection $\tilde{\xi}_0, \xi_0$ satisfying $|\tilde{\xi}_0|, |\xi_0| \in [\frac{|\theta_0|}{11}, 2|\theta_0|]$ and the center z_0 on a line passing through $\Gamma_a^+(t_0)$ and perpendicular to $[\Gamma_a^+]$ at $\Gamma_a^+(t_0)$. Let r_0 denote the radius in this case. On some neighborhood of $\Gamma_a^+(t_0)$, the curve Γ_a^+ can be suitably represented by a graph of a function as shown in the sequel. Let us define

$$\begin{aligned} x(t) &= e^{a(t+t_0)} \cos(t+t_0) - e^{at_0} \cos(t_0), \quad t \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}), \\ y(t) &= e^{a(t+t_0)} \sin(t+t_0) - e^{at_0} \sin(t_0), \quad t \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \end{aligned}$$

The choice $t_0 = -\arctan(\frac{1}{a})$ implies $\cos t_0 = -a \sin t_0$,

$$\sin t_0 = -\sin(-t_0) = -\sqrt{\frac{\tan^2(-t_0)}{1+\tan^2(-t_0)}} = -\sqrt{\frac{\frac{1}{a^2}}{1+\frac{1}{a^2}}} = -\frac{1}{\sqrt{1+a^2}}, \tag{38}$$

$$\cos(t+t_0) = \cos t_0 \cos t - \sin t_0 \sin t = -\sin t_0 (a \cos t + \sin t) = \frac{a\cos t + \sin t}{\sqrt{1+a^2}} \tag{39}$$

and

$$\sin(t+t_0) = \sin t_0 \cos t + \cos t_0 \sin t = -\sin t_0 (a \sin t - \cos t) = \frac{a \sin t - \cos t}{\sqrt{1+a^2}}\,. \tag{40}$$

Hence

$$x'(t) = \frac{d}{dt}\Big(\frac{e^{at_0}}{\sqrt{1+a^2}}(e^{at}(a\cos t + \sin t))\Big) = \frac{e^{a(t+t_0)}}{\sqrt{1+a^2}}(1+a^2)\cos t$$

and

$$y'(t) = \frac{d}{dt}\Big(\frac{e^{at_0}}{\sqrt{1+a^2}}(e^{at}(a\sin t - \cos t))\Big) = \frac{e^{a(t+t_0)}}{\sqrt{1+a^2}}(1+a^2)\sin t.$$

Therefore we see that we can consider $x \mapsto y$ as a function $f : (x(-\frac{\pi}{2}), x(\frac{\pi}{2})) \mapsto \mathbb{R}$ with $f'(x) = \tan t$, where t is such that $x = x(t)$.

Next, let us show that the function f satisfies the assumptions of Lemma 2.1, the version from Remark 2.2. There is $\tilde{K} > 0$ large enough such that for $a > \tilde{K}$ we have $(-\frac{\pi}{2}, \frac{\pi}{2}) \supset [-\frac{12}{5a}, \frac{1}{a}]$ and $|f'| \leq \frac{1}{4}$ on $[x(-\frac{12}{5a}), x(\frac{1}{a})]$. Next, using $x(0) = 0$ and $f'(0) = f'(x(0)) = \tan 0 = 0$, one can easily check that the first coordinate of the center z_0 is the same as the first coordinate of the point $[x(0), f(x(0))]$.

It remains to check condition (11). For $t \in [-\frac{12}{5a}, \frac{1}{a}]$ let us define

$$\begin{aligned}
\Psi_a(t) &= 2|x(t)|\sqrt{1 + f'^2(x(t))} \\
&\qquad - (1 + \sqrt{1 + f'^2(x(t))})\mu_f(\{(u,v) : u \in I(0, x(t)), v \in \mathbb{R}\}) \\
&= \operatorname{sgn} t\Big(2(e^{a(t+t_0)}\cos(t + t_0) - e^{at_0}\cos t_0)\sqrt{1 + \tan^2 t} \\
&\qquad - (1 + \sqrt{1 + \tan^2 t})\sqrt{1 + \frac{1}{a^2}}(e^{a(t+t_0)} - e^{at_0})\Big),
\end{aligned}$$

where we have used (7). From (38), (39), (40), $\sqrt{1 + \tan^2 t} = \frac{1}{|\cos t|} = \frac{1}{\cos t}$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos t_0 = -a \sin t_0 = \frac{a}{\sqrt{1+a^2}}$ and $a\sqrt{1 + \frac{1}{a^2}}\sqrt{1 + a^2} = 1 + a^2$ we obtain

