On Manes' countably compact, countably tight, non-compact spaces

James Dabbs

Abstract. We give a straightforward topological description of a class of spaces that are separable, countably compact, countably tight and Urysohn, but not compact or sequential. We then show that this is the same class of spaces constructed by Manes [*Monads in topology*, Topology Appl. **157** (2010), 961–989] using a category-theoretical framework.

Keywords: countably compact, countably tight, p-compact, p-sequential

Classification: 54D30, 54A10

1. Introduction

In [2], Nyikos asked several questions related to the existence of separable, countably compact, countably tight spaces that are not compact. In [3], Nyikos and Vaughn constructed a Hausdorff such a space in ZFC, but their example was not Urysohn[1] and so certainly not regular. Dow [4] constructed a compact, Hausdorff, non-sequential such space under the added hypothesis that $2^{\omega} = 2^{\omega_1}$. Manes [1] later constructed a class of separable, Urysohn, countably compact, countably tight, non-compact, non-sequential spaces in ZFC, using the category-theoretical concept of a monad. The question of whether or not a regular such space exists in ZFC remains open. Our aim is to give a purely topological construction of the class of spaces studied by Manes.

2. Preliminaries

Throughout, p will denote an arbitrary element of $\beta\omega \setminus \omega$.

Definition 1. Given a sequence x_n in a space X, $x = p\text{-}\lim x_n$ if for every open $O \subset X$ with $x \in O$, $\{n \mid x_n \in O\} \in p$.

p-limits provide a natural way to generalize sequential properties. The following definitions are well known (see [5] for more details):

Definition 2. A space X is p-compact if for every sequence x_n in X, $p\text{-}\lim x_n$ exists and is in X.

[1] A space is Urysohn if any two distinct points have neighborhoods with disjoint closures.

Definition 3. A space X is p-sequential if for every non-closed $A \subset X$ there is some $x \in X \setminus A$ and sequence x_n in A with $x = p\text{-}\lim x_n$.

It is routine to verify [5] that every compact space is p-compact for any p and that every p-compact space is limit-point compact (and thus countably compact, provided it is T_1). It can also be shown that any p-sequential space is countably tight using a straightforward variation of the usual proof that every sequential space is countably tight, included here for completeness:

Proposition 2.1. *Any p-sequential space is countably tight.*

Proof: Let X be p-sequential, $A \subset X$ and $x \in \overline{A} \setminus A$.

Define $A' = \{y \in X \mid y = p\text{-}\lim a_n \text{ for some sequence } a_n \text{ in } A\}$. Let $A_0 = A$ and $A_1 = A'$, $A_{\alpha+1} = A'_\alpha$ for successor ordinals $< \omega_1$, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for limit ordinals $\leq \omega_1$.

If x_n is a sequence in A_{ω_1} then $\{x_n\} \subset A_\alpha$ for some least α and so $p\text{-}\lim x_n \in A_{\alpha+1} \subset A_{\omega_1}$. Thus A_{ω_1} is p-sequentially closed and, since X is p-sequential, closed.

Thus $x \in A_{\omega_1}$. We claim that for any $y \in A_{\omega_1}$, there is a countable $B \subset A$ so that $y \in \overline{B}$. If $y \in A_1$ then there is a sequence x_n in $A_0 = A$ so that $y = p\text{-}\lim x_n$ and so $y \in \overline{\{x_n\}}$. Suppose that for every $\beta < \alpha$ and $y \in A_\beta$ there is a countable $B_y \subset A$ with $y \in \overline{B_y}$. Let $y \in A_\alpha$. If $y \in A_\beta$ for some $\beta < \alpha$ then there is such a B_y by assumption. So suppose not. Then by construction α must be a successor ordinal $\alpha = \gamma + 1$ so $y = p\text{-}\lim x_n$ for a sequence x_n in A_γ. Then if $B_y = \bigcup_{n\in\omega} B_{x_n}$, $y \in \overline{B_y}$ as required. □

3. Construction

We will show that the set of p-compact subsets of any space is closed under arbitrary intersections and finite unions. Thus the p-compact subsets of a space form the closed sets of a new topology on that space. The example we seek will be a topology generated by the p-compact subsets of $\beta\omega$.