$$\begin{aligned}
\Psi_a(t) &= e^{at_0}\operatorname{sgn} t\Big(2\Big(e^{at}\frac{1}{\sqrt{1 + a^2}}(a\cos t + \sin t) - \frac{a}{\sqrt{1 + a^2}}\Big)\frac{1}{\cos t} \\
&\qquad\qquad - \Big(1 + \frac{1}{\cos t}\Big)\sqrt{1 + \frac{1}{a^2}}(e^{at} - 1)\Big) \\
&= \frac{e^{at_0}\operatorname{sgn} t}{a\sqrt{1 + a^2}\cos t}\Big(2ae^{at}(a\cos t + \sin t) - 2a^2 - (1 + \cos t)(1 + a^2)(e^{at} - 1)\Big) \\
&= \frac{e^{at_0}\operatorname{sgn} t}{a\sqrt{1 + a^2}\cos t}\Big(e^{at}(2a^2\cos t + 2a\sin t - (\cos t + 1)(1 + a^2)) \\
&\qquad\qquad - 2a^2 + (\cos t + 1)(1 + a^2)\Big) \\
&= \frac{e^{at_0}\operatorname{sgn} t}{a\sqrt{1 + a^2}\cos t}\Phi_a(t).
\end{aligned}$$

Hence Lemma 5.3 implies $\Psi_a(t) \geq 0$ on $[-\frac{12}{5a}, \frac{1}{a}]$ provided $a > K_5 = \max(K_4, \tilde{K})$. This proves inequality (11) on $[x(-\frac{12}{5a}), x(\frac{1}{a})]$. Further, we can see that the curve $(x(t), y(t)) + \Gamma_a^+(t_0)$, $t \in [-\frac{12}{5a}, \frac{1}{a}]$, parameterizes the set

$$M = \{\Gamma_a^+(t) : t \in [t_0 - \tfrac{12}{5a}, t_0 + \tfrac{1}{a}]\} = [\Gamma_a^+] \cap \Big\{x \in \mathbb{R}^2 : |x| \in \Big[e^{-\frac{12}{5}}|\theta_0|, e^1|\theta_0|\Big]\Big\}.$$

Hence, as $\exp(-\frac{12}{5}) < \frac{1}{11} \leq \frac{|\tilde{\xi}_0|}{|\theta_0|}$ and $\exp 1 > 2 \geq \frac{|\xi_0|}{|\theta_0|}$, we have $\xi_0, \tilde{\xi}_0 \in M$. Therefore Lemma 2.1 and Remark 2.2 imply that μ_a is monotone at (z_0, r_0). Thus, the self-similarity of logarithmic spirals gives that μ_a is monotone at (z, r). $\square$

Proof of Proposition 5.1: Let us recall that $\mu_a = \mu_a^+ + \mu_a^-$. The monotonicity at (z,r) for μ_a^+ follows from Lemma 5.2 and Lemma 5.4. Next, the symmetry between Γ_a^+ and Γ_a^- gives the same for μ_a^-. Finally, the super-additivity of the lower derivative $\underline{\mathrm{D}}_{\mathrm{r}}$ concludes the proof. $\square$

Proof of Theorem 1.3: Fix $K > \max(K_1, K_2)$ large enough so that

$$\sqrt{1+\frac{1}{K^2}} < 1+\varepsilon. \tag{41}$$

For $z \in S(1)$ and $r \geq \frac{9}{10}$, the monotonicity at (z,r) of $\mu_a + \mathcal{H}^1 \llcorner (L_1 \cup L_2)$ follows from Proposition 4.1. If $z \in S(1)$ and $r \leq \frac{9}{10}$, then we use Proposition 5.1. In any other case, with $z \neq (0,0)$, the monotonicity at (z,r) follows from above by the self-similarity of the logarithmic spirals. Finally, if $z = (0,0)$ and $r > 0$, then the monotonicity at (z,r) is easily obtained from (8) and Lemma 2.3. The non-unique tangential behavior follows from Proposition 3.1. The density properties easily follow from the definitions of Γ_a^+, Γ_a^-, L_1 and L_2, from (8) and from (41). $\square$

6. Necessity of compensation

In this section, we show that the measure μ_a is not monotone by itself for any $a > 0$. Further, since for any line L, the measure $\mathcal{H}^1 \llcorner L$ does not provide any compensation for balls centered on L (see (14) with $d = 0$), only one line cannot be a sufficient compensation for the monotonicity.