First, a useful lemma:

Lemma 3.1. *If x_n, x'_n are two sequences in X and $\{n \mid x_n = x'_n\} \in p$, then $p\text{-}\lim x_n = p\text{-}\lim x'_n$ (if either limit exists).*

Proof: Let $\{n \mid x_n = x'_n\} = B \in p$. Suppose without loss of generality that $p\text{-}\lim x_n = x$. Then for any open $O \subset X$ with $x \in O$, $\{n \mid x_n \in O\} \in p$. Thus $\{n \mid x'_n \in O\} \supset \{n \mid x'_n = x_n \wedge x_n \in O\} = B \cap \{n \mid x_n \in O\} \in p$. So $x = p\text{-}\lim x'_n$. □

Note that the p-compactness is preserved by finite unions and arbitrary intersections:

Proposition 3.2. *If $C_1, C_2 \subset X$ are p-compact, $C_1 \cup C_2$ is p-compact.*

Proof: Let x_n be a sequence in $C_1 \cup C_2$. Let $B_i = \{n \mid x_n \in C_i\}$. Since p is an ultrafilter, assume without loss of generality that $B_1 \in p$. Fix an arbitrary

$z \in C_1$ and let $x'_n = x_n$ for all $n \in B_1$ and $x'_n = z$ otherwise. Then x'_n is a sequence in C_1. Let $x = p\text{-}\lim x'_n$. Then x exists and $x \in C_1$ by assumption and $x = p\text{-}\lim x_n$ by 3.1. Thus $p\text{-}\lim x_n \in C_1 \cup C_2$. $\square$

Proposition 3.3. *If $C_i \subset X$ is p-compact for each i, $\bigcap C_i$ is p-compact.*

PROOF: If x_n is a sequence in $\bigcap C_i$, then x_n is a sequence in C_i for each i so $p\text{-}\lim x_n \in C_i$ for each i. $\square$

Definition 4. Given a space X and an ultrafilter p, let $X_p = X$ as a set and define a topology on X_p by letting all p-compact subsets of X be closed in X_p (along with X if X was not p-compact).

Now, consider the relationship between the topologies X and X_p:

Proposition 3.4. *If X is p-compact and $C \subset X$ is closed, then C is p-compact.*

PROOF: If x_n is a sequence in C then $x = p\text{-}\lim x_n$ exists in X and for every open $O \subset \beta\omega$ with $x \in O$, $\{n \mid x_n \in O\} \in p$. In particular, $O \cap C$ is non-empty and $x \in \overline{C} = C$. Thus C is p-compact. $\square$

Corollary 3.5. *If X is p-compact, the topology on X_p is finer than the usual topology on X.*

Proposition 3.6. *Let x_n be a sequence in X_p. If $p\text{-}\lim_X x_n$ exists, then $p\text{-}\lim_{X_p} x_n = p\text{-}\lim_X x_n$.*

PROOF: Let $x = p\text{-}\lim_X x_n$. Suppose $x \neq p\text{-}\lim_{X_p} x_n$. Then there is an $O \subset X_p$ open in X_p with $x \in O$ but $\{n \mid x_n \in O\} \notin p$. Thus $\{n \mid x_n \notin O\} \in p$. Fix a $z \in X_p \setminus O$. Define $x'_n = x_n$ if $x_n \in X_p \setminus O$ and $x'_n = z$ otherwise. Then x'_n is a sequence in $X_p \setminus O$ and since O is open, $X_p \setminus O$ is closed in X_p and thus p-compact in X. Thus $p\text{-}\lim_X x'_n \in X \setminus O$. But $x_n = x'_n$ for all $n \in \{n \mid x_n \notin O\}$ so by 3.1, $p\text{-}\lim_X x_n = p\text{-}\lim_X x'_n \notin O$, a contradiction. $\square$

Corollary 3.7. *If X is p-compact then X_p is p-compact.*

Proposition 3.8. *If $A \subset X_p$, let $A' = \{x \mid x = p\text{-}\lim a_n$ for some sequence a_n in $A\}$ and define A_α inductively by $A_0 = A$, $A_{\alpha+1} = A'_\alpha$ for successor ordinals and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for limit ordinals. Then $\mathrm{cl}_{X_p}(A) = A_{\omega_1}$.*

PROOF: By definition,

$$\begin{aligned}\mathrm{cl}_{X_p}(A) &= \bigcap\{C \subset X : C \ \text{ closed in } \ X_p,\ A \subset C\}\\ &= \bigcap\{C \subset X : C\ p\text{-compact in } \ X,\ A \subset C\}.\end{aligned}$$

Note that if x_n is a sequence in A_{ω_1} then by construction there is some $\alpha < \omega_1$ so that x_n is a sequence in A_α. Thus $p\text{-}\lim x_n \in A_{\alpha+1}$ and so A_{ω_1} is p-compact. Thus $\mathrm{cl}_{X_p}(A) \subset A_{\omega_1}$.