Proposition 6.1. *Assume $a > 0$ and $z \in \mathbb{R}^2 \setminus \{0\}$. Then there exists $r > 0$ such that $\varrho \mapsto \frac{\mu_a B(z,\varrho)}{\varrho}$ is decreasing on some neighborhood of r.*

Let us start with some preliminary work. In this section, ξ_j, η_j, etc. no longer denote the j-th coordinate of a point but the j-th member of a sequence.

Lemma 6.2. *Let $a > 0$ and $z \in S(1)$. Then there is $r_0 > 1$ with the following property:*
If $r \geq r_0$, then $S(z,r) \cap \operatorname{spt} \mu_a = \{\xi, \eta\}$, where $\xi \in S(z,r) \cap [\Gamma_a^+]$ and $\eta \in S(z,r) \cap [\Gamma_a^-]$, the functions $r \mapsto |\xi|$ and $r \mapsto |\eta|$ are increasing and continuously differentiable on (r_0, ∞) and satisfy (19), (21), (22) and (24).

Proof: The proof is similar to the proof of Lemma 4.6. $\square$

Lemma 6.3. *Assume $a > 0$, $z \in S(1)$. Then there is a sequence of radii $\{r_j\}_{j=1}^{\infty}$ such that $r_j \geq r_0$ ($r_0 > 1$ is given by Lemma 6.2), $r_j \to \infty$ and the points of intersection $\xi_j = \Gamma_a^+(\tau_j) \in S(z,r_j) \cap [\Gamma_a^+]$ and $\eta_j = \Gamma_a^-(\sigma_j) \in S(z,r_j) \cap [\Gamma_a^-]$ satisfy $\cos\psi_j = 1$ and $0 < \varphi_j - \psi_j \leq \frac{2}{a(r_j-1)}$ for all $j \in \mathbb{N}$.*

Proof: Applying Lemma 6.2 to $r = r_0$ we obtain a unique $t_0 \in \mathbb{R}$ such that $\Gamma_a^-(t_0) \in S(z,r) \cap [\Gamma_a^-]$. Given $j \in \mathbb{N}$, find $t_j \in [|t_0| + \frac{2}{a} + 2\pi(j-1), |t_0| + \frac{2}{a} + 2\pi j)$

such that $\cos(t_j - \vartheta) = 1$ and set $r_j = |\Gamma_a^-(t_j) - z|$. Thus $\Gamma_a^-(t_j) = -(r_j-1)z$ and

$$\begin{aligned} r_j &= |\Gamma_a^-(t_j) - z| \geq |\Gamma_a^-(t_j)| - |z| = e^{at_j} - 1 \\ &\geq e^{a|t_0|+2} - 1 \geq e^{a|t_0|} + 1 \geq |\Gamma_a^-(t_0)| + |z| \geq r_0. \end{aligned}$$

Lemma 6.2 gives that ξ_j and η_j are well defined, $\sigma_j = t_j$, $\eta_j = -(r_j-1)z$ and

$$|\xi_j| \geq |\xi_j - z| - |z| = r_j - 1 = |\eta_j|.$$

Hence $\tau_j \geq \sigma_j$. On the other hand $|\xi_j| \leq |\xi_j - z| + |z| = r_j + 1$. Therefore

$$\begin{aligned} &0 \leq \varphi_j - \psi_j = \tau_j - \sigma_j = \frac{1}{a}\ln|\xi_j| - \frac{1}{a}\ln|\eta_j| \leq \frac{1}{a}\ln\Big(\frac{r_j+1}{r_j-1}\Big) \\ &= \frac{1}{a}\ln\Big(1 + \frac{2}{r_j-1}\Big) \leq \frac{2}{a(r_j-1)}. \end{aligned}$$

Finally, if $\varphi_j = \psi_j$, we have $\tau_j = \sigma_j$. Thus $\xi_j = -\eta_j = (r_j-1)z$. Therefore

$$r_j = |\xi_j - z| = |(r_j-1)z - z| = |(r_j-2)z| = |r_j - 2|.$$

Hence from $r_j \geq r_0 > 1$ we obtain a contradiction. This implies $\varphi_j > \psi_j$ and we are done. □

Proof of Proposition 6.1: From the self-similarity of the logarithmic spirals we see that it is enough to consider $z = (\cos\vartheta, \sin\vartheta)$, $\vartheta \in [0, 2\pi)$, only. From Lemma 6.2 and (7) for any $r > r_0$ we have

$$\begin{aligned} \frac{\partial}{\partial r}\frac{\mu_a B(z,r)}{r} &= \frac{1}{r^2}\Big(r\frac{\partial}{\partial r}\mu_a B(z,r) - \mu_a B(z,r)\Big) \\ &= \frac{1}{r^2}\sqrt{1+\frac{1}{a^2}}\Big(\frac{\partial|\xi|}{\partial r}r - |\xi| + \frac{\partial|\eta|}{\partial r}r - |\eta|\Big). \end{aligned} \tag{42}$$