Conversely, suppose $A_\beta \subset \mathrm{cl}_{X_p}(A)$ for all $\beta < \alpha$. If α is a limit ordinal then $A_\alpha \subset \mathrm{cl}_{X_p}(A)$ trivially. If not, then $\alpha = \gamma + 1$ and for any $a \in A_\alpha$, $a = p\text{-}\lim x_n = p\text{-}\lim_{X_p} x_n$ for some sequence x_n in A_γ. Thus for any O open in X_p

with $a \in O$, $\{n \mid x_n \in O\} \in p$ so $A_\gamma \cap O$ is infinite and $a \in \mathrm{cl}_{X_p}(A)$. Thus $A_{\omega_1} \subset \mathrm{cl}_{X_p}(A)$. □

Corollary 3.9. $|\,\mathrm{cl}_{X_p}(A)| \leq |A|^\omega$.

Proposition 3.10. *If X is p-compact, then X_p is p-sequential.*

PROOF: Let $A \subset X_p$ be non-closed. Then $A \subset X$ is not p-compact. So by definition there is a sequence x_n in A so that $x = p\text{-}\lim x_n \notin A$. □

The particular example we seek is obtained by applying this construction to $\beta\omega$:

Proposition 3.11. *$\beta\omega_p$ is Urysohn.*

PROOF: By 3.5, the topology on $\beta\omega_p$ is finer than the topology on $\beta\omega$ and since $\beta\omega$ is Urysohn, so is $\beta\omega_p$. □

Proposition 3.12. *$\beta\omega_p$ is not compact.*

PROOF: $\beta\omega_p$ is countably tight by 3.10 and 2.1, so $\beta\omega_p \not\cong \beta\omega$. Thus the topology on $\beta\omega_p$ is strictly finer and so $\beta\omega_p$ cannot be compact. □

Proposition 3.13. *$\beta\omega_p$ contains no non-trivial convergent sequences.*

PROOF: Since $\beta\omega_p$ has a finer topology, the inclusion map $i : \beta\omega_p \to \beta\omega$ is continuous. Thus if $x_n \to x$ in $\beta\omega_p$, $i(x_n) \to i(x)$ in $\beta\omega$ and thus x_n is eventually constant. □

Corollary 3.14. *$\beta\omega_p$ is not sequential.*

PROOF: If $F \subset \beta\omega_p$ and x_n is a sequence in F with $x_n \to x$ then $x = x_m$ for some m so $x \in F$. Thus every subset of $\beta\omega_p$ is sequentially closed. But $\beta\omega_p$ is not discrete ($\omega \subset \beta\omega_p$ is not closed), so $\beta\omega_p$ is not sequential. □

Thus $\beta\omega_p$ is a countably compact, countably tight, Urysohn, non-compact, non-sequential space. It is not separable though: by 3.9, $|\,\mathrm{cl}_{\beta\omega_p}(A)| \leq |A|^\omega < |\beta\omega|$ for any countable A. However:

Proposition 3.15. *If $A \subset \beta\omega_p$ is countable, $X = \mathrm{cl}_{\beta\omega_p}(A)$ is a separable, Urysohn, countably compact, countably tight, non-compact, non-sequential subspace of $\beta\omega_p$.*

PROOF: Separability is trivial. Since X is a closed subset of $\beta\omega_p$, X is Urysohn, countably compact and countably tight. X is not discrete so as in 3.14, X is not sequential.

If X is compact then $i(X) \subset \beta\omega$ is a compact and thus closed subset of $\beta\omega$ and so $i(X)$ contains a homeomorphic copy of $\beta\omega$. Since $\beta\omega$ contains weak P-points [6], $i(X)$ is not countably tight. But i is a homeomorphism onto its image and X is countably tight, a contradiction. Thus X is not compact. □

4. Monads

In [1], Manes defines a monad as a triple $(T, \eta, (\cdot)^{\#})$ where T is a functor from the category of sets to itself, $\eta_X : X \to TX$ for all sets X and if $f : X \to TY$ then $f^{\#} : TX \to TY$ subject to the conditions

(1) $f^{\#}\eta_X = f$,
(2) $(\eta_X)^{\#} = \mathrm{id}_{TX}$,
(3) $(g^{\#}f)^{\#} = g^{\#}f^{\#}$ for any set $f : X \to TY$ and $g : Y \to TZ$.

The prototypical example of a monad is the Stone-Čech compactification $(\beta, \eta, (\cdot)^{\#})$ where $\eta_X : X \to \beta X$ is the usual inclusion and $f^{\#} : \beta X \to \beta Y$ by

$$f^{\#}(\mathcal{F}) = \{B \subset Y \mid \{x \in X \mid B \in f(x)\} \in \mathcal{F}\}.$$

It is straightforward to verify that this definition of a monad satisfies the listed properties and that this definition is equivalent to the standard definition [7] in terms of (T, η, μ) by letting $\mu_X = (\mathrm{id}_{TX})^{\#} : TTX \to TX$. A subfunctor $T \subset \beta$ will generate a submonad $(T, \eta, (\cdot)^{\#})$ of $(\beta, \eta, (\cdot)^{\#})$ provided that for all sets X, $\eta_X(X) \subset TX$ and for all maps $f : X \to \beta Y$ with $f(X) \subset TY$, $f^{\#}(TX) \subset TY$.

Given a function $f : X \to Y$, the functor β takes f to the induced map $\beta f : \beta X \to \beta Y$. Given an ultrafilter $r \in \beta X$, we let fr denote the ultrafilter $(\beta f)(r) = \{Z \subset Y \mid f^{-1}(Z) \in r\}$.

For a fixed ultrafilter $r \in \omega^*$, Manes considers the subfunctor

$$G_r X = \{fr \mid f : \omega \to X\} \subset \beta X$$

and the monad T_r generated by G_r (i.e., the smallest submonad of β so that $G_r X \subset T_r X$ for all sets X). Note first that $G_r\omega$ is a familiar object:

Proposition 4.1. *Using the notation from* 3.8, $G_r\omega = \omega'$.

PROOF: Let $f : \omega \to \omega$. Then for any basic open set $\overline{O} \subset \beta\omega$ with $fr \in \overline{O}$, $O \in fr$. Thus by definition, $f^{-1}(O) = \{n \mid f(n) \in O\} \in r$. Thus $fr = r\text{-}\lim f(n)$. □

To describe $T_r\omega$, we observe the following: since T_r is a subfunctor of β, given $f : A \to T_r\omega$, the function $T_r f : T_r A \to T_r(T_r\omega)$ is the restriction of βf to $T_r A$. Also since $(T_r, \eta, (\cdot)^{\sharp})$ is a submonad of $(\beta, \eta, (\cdot)^{\sharp})$, if $h : T_r\omega \to T_r\omega$ and $\mathcal{F} \in T_r(T_r\omega)$, then $h^{\sharp}(\mathcal{F}) = \{D \subset \omega \mid \{x \in T_r\omega \mid D \in h(x)\} \in \mathcal{F}\}$.

Proposition 4.2. *For any $A \subset T_r\omega$ and $g : \omega \to A$, if $i : A \to T_r\omega$ is the inclusion map,* $\mathrm{id}^{\#}_{T_r\omega}\big((T_r i)(gr)\big) = r\text{-}\lim g(n)$ *(with limit taken in $\beta\omega$).*

PROOF: By definition,

$$\mathrm{id}^{\#}_{T_r\omega}\big((T_r i)(gr)\big) = \{D \subset \omega \mid \{x \in T_r\omega \mid D \in x\} \in (T_r i)(gr)\}$$

and

$$\{x \in T_r\omega \mid D \in x\} \in T_r i(gr) \iff$$
$$\exists C \in gr(C \subset \{x \in T_r\omega \mid D \in x\}) \iff$$

$$\exists C \in gr(\forall y \in C(y \in \{x \in T_r\omega \mid D \in x\})) \iff$$

$$\exists C \in gr(\forall y \in C(D \in y)) \iff$$

$$\exists B \in r \wedge \exists C \supset g(B)(\forall y \in C(D \in y)) \iff$$

$$\exists B \in r(\forall y \in g(B)(D \in y)).$$

The last equivalence following from taking $C = g(B)$. Thus $D \in \mathrm{id}^{\#}_{T_r\omega}\big((T_r i)(gr)\big)$ $\iff$ $\exists B \in r$ with $D \in \bigcap g(B)$, and so $\mathrm{id}^{\#}_{T_r\omega}\big((T_r i)(gr)\big) = \bigcup_{B\in r} \cap g(B)$.