From (19), (21), (22), (24), (42) and $\cos\psi_j = 1$ we obtain

$$\frac{\partial}{\partial r}\frac{\mu_a B(z,r)}{r}\Big|_{r=r_j} = \frac{1}{r_j^2}\frac{\sqrt{1+\frac{1}{a^2}}}{\sqrt{\cos^2\varphi_j + r_j^2 - 1} + \frac{\sin\varphi_j}{a}}\Phi(r_j),$$

where

$$\begin{aligned}\Phi(r_j) &= \Big(\sqrt{\cos^2\varphi_j + r_j^2 - 1} + \frac{\sin\varphi_j}{a}\Big)\Big(\frac{r_j^2}{\sqrt{\cos^2\varphi_j + r_j^2 - 1} + \frac{\sin\varphi_j}{a}}\\
&\qquad - \big(\cos\varphi_j + \sqrt{\cos^2\varphi_j + r_j^2 - 1}\big) - r_j + (-1 + r_j)\Big)\\
&= r_j^2 + \Big(1 - \cos\varphi_j - \sqrt{\cos^2\varphi_j + r_j^2 - 1}\Big)\Big(\sqrt{\cos^2\varphi_j + r_j^2 - 1} + \frac{\sin\varphi_j}{a}\Big)\\
&= r_j^2 + (1 - \cos\varphi_j)\sqrt{\cos^2\varphi_j + r_j^2 - 1} + (1 - \cos\varphi_j)\frac{\sin\varphi_j}{a} - \cos^2\varphi_j - r_j^2\\
&\qquad + 1 - \sqrt{\cos^2\varphi_j + r_j^2 - 1}\,\frac{\sin\varphi_j}{a}\,.\end{aligned}$$

Hence it is enough to show that $\Phi(r_j) < 0$ for j large enough. As $r_j - 1 \le \sqrt{\cos^2\varphi_j + r_j^2 - 1} \le r_j$, $1 - \cos^2\varphi_j = (1 + \cos\varphi_j)(1 - \cos\varphi_j) \le 2(1 - \cos\varphi_j)$ and $\sin\varphi_j \ge 0$ for j large, we obtain

$$\begin{aligned}\Phi(r_j) &\le (1 - \cos\varphi_j)r_j + \frac{\sin\varphi_j}{a} + 2(1 - \cos\varphi_j) - (r_j - 1)\frac{\sin\varphi_j}{a}\\
&= (1 - \cos\varphi_j)(r_j + 2) - (r_j - 2)\frac{\sin\varphi_j}{a}\,.\end{aligned}$$

Recall that we have $\cos\psi_j = 1$ by Lemma 6.3. Hence ψ_j is a multiple of 2π. Moreover, as $1 - \cos t = 2\sin^2(\frac{t}{2})$, we have

$$1 - \cos\varphi_j = 1 - \cos(\varphi_j - \psi_j) = 2\sin^2\Big(\frac{\varphi_j - \psi_j}{2}\Big).$$

Thus from $\varphi_j - \psi_j \in (0, \frac{2}{a(r_j - 1)}]$ (which comes from Lemma 6.3) and $\frac{t}{2} \le \sin t \le t$ on $[0, \frac{\pi}{2}]$ we obtain for r_j sufficiently large

$$\begin{aligned}\Phi(r_j) &\le 2(r_j + 2)\sin^2\Big(\frac{\varphi_j - \psi_j}{2}\Big) - (r_j - 2)\frac{\sin(\varphi_j - \psi_j)}{a}\\
&\le 4r_j\Big(\frac{\varphi_j - \psi_j}{2}\Big)^2 - \frac{r_j}{2}\frac{\varphi_j - \psi_j}{2a} = r_j(\varphi_j - \psi_j)\Big(\varphi_j - \psi_j - \frac{1}{4a}\Big)\\
&\le r_j(\varphi_j - \psi_j)\Big(\frac{2}{a(r_j - 1)} - \frac{1}{4a}\Big) < 0.\end{aligned}$$

Thus $\frac{\partial}{\partial r}\frac{\mu_a B(z,r)}{r}$ is negative on some neighborhood of r_j for j large enough. □

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