Given any basic open $\overline{O} \subset \beta\omega$ containing $\bigcup_{B\in r} \cap g(B)$, $O \in \bigcup_{B\in r} \cap g(B)$ so there is some $B \in r$ so that $O \in x$ for every $x \in g(B)$. Thus $B \subset \{n \mid O \in g(n)\} = \{n \mid g(n) \in \overline{O}\}$ and so $\{n \mid g(n) \in \overline{O}\} \in r$ and $\bigcup_{B\in r} \cap g(B) = r\text{-}\lim g(n)$ as required. □

Corollary 4.3. *For any $g : \omega \to T_r\omega$, $\mathrm{id}^{\#}_{T_r\omega}(gr) = r\text{-}\lim g(n)$.*

Proof: Since $T_r(\mathrm{id}_{T_r\omega}) = \mathrm{id}_{T_rT_r\omega}$. □

Proposition 4.4. *As a set, $T_r\omega = \mathrm{cl}_{\beta\omega_r}(\omega)$.*

Proof: Since T_r is a monad and $\mathrm{id}_{T_r\omega} : T_r\omega \to T_r\omega$, $\mathrm{id}^{\#}_{T_r\omega}(T_rT_r\omega) \subset T_r\omega$. For any sequence x_n in $T_r\omega$, let $g : \omega \to T_r\omega$ by $n \mapsto x_n$, so $gr \in T_rT_r\omega$ and $\mathrm{id}^{\#}_{T_r\omega}(gr) = r\text{-}\lim x_n \in T_r\omega$. Thus $T_r\omega$ is r-compact and $T_r\omega \supset \mathrm{cl}_{\beta\omega_r}(\omega)$.

On the other hand, taking $A_0 = \omega$ and letting A_α be defined as in 3.8, $A_1 = A' = G_r\omega \subset T_r\omega$ by definition. If $A_\beta \subset T_r\omega$ for all $\beta < \alpha$, then if α is a limit ordinal, $A_\alpha \subset T_r\omega$ trivially. If not, let $\alpha = \gamma + 1$ and $x \in A_\alpha$. Then there is a sequence x_n in A_γ with $x = r\text{-}\lim x_n$. Let $g : \omega \to A_\gamma \subset T_r\omega$ by $n \mapsto x_n$. Then $x = r\text{-}\lim x_n = \mathrm{id}^{\#}_{T_r\omega}(gr) \in T_r\omega$. Thus $A_\alpha \subset T_r\omega$ for all $\alpha \le \omega_1$. □

Definition 5. Following [1], a subset $A \subset T_r\omega$ is closed if it is a subalgebra, that is, if there is a map ξ_0 rendering the following diagram commutative:

$$\begin{tikzcd} T_rA \arrow[r, "T_ri"] \arrow[d, "\xi_0"] & T_rT_r\omega \arrow[d, "\mathrm{id}^{\#}_{T_r\omega}"] \\ A \arrow[r, "i"] & T_r\omega \end{tikzcd}$$

Proposition 4.5. *The topologies on $\mathrm{cl}_{\beta\omega_r}(\omega)$ and $T_r\omega$ coincide.*

Proof: Such a ξ_0 will exist only if $\mathrm{id}^{\#}_{T_r\omega}\big((T_ri)(T_rA)\big) \subset A$, but if this is true, then $\forall gr \in T_rA$, $\mathrm{id}^{\#}_{T_r\omega}\big((T_ri)(gr)\big) = r\text{-}\lim g(n) \in A$. Thus A is closed if and only if it is r-compact. □

References

[1] Manes E., *Monads in topology*, Topology Appl. **157** (2010), 961–989.
[2] Nyikos P., *Classic problems* - 25 *years later* (*part* 2), Topology Proc. **27** (2003), 365–378.
[3] Nyikos P., Vaughn J., *The Scarborough-Stone problem for Hausdorff spaces*, Topology Appl. **44** (1992), 309–316.
[4] Dow A., *A countably compact, countably tight, non-sequential space*, Proceedings of the 1988 Northeast Conference on General Topology and Applications, Dekker, New York, 1990, pp. 71–80.
[5] Kombarov A., *Compactness and sequentiallity with respect to a set of ultrafilters*, Moscow Univ. Math. Bull. **40** (1985), 15–18.
[6] van Mill J., *An introduction to $\beta\omega$*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 503–567.
[7] Mac Lane S., *Categories for the working mathematician*, Graduate Texts in Mathematics, 5, Springer, New York-Berlin, 1971.

Department of Mathematics, Auburn University, Auburn, AL 36830, USA
E-mail: jcd0013@auburn.edu

(*Received* September 21, 2010, *revised* July 10, 2011